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RALPH D. JAMES, *Editor*

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THE TEACHING OF CONCRETE MATHEMATICS

JOHN W. TUKEY, Princeton University and Bell Telephone Laboratories, Inc.

1. Introduction. One syndrome must, from time to time, disturb the sleep of all concerned with the applications of mathematics,—a syndrome never discussed in open meeting, perhaps because of its sensitivity. It seems to be generally agreed that "applied mathematics" is more difficult than "pure mathematics" in requiring more maturity and more years of study before useful results are attained. Today's leaders in "applied mathematics" were mainly trained in "pure mathematics." Yet from the point of view of *research potential and related intellectual ability* the students who *study* in "applied" fields do not compare in strength with those who go into "pure" mathematics! Is this not a paradoxical situation?

One can try to make the situation appear less paradoxical by going further, and asserting that: "Just as today's leaders in the applied fields have come mainly by conversion from the pure, so too will tomorrow's!" (A statement which is undoubtedly true for tomorrow!) But what of the day after tomorrow? Should conversion be inevitable? If it is not, as the writer believes, then the answer must lie in the early training of our students.

Two causes deflect students from the "applied" to the "pure" today:

(1) a feeling among teachers that the "applied" is beneath the "pure,"
(2) a failure to present the "applied" so that it is as intellectually stimulating as the "pure."

Given the temperaments and intellectual orientations of collegiate teachers of mathematics, it is clear that (1) is an inevitable consequence of (2) and that direct (or slanting) attacks on (1) are useless. To improve the situation we must deal with (2), when (1) will, more or less slowly, take care of itself.

How then, may we make "applied" or "concrete" mathematics more stimulating? Many ideas may be needed in the long run, but here are some which appeal strongly to the writer:

(a) we may strive to develop the areas of formulation and approximation, where applied mathematics has failed to heed the admonition "physician, heal thyself."

Success here could give us something of a truly mathematical nature worthy of being taught as applied mathematics. At the best, this is a long-range program (and some may term it visionary)—it is discussed briefly in Section 7.

(b) we may introduce more generality into each stage of the teaching of concrete mathematics—for example, after meeting one or two expansions into eigenfunctions, we may give a nonrigorous introduction to eigenfunction expansions in general.

This program could begin tomorrow, or even today, and needs no detailed spell-

ing out here. It will only succeed, however, if it is focussed on concepts rather than rigor. Usually the physicist or other user will not only omit all rigor in his justified haste to treat a particular point, but he will omit or gloss over many concepts of general interest and help. It is for the mathematician to introduce the student to the majority of the concepts first, letting the rigor wait till its appropriate time. New concepts should be injected into the student as gently as possible!

(c) we may emphasize the study of computational procedures in their own right by discussing their general properties—the mathematics of computation—rather than merely grinding through them.

All of these changes require teaching time and student's time. If we are realistic, we must find time, or at any rate most of it, by saving it elsewhere. Where is time now occupied? With the mechanics of computation, numerical and algebraic. How can it be freed? By reorienting our attitude toward computation—by trying to make it less of a road block.

The key to the immediate attack, then, lies in our attitude and practices concerning computation, taken in a most general sense, and its techniques.

As we succeed with such a program, shifting emphasis from avoidable labor of computation to broader concepts on the one hand and the mathematics of computation on the other, we shall be teaching better 20th-century mathematics—better for mathematicians to teach—and better for students to learn.

2. Attitudes toward computation. Computation may be numerical or "algebraic" where the latter term seems in practice to cover all forms of systematic manipulation which are not merely numerical—polynomials, trigonometric functions, indefinite integration, summation (not summability) of series, tensors and logic, to name a few, all have algebras in the sense of orderly procedures of computation. In their essentials, the practices of these diverse forms of computation are the same. Given input data, one performs more-or-less-or-much-less routine operations with the intent of reaching output results of a predetermined form. Interest centers in the certainty, efficiency, and ease of manipulation of the operations. A certain amount of practice is useful, both to promote understanding (which is not helped appreciably by still more extended practice) and to provide a little facility of manipulation (usually a little suffices).

Numerical computation, through the centuries, has often faced up to reality and made things easier. The use of logarithmic tables, even by those who do not know how to recompute them, and of desk calculators and, now, electronic calculators, even by those who cannot repair them, has been a commonplace. Today the "software" comprising the carefully planned interpretive routines, compilers, and other aspects of automative programming are at least as important to the modern electronic calculator as its "hardware" of tubes, transistors, wires, tapes and the like. When a student or a user begins to use an electronic calculator, we do not ask him to learn all the details of the automatic programming—and surely not to learn why these details were chosen instead of

others. A few students and users will develop slowly into designers or programmers, but their number will be few and their treatment special. Let us look to the analogy in all forms of computation.

Throughout computation, since the usual student will not compute steadily, but rather occasionally, even if he continues to compute all his life, emphasis is best laid on methods which are easy to remember—and, more importantly still, easy to relearn when forgotten. We are really concerned with the continuing capabilities of the student, including those which require supplementation by some re-study when used, rather than with student behavior in a final examination.

There are some students (but how few!) who will go on to compute steadily. They require special training, but their needs should not prejudice the training of the larger student body. We do the specialists no injustice to teach them the easy-to-remember way first, even though it may take 10%, or 20%, or 50%, or even 100% longer than the fanciest method when in steady use. This will not dull their interest in, and appreciation of, the fast, hard-to-remember methods. But if we teach the hard-to-remember method first, the occasional computer will never get to the easy-to-remember method, never have a method he can use when he meets a real problem, and thus never solve the problem.

3. An example from numerical computation. But, some may say, teaching easy-to-remember methods means teaching technique, and ideas and concepts will suffer. This is not so. Let us take Aitken's method of interpolation, [**3**], [**4**], as an example. The basic problem is to pass a polynomial through given points (x_1, y_1), (x_2, y_2), (x_3, y_3), and so on. If $P(x)$ with numerical subscripts represents a polynomial passing through the points indicated by the subscripts (thus $P_{124}(x)$ passes through points 1, 2 and 4), and if "$-$" stands for any collection of subscripts other than i or j then

$$[(x - x_i)P_{-j}(x) + (x_j - x)P_{-i}(x)]/(x_j - x_i)$$

passes through all the points corresponding to "$-$" and through points i and j. The argument which shows this is simple, direct, and truly mathematical. Hence we may define

$$(x_j - x_i)P_{-ij}(x) = (x - x_i)P_{-j}(x) + (x_j - x)P_{-i}(x)$$

and starting with

$$\begin{aligned} P_g(x) &= y_g, \\ P_{gh}(x) &= [(x - x_g)y_h + (x_h - x)y_g]/(x_h - x_g) \\ &= \frac{y_h - y_g}{x_h - x_g}\, x + \frac{x_h y_g - x_g y_h}{x_h - x_g}, \end{aligned}$$

obtain all the interpolating polynomials we may desire.

If all we wish is the value of the interpolating polynomial at a given x, then the process reduces itself to successive linear interpolations; thus, for $x=6.1$ and a particular array of x_i and y_i, we have:

i	x_i	x_i-x	$y_i=p_i$	p_{1i}	p_{12i}	p_{123i}	p_{1234i}
1	7	0.9	.84510				
2	5	−1.1	.69897	.77935			
3	8	1.9	.90309	.79391	.78469		
4	4	−2.1	.60206	.77219	.78723	.78580	
5	10	3.9	1.0000	.79863	.78359	.78573	.78578

Every computation here is straightforward linear interpolation or extrapolation. Thus

$$.77935 = \frac{(0.9)(.69897) + (1.1)(.84510)}{2.0},$$

$$.78723 = \frac{(-1.1)(.77219) + (2.1)(.77935)}{1.0},$$

and so on. Undoubtedly this is the easiest of all polynomial interpolation schemes to learn or relearn. (Even though, in the hands of the professional computer, it may be a little slow by comparison with some other schemes, its nearly-iterative and checking features are quite valuable.)

It is easy to learn, yet its teaching is not mainly teaching technique. What has to be taught, in order that its functioning be understood is not technique, but rather (i) that there will be an interpolating polynomial, (ii) that linear interpolation between two equal values returns the same value, (iii) that linear interpolation at an endpoint returns the given value. The algorithm for the interpolating polynomial now follows, and from it the numerical algorithm. After learning a few things of mathematical content (he may even be led to try other operations in place of linear interpolation) the student is equipped to do polynomial interpolation of any order, direct or inverse (for no properties of the spacing of the x_i were used), without the need to recall or look up any coefficients.

The more computation that we can teach in such a form, the better—both for applied mathematics and for pure mathematics.

4. An example from algebraic computation. Formal integration is another example of computation. It points a road we should travel, quite a different road from that just indicated. Some teachers of calculus seem to fear integral tables, apparently feeling that their students should not only be able to develop all the elementary formulas, but should have had to do each several times! What is this but teaching unnecessary technique? In Newton's day these formulas were new and interested mathematicians. Today they are of use, rather than interest. So why should we not strive to make them useful? This means learning how to use integral tables, rather than how to derive them.

Some fear the development of "handbook engineers"—persons who cannot operate without handbooks. This fear is complex; *i.e.*, it has both real and imaginary parts. The ex-student who cannot integrate $\int x^7 dx$ or $\int \sqrt{1-x^2}x dx$ with relative ease would be a discredit to the mathematics department, but need inability to find $\int (1+x^2)^{-3/2}dx$ or $\int \sin^2\theta \cos^4\theta d\theta$, without either a handbook or considerable pain, matter? The writer cannot see that it does. (He himself can work out the transformations, but would rather walk across the hall and borrow an integral table. Does this make him less of a mathematician—or only less of a computer?)

If the time that would otherwise be spent in learning how to derive integration formulas were diverted, not away from "mathematics," but to the introduction of additional mathematical ideas, we should make a great gain. In part, this could be done within the integral tables themselves. Consider the treatment of $\int x^n f(x)dx$ for some relatively simple $f(x)$ for which all such integrals are elementary. The most classical integral tables gave formulas for $n=1$, $n=2$ and usually a reduction formula for lowering n by one or two units. Then came tables which gave the explicit result for $n=1$, $n=2$, $n=3$, and $n=4$ before the user had to resort to the reduction formula. Next perhaps to $n=6$, and so on. The more extended tables are more useful, but some mathematicians find them overelaborate.

Why has no one taken the logical next steps? First, the value of $\int x^n f(x)dx$ will be a finite series. We do have notation, including the use of summation signs, with which to represent finite series. Can we not give the worked-out form for $\int x^n f(x)dx$ for general n in nearly every case?

Would this not be much more useful than a reduction formula? Why should user after user have to go through the same operations to deduce the same finite series from the reduction formula? Not only might we save labor by giving the finite series once and for all, but it is quite likely that we might hint successfully, to students and to users, that generality can mean less work. As mathematicians we should favor such hints.

Second, there is a deeper opportunity. Our integral tables give $\int x^n f(x)dx$, and in some applications we may get such expressions, but in others the user is concerned with $\int P(x)f(x)dx$ where $P(x)$ is a particular polynomial. Once we realize that $\int x^n f(x)dx$ corresponds to a finite series, it follows that $\int P(x)f(x)dx$ is also a finite (double) series. If we invert the order of summation, we shall usually find the answer to be a finite series, each of whose coefficients is of the form $b_{j0}a_0 + \cdots + b_{jm}a_m$, where $P(x) = a_0 + a_1x + \cdots + a_mx^m$. In more abstract terms, the vector of final coefficients is obtained from the vector u_i of polynomial coefficients by multiplication with a constant matrix b_{ji} with numerical entries. We need not use those fearsome words—but we can tabulate the numerical values of the b_{ji} and explain how to use them. To the extent that students and users make use of tables of b_{ji}, they are being introduced to the practice of matrix computation, and, implicitly, to the idea of a linear transformation.

We could use an integral table, were it rightly constructed, to introduce its users to such important mathematical ideas and notations as matrices, summa-

tion sign technique, and linear transformations. If these things come in as an alternative method—one not taught in class, but acceptable in home work or examination—as an alternative method which *saves work*, they will have by far the greatest chance of penetrating the indifference of the student or user not yet awakened to mathematics.

If a good table of indefinite integrals on this pattern takes 500 handbook-sized pages—with a thumb-index and keys like a book on birds or fishes—should we complain? Or even feel badly? The writer has a waiting, gaping vacancy on his desk for such a volume, which might appropriately be called "Integral Tables for the Occasional Integrator"—as do many of his colleagues. The student need not have to have all 500 pages—we can make up a student's 50-page version, containing the first 30 pages, one page in 7 for the next 70, and one page in 40 thereafter (it would not harm the student or weaken the effect if its paging showed the gaps). But the user would have a place for all 500 pages. If the table were laid out and explained properly, a student who had once learned to use it could relearn readily, as an ex-student, just what he needed in a specific situation. Such a person would not be looked down upon as a "handbook engineer," but rather looked up to as a user of mathematics who was controlling his computational problems, and not letting them control him.

5. Some nonexistent examples. There are other sorts of computation—and most of them lack even the beginnings of the tables and handbooks which would make their use easier, or even easy. A complete catalog would be lengthy, but some examples may be illuminating.

The National Bureau of Standards, with support from the National Science Foundation, has undertaken the preparation of a table of functions. Some think of this as a revision of Jahnke and Emde's table [1], now nearly 50 years old, but others think of it as more nearly the first approximation to a "Numerical Tables for the Occasional Figurer." It may well show a substantial amount of this last aspect, and, to the extent it does, it will tend to make numerical computation less of a bar to the applications of mathematics.

Consider ordinary differential equations. What is "ordinary" about them from the user's standpoint? The writer knows of but one table of solutions [2], and has no reason to be tremendously encouraged about its usefulness. Yet there are a number of possibilities. Our books on intermediate differential equations discuss the reduction of second-order linear equations to standard form—yet, though a number of second-order linear equations have been solved, who has ever seen a table of solutions for such equations after reduction to standard form. Why should there not be such a table?

Our books on elementary differential equations have followed for 60 years the mold of Murray's book ([5], 1897)—a book written but two years after Cantor introduced the union of two sets as a formal operation. Such books contain some relatively widely useful methods of solution, and some special methods which amused 19th-century mathematicians. Today they tend to say a little

more about solutions in series, and, even, numerical solutions. If we had a "Differential Equation Table for the Occasional Integrator," we could condemn the minor techniques to the lumber room, and instead teach vastly more important, suggestive, and stimulating topics such as solution in series and numerical solution.

Modern mathematics devotes a great part of its attention to linear expansions of one sort or another. And many linear expansions are of great practical importance. The process of finding coefficients can almost always be regarded as a process of biorthogonal expansion—yet who has seen even a little table of biorthogonal expansions, to say nothing of the "Biorthogonal Expansion Tables for the Occasional Expander."

Let the reader continue the list.

6. Sources. Where are the better integral tables, the usable differential equation tables, the first biorthogonal expansion tables, and all the others, to come from? The writer does not know, but—there are many members of the Mathematical Association of America who are competent mathematicians, fitted to do original work, yet not stimulated enough by current abstract mathematics to be carrying on important research. If only a small portion of them were to consider the interest and reward associated with trying to make these forms of computation simple, general, *and* easy to learn or relearn, there would be hands enough to do much. (Many of these tasks could well be done cooperatively by substantial groups.)

The successful completion of such tasks would do much to aid the healthy and mutually supporting growth of pure and applied mathematics in America—let us hope that they will be completed.

7. Formulation and approximation. Finally, a word about two areas where we have not explored far enough to see which way we should follow—but which we should clearly attack and exploit—formulation and approximation.

It is agreed that formulation of the problem is usually the most important stage in "applied mathematics," just as insight into what theorem is true and (probably) provable is often the most important stage in "pure mathematics." In each case the formation of new concepts or the refinements of old concepts is likely to be an essential step. Insofar as a concept-former is a philosopher, all mathematicians need to be philosophers (of a very special sort).

The formulation of the problem is of the essence—yet who has studied the problem of formulation, who has tried to explain it to the student? Pólya wrote "How to Solve It"; who will now write "How to Formulate It"? Probably no one person can do it; many must work together, almost all of whom must be mathematicians—though they will usually have other skills as well. Studying the problem of formulation, formulating better and better approximations to it, finding useful concepts for its treatment—these are tasks for "applied mathematicians" skilled in formulation.

It is easy to argue that a book on "How to Formulate It" will be empty

of real intellectual content. It would have been easy to argue 30 years ago that a book on the theory of games would lack intellectual content. In the past 30 years many relevant and useful concepts *have* been formulated in game theory. In the next 30 years many relevant and useful concepts *could* be formulated in formulation theory. Perhaps it is time to start, even if the task may be harder.

If we tried to write "How to Formulate It" today we would strike mainly questions: Where can we find examples to adequately set forth the principles? (What are the principles, anyway?). What kind of exercises can be used as homework? And so on, and on, and on. Yet, if "applied mathematics" is to grow properly, if there is to be something teachable and worthy of the name "applied mathematics," someone must tackle this problem—and eventually there must be developed a technique of wide usefulness and acceptability for teaching. This will not be easy, but it is badly needed.

What of approximation? Why is it paired with formulation? It, too, is a major stage in "applied mathematics," a matter of tactics rather than strategy perhaps, but surely a major stage. Without good approximations we should be lost, yet who knows what concepts are important in approximation? (From a routine mathematics course one would feel that taking the first terms of some series, power or Fourier or perhaps something more complex, was the natural approximation when only a few terms were permissible, yet this is often very wrong! How many connect the $(C, 1)$ summability of most Fourier series with practical approximations?) Yet a reasonable number of concepts have already been isolated, and are to be found by looking in corners. Many more concepts are undoubtedly near the surface. A discussion of approximation from the point of view of concepts and principles rather than labor would undoubtedly bring out much that was worthwhile. Here is another area for the formulators!

Who will begin to study approximation theory, rather than the theory of various specific approximations? Who will try to collect the important concepts in approximation, and try to add to them? What will be the results?

The suggestion of this section is merely this: Just as there is an applied mathematics of games, genetics, and mechanics, so there should be an applied mathematics (at least in terms of concepts, perhaps with techniques and operations) of the applications of mathematics. When there is, mathematicians will be able to teach "the applications of mathematics." At present only individual applications can be taught (and it is not likely to be too good for pure mathematicians to teach applications).

8. Summary. In brief we have said and argued that:

(1) the teaching of any form of computation should be directed (in relatively elementary or general courses) toward the occasional computer rather than the steady computer;

(2) this requires emphasis on simple, easy-to-learn-or-relearn methods;

(3) such emphasis need not mean emphasis on techniques, but rather may, and should, bring in more mathematical ideas than its converse, as exemplified by Aitken's interpolation procedure;

(4) the use of integral tables in teaching elementary calculus should be greatly broadened;

(5) integral tables could be redesigned to bring in the beginnings of matrices, finite summation, *etc.*, as methods of saving labor;

(6) many other forms of table, *e.g.*, solutions of differential equations in standard form, biorthogonal expansions, are badly needed;

(7) there are mathematicians competent to develop such tables who the writer feels would gain enjoyment and satisfaction from their preparation;

(8) a reduction in the labor of computation is the only visible way of finding the time and effort to make the study of computation more rewarding;

(9) there is much to be done in connection with the development of applied mathematics of formulation and approximation.

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ON THE LIMITING EQUILIBRIUM OF n MASSES

D. J. BENNEY, Canterbury University College, New Zealand

1. The general problem. A rigid framework rests on a rough horizontal plane (coefficient of friction μ), being in contact with the plane at the $n+1$ points $A_0, \cdots, A_n$. Let the normal reactions at these points be $W_0, \cdots, W_n$, respectively. A horizontal force P is applied at A_0 making an angle θ with a fixed direction, the magnitude of P being gradually increased until equilibrium is about to be disturbed. The initial displacement of the framework can be represented as a rotation about an instantaneous center I. Our problem is to investigate possible positions for I, and, in particular, to consider whether I may coincide with any of the points A_r. If I coincides with A_r then friction will be limiting at each point A_s, $s\neq r$, while if I is distinct from A_r the friction will be limiting at all points A_s.

We consider two problems of this type [1].

2. (i) Straight rod. A light rigid rod A_0A_n rests on a rough horizontal plane, being in contact with the plane at the $n+1$ equidistant points $A_0, \cdots, A_n$. At each point A_s a particle of weight W is fixed to the rod. The force P is applied at A_0 in a direction making an angle θ with A_nA_0.

Firstly we consider the possibility of the initial position of the instantaneous center I coinciding with A_r. Let X and Y be the components of friction at A_r and let $A_{s-1}A_s = 2a$, $s=1, \cdots, n$. For equilibrium we have

$$X = P\cos\theta, \tag{2.1}$$

$$Y = P\sin\theta + \mu W(n-2r), \tag{2.2}$$

$$P\sin\theta = (K/r)\mu W, \tag{2.3}$$

where

$$K = \tfrac{1}{2}[r(r+1) + (n-r)(n-r+1)]. \tag{2.4}$$

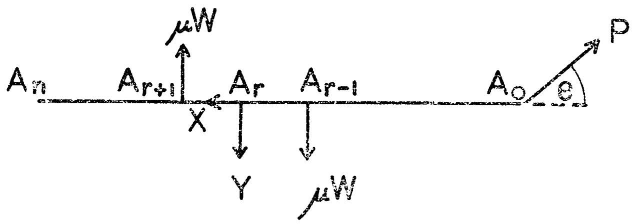

FIG. 1

After eliminating P from (2.1), (2.2), (2.3), we obtain

$$X = \mu W(K/r)\cot\theta, \qquad Y = \mu W[(K/r) + n - 2r].$$

In order that equilibrium should be broken by a rotation about A_r, we require $X^2+Y^2<\mu^2W^2$. This inequality is satisfied provided

$$\operatorname{cosec}^2\theta < \frac{4r[r-n(n-2r)(n-r+1)]}{(2r^2-2nr+n^2+n)^2}\cdot \tag{2.5}$$

Since $\operatorname{cosec}^2\theta \geqq 1$ for real θ, we find that the condition on r is that

$$f(r^2) = r^4 - r^2[n(n+1)+1] + [n(n+1)/2]^2 < 0, \tag{2.6}$$

where r can take the values $0, \cdots, n$.

On solving $f(r^2)=0$, we obtain the two roots $r=\frac{1}{2}(\sqrt{1+2\lambda}\pm 1)$, where $\lambda = n(n+1)$. Thus, as the difference of the roots is unity, there will, in general, be only one integer between these roots and so rotation will be possible about just one A_r. Hence for a certain range of θ about $\pi/2$, determined by (2.5), we will have I coinciding with A_r, where r is the integer between $\frac{1}{2}(\sqrt{1+2\lambda}-1)$ and $\frac{1}{2}(\sqrt{1+2\lambda}+1)$.

(ii) Exceptional cases. It may happen that $\frac{1}{2}(\sqrt{1+2\lambda}+1)$ is an integer. If this is so, we will have $f(r^2)<0$ only between consecutive integer values and hence no possible A_r can be found. Hence an exceptional case will arise if the equations $r=\frac{1}{2}(\sqrt{1+2\lambda}+1)$, $\lambda=n(n+1)$, possess positive integer solutions.

If $s=\sqrt{1+2\lambda}$ then $r=\frac{1}{2}(s+1)$; and so

$$s^2 = 1 + 2n(n+1) = \tfrac{1}{2}(2n+1)^2 + \tfrac{1}{2} = \tfrac{1}{2}(y^2+1),$$

where $y=2n+1$; *i.e.*, $y^2=2s^2-1$, where $r=\frac{1}{2}(s+1)$, $n=\frac{1}{2}(y-1)$. The general solution of this diophantine equation is

$$s = \frac{1}{2\sqrt{2}}\left[(\sqrt{2}+1)^{2m+1} + (\sqrt{2}-1)^{2m+1}\right],$$

$$y = \frac{1}{2}\left[(\sqrt{2}+1)^{2m+1} - (\sqrt{2}-1)^{2m+1}\right],$$

where m is an integer. It is readily found that the first few integer solutions are

s	y	$n=\frac{1}{2}(y-1)$	$r=\frac{1}{2}(s+1)$
5	7	3	3
29	41	20	15
169	239	119	85
985	1393	696	493

Further exceptional values of n and r can be most easily calculated by using the fact that both s and y satisfy the difference equation $U_p=6U_{p-1}-U_{p-2}$. When n has one of the values 3, 20, 119, 696, *etc.*, it is impossible for equilibrium to be disturbed by rotation about an A_r.

(iii) Simple cases. By using (2.5) it can readily be shown that the following results are valid:

$n=1$: Rotates about A_1 for $\operatorname{cosec}^2\theta<2$, *i.e.*, $\pi/4<\theta<3\pi/4$.
$n=2$: Rotates about A_2 for $\operatorname{cosec}^2\theta<4/3$, *i.e.*, $\pi/3<\theta<2\pi/3$.
$n=3$: Exceptional case; no rotation possible about an A_r.
$n=4$: Rotates about A_3 for $\operatorname{cosec}^2\theta<57/49$.

(iv) Uniform rod. If we put $r/n=\lambda$ in (2.5) and (2.6) and let $n\to\infty$ while λ remains finite, we obtain that $\lambda=1/\sqrt{2}$ if $\theta=\pi/2$. This is the familiar result for a uniform rod.

(v) Locus of the instantaneous center. Let us now suppose that each point A_r slips initially so that I will be distinct from an A_r. Let A_sI make an angle α_s with A_nA_0.

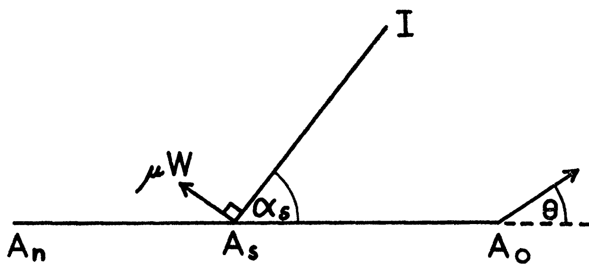

FIG. 2

Taking moments about A_0, we have $\sum_{s=1}^{n} 2as\mu W \cos \alpha_s = 0$, *i.e.*, $\sum_{s=1}^{n} s \cos \alpha_s = 0$. This equation determines the locus of I. Further, this equation shows that I lies on a certain line of force due to collinear point charges s at A_s, $s = 1, \cdots, n$; for $\sum_{s=1}^{n} e_s \cos \alpha_s =$ constant gives the equations of all lines of force due to a set of collinear point charges e_s, $s = 1, \cdots, n$. [2].

If each $\alpha_s = \pi/2$, we get a possible position for I so that the locus of I coincides with that line of force having an asymptote perpendicular to the rod.

As a check on our previous work, let us suppose that the curve $\sum_{s=1}^{n} s \cos \alpha_s = 0$ passes through A_r. Then $\alpha_s = \pi$ for $s < r$, $\alpha_s = \beta_r$ for $s = r$, and $\alpha_s = 0$ for $s > r$ must satisfy the equation. Substitution requires that $\cos \beta_r = r - n(n+1)/(2r)$ and, expressing the condition that β_r be real, we obtain the condition

$$r^4 - r^2[n(n+1) + 1] + [n(n+1)/2]^2 \leqq 0.$$

This, apart from the possibility of equality, is merely (2.6). When equality occurs we again have the exceptional case and we get $\beta_r = \pi$, $\beta_{r+1} = 0$. This means that the locus of I is the curve separating the lines of force leaving A_r and A_{r+1}, and cutting A_nA_0 at the neutral point N between A_r and A_{r+1}.

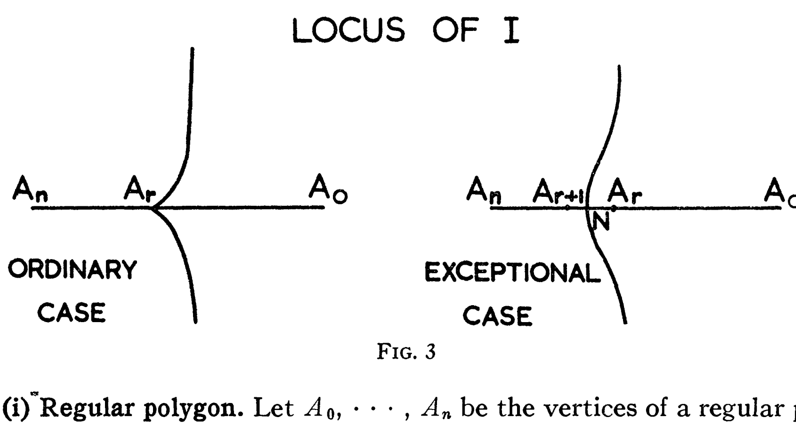

FIG. 3

3. (i) Regular polygon. Let $A_0, \cdots, A_n$ be the vertices of a regular polygon formed by n rigidly jointed light rods. A weight W is placed at each vertex and the polygon rests on a rough horizontal plane (coefficient of friction μ), being in contact with the plane at each vertex. A gradually increasing horizontal force

P is applied at A_0. The direction of P being varied, we wish to consider under what conditions equilibrium will be broken by rotation about a vertex.

Let O be the center of the polygon and suppose that the initial rotation is about A_r, where $r \leqq n/2$ (symmetry about OA_0 will enable us to deduce corresponding results when $r \geqq n/2$). Let the force P make an angle θ with A_rA_0. Let $0 < \theta < \pi$ so that if an initial rotation about A_r is possible, it will be clockwise.

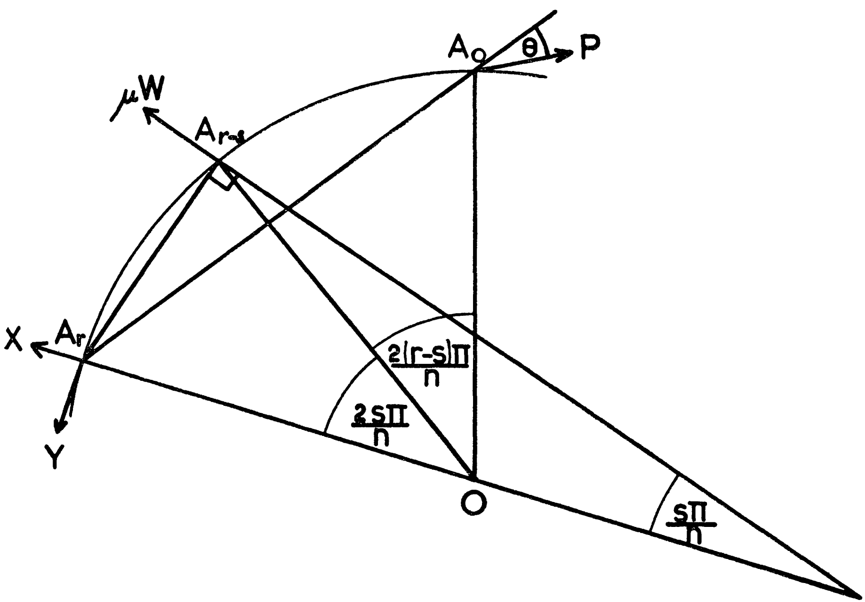

FIG. 4

If X and Y are the friction components at A_r, along and perpendicular to OA_r, we have the following three conditions for equilibrium:

$$X = P \sin\left(\theta + \frac{r\pi}{n}\right) - \mu W \sum_{s=1}^{n-1} \cos\frac{s\pi}{n}, \text{ i.e., } X = P \sin\left(\theta + \frac{r\pi}{n}\right), \tag{3.1}$$

$$\begin{aligned} Y &= P \cos\left(\theta + \frac{r\pi}{n}\right) + \mu W \sum_{s=1}^{n-1} \sin\frac{s\pi}{n}, \text{ i.e.,} \\ Y &= P \cos\left(\theta + \frac{r\pi}{n}\right) + \mu W \cot\frac{\pi}{2n}, \end{aligned} \tag{3.2}$$

$$P \sin\theta \sin\frac{r\pi}{n} = \sum_{s=1}^{n-1} \mu W \sin\frac{s\pi}{n}, \text{ i.e., } P \sin\theta \sin\frac{r\pi}{n} = \mu W \cot\frac{\pi}{2n}. \tag{3.3}$$

Using (3.3) to eliminate P from (3.1) and (3.2), we obtain

$$X = \mu W \cot\frac{\pi}{2n}\left(\cot\theta + \cot\frac{r\pi}{n}\right), \qquad Y = \mu W \cot\frac{\pi}{2n}\left(\cot\theta \cot\frac{r\pi}{n}\right).$$

For an initial rotation about A_r we require $X^2+Y^2<\mu^2W^2$. Expressing this condition we find that we require

$$(3.4)\qquad f(\theta) = \cot^2\theta + 2\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}\cot\theta + \cos^2\frac{r\pi}{n} - \sin^2\frac{r\pi}{n}\tan^2\frac{\pi}{n} < 0,$$

where $0<\theta<\pi$.

If, on the other hand, $-\pi<\theta<0$, we merely need to change the sign of θ and μ in our previous equations. Therefore, the overall condition for an initial rotation about the vertex A_r is that

$$(3.5)\qquad f(\theta) < 0, \qquad -\pi < \theta < \pi.$$

(ii) Vertices possible as initial instantaneous centers. Clearly, $f(\theta)>0$ for θ small. Further, $f(\theta)=0$ has imaginary roots if (using the fact that $r\leqq n/2$)

$$(3.6)\qquad \cos^2\frac{r\pi}{n} > \sin\frac{r\pi}{n}\tan\frac{\pi}{2n}\cdot$$

Thus no rotation is possible about A_r if r satisfies the inequality (3.6); but if (3.6) is not satisfied, then I will coincide with A_r for a certain range of θ determined by (3.5).

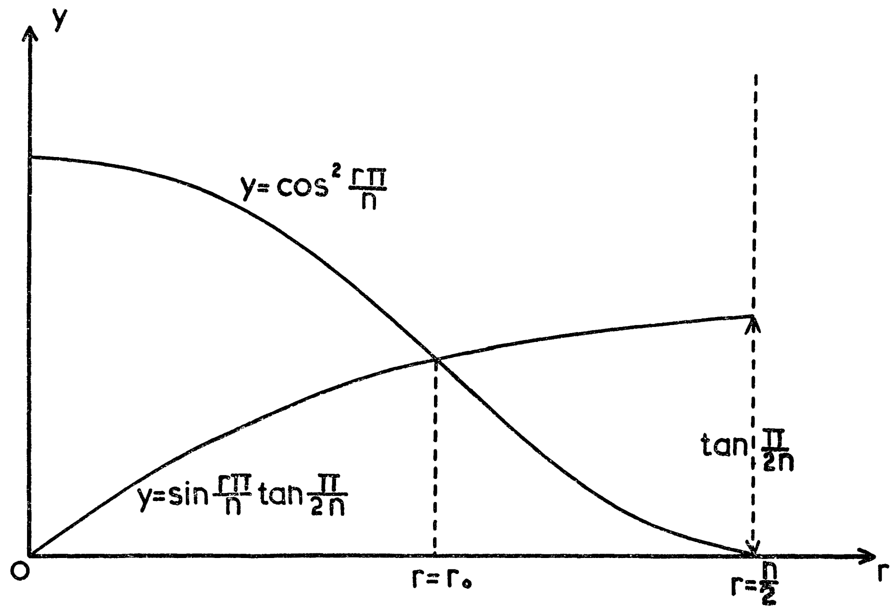

FIG. 5

Consider the graphs of $y=\cos^2(r\pi/n)$ and $y=\sin(r\pi/n)\tan(\pi/2n)$ for r between 0 and $n/2$. If r_0 is the value of r satisfying the equation $\cos^2(r\pi/n)$

$=\sin(r\pi/n)\tan(\pi/2n)$, then, solving, we find that

$$r_0 = (n/\pi)\sin^{-1}(\sqrt{1+\alpha^2}-\alpha),$$

where $\alpha=\frac{1}{2}\tan(\pi/2n)$. This equation has the series solution

$$r_0 = \frac{n}{2} - \sqrt{\frac{n}{2\pi}}\left[1 - \frac{1}{12}\left(\frac{\pi}{2n}\right) + \frac{113}{1440}\left(\frac{\pi}{2n}\right)^2 - \cdots\right].$$

Thus we conclude that I may coincide with A_r if and only if $r > r_0$.

(iii) Uniform circular ring. If we put $r/n=\lambda$ in (3.5) and (3.7) and let $n\to\infty$ while λ remains finite, we obtain $\lambda=\frac{1}{2}$ if $\theta=\pi/2$. This is the well-known result for a uniform circular ring that, if P is applied tangentially, the ring commences to rotate about the diametrically opposite point.

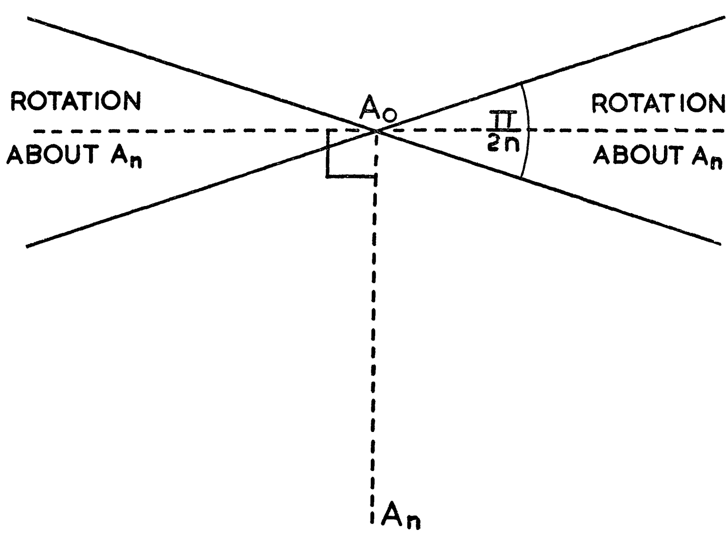

FIG. 6

(iv) Even number of vertices. Let us replace n by $2n$ in the equations. We consider the range of θ for which rotations are possible about A_n, *i.e.*, about the vertex opposite to A_0. Equation (3.5) requires that $\cot^2\theta < \tan^2(\pi/(4n))$. Thus, a rotation about A_n is always possible and will, in fact, occur if

$$\frac{\pi}{2} - \frac{\pi}{4n} < \theta < \frac{\pi}{2} + \frac{\pi}{4n}\,.$$

For a rod, square, and hexagon, the opposite vertex is the only possible vertex about which a rotation may occur; but for the octagon, neighboring vertices are also possible positions for I.

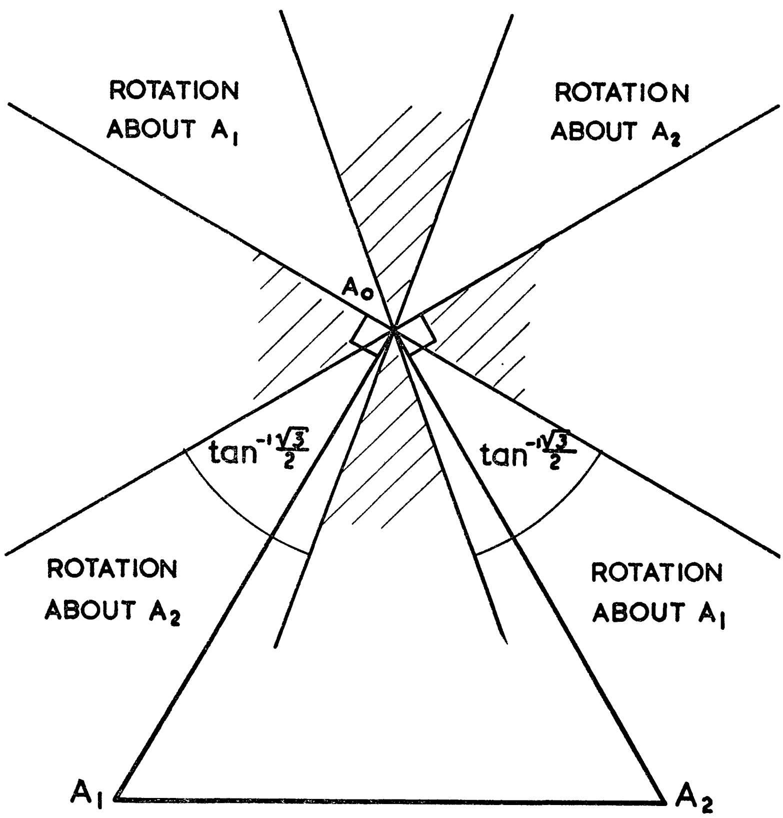

FIG. 8

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1. E. J. Routh, Analytic Statics, vol. I, Cambridge, 1896, pp. 124, 134.
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DIVIDED DIFFERENCES IN COMPLEX FUNCTION THEORY

ERBEN COOK, JR., University of Connecticut

1. Divided differences have been used to prove the existence and uniqueness of solutions to certain boundary value problems but have been more or less neglected in the usual developments of complex function theory. The purpose of this paper is to show how divided differences may be used to advantage in complex function theory.

The proof of the divided difference theorem is not much more complicated

than that of its special case, Taylor's theorem with remainder. Both proofs are essentially the same. Divided differences, although not mentioned as such, appear in many of the standard complex variables texts in connection with the proof of Taylor's theorem. They also give a very simple form for writing the remainder.

The developments of an analytic function in the forms corresponding to those most common for polynomials and rational functions are also presented here to illustrate further the use of divided differences in complex variables.

The present development presupposes the following five theorems:

1.0. *If $f(z)$ is continuous in a region Ω and γ is any arc in Ω, then $\int_\gamma f(z)dz$ depends only on the end points of γ if and only if $f(z)$ is the derivative of a function which is analytic in Ω.**

1.1. CAUCHY'S INTEGRAL FORMULA.† *If $f(z)$ is analytic in a region Ω and γ is any closed curve in Ω which is homotopic to a point with respect to Ω, then*

$$n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z},$$

$n(\gamma, z)$ being the index (winding number) of γ with respect to z.

1.2. *If $f(z)$ is analytic in a region Ω then $f'(z)$ is analytic in Ω.*

1.3. *If $R(z)$ is a rational function with a zero of order greater than one at ∞ and with its poles at the set of points $\{a_i\}$, $i=1, \cdots, n$, then $\int_\gamma R(z)dz=0$ for every closed curve γ for which $n(\gamma, a_i)=n(\gamma, a_1)$ for $i=2, \cdots, n$.*

This follows from the fact that $R(z)=N(z)/\prod_{i=1}^n (z-a_i)^{h_i}$, with $N(z)$ a polynomial of degree less than $h_1+\cdots+h_n-1$, has the partial fraction representation

$$R(z) = \sum_{k=1}^{n}\sum_{i=1}^{h_k} \frac{b_{k,i}}{(z - a_k)^i}$$

with $\lim_{z\to\infty} zR(z) = \sum_{k=1}^n b_{k,1}=0$. Hence

$$\int_\gamma R(z)dz = \sum_{k=1}^{n}\int_\gamma \frac{b_{k,1}dz}{z - a_k} = \sum_{k=1}^{n} n(\gamma, a_k)b_{k,1} = \sum_{k=1}^{n} n(\gamma, a_1)b_{k,1} = 0.$$

1.4. REMAINDER THEOREM. *If $F(z)$ is analytic in a region Ω and $a\in\Omega$, then there exists a function $G(z)$ which is analytic in Ω and for which $F(z)=(z-a)G(z)+F(a)$.*

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* Lars V. Ahlfors, Complex Analysis, New York, 1953, the last statement on p. 86.

† The wording is based on that used by F. M. Stewart in an address at the meeting of the Northeastern Section of the Mathematical Association of America, November 24, 1956.

DEFINITION. *The function*

$$G(z) = \begin{cases} \dfrac{F(z) - F(a)}{z - a}, & z \neq a, \\ F'(a), & z = a, \end{cases}$$

usually denoted by $F(a, z)$, is the first divided difference of $F(z)$ with respect to a and z.

If $f(z)$ is a function which is analytic in Ω and $\{a_i\}$, $i=1, 2, \cdots$, is a sequence of points in Ω, then $f(a_1, \cdots, a_n, z)$, the nth divided difference of $f(z)$ with respect to $a_1, \cdots, a_n, z$, is defined inductively by the recurrence relation

$$f(a_1, \cdots, a_{n-1}, z) = (z - a_n)f(a_1, \cdots, a_n, z) + f(a_1, \cdots, a_{n-1}, a_n).$$

2. 2.0. THE DIVIDED DIFFERENCE THEOREM. *If $f(z)$ is analytic in a region Ω and $\{a_i\}$, $i=1, 2, \cdots$, is a sequence of points in Ω, then, for each $n>1$,*

$$\text{(a)} \qquad f(z) = \sum_{i=1}^{n} f(a_1, \cdots, a_i) \prod_{j=1}^{i-1} (z - a_j) + f(a_1, \cdots, a_n, z) \prod_{j=1}^{n} (z - a_j)$$

with $f(a_1, \cdots, a_n, z)$ analytic in Ω and

$$\text{(b)} \qquad n(\gamma, z)f(a_1, \cdots, a_n, z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)\prod_{i=1}^n (\zeta - a_i)}$$

for every closed curve γ in Ω which is homotopic to a point with respect to Ω and for which $n(\gamma, a_i) = n(\gamma, z)$ for $i=1, \cdots, n$.

Proof. The first part of the theorem follows inductively from the definition of $f(a_1, \cdots, a_n, z)$ and from the remainder theorem.

As to the rest of the theorem, 1.1 and 2.0 (a) give

$$\begin{aligned} n(\gamma, z)f(a_1, \cdots, a_n, z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(a_1, \cdots, a_n, \zeta)d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta) - \sum_{i=1}^n f(a_1, \cdots, a_i)\prod_{k=1}^{i-1}(\zeta - a_k)d\zeta}{(\zeta - z)\prod_{k=1}^n (\zeta - a_k)}, \end{aligned}$$

while, from 1.3 one has

$$\int_\gamma \frac{\sum_{i=1}^n f(a_1, \cdots a_i)\prod_{k=1}^{i-1}(\zeta - a_k)d\zeta}{(\zeta - z)\prod_{k=1}^n (\zeta - a_k)} = 0.$$

Hence 2.0 (b) holds.

2.1. PERMUTATIONS. *If $\{b_i\}$, $i=1, \cdots, n$ is any permutation of $\{a_i\}$, $i=1, \cdots, n$, then $f(b_1, \cdots, b_n) = f(a_1, \cdots, a_n)$.*

This follows immediately from the integral representation of the $(n-1)$th divided difference.

2.2. LAGRANGE INTERPOLATION FORMULA. *If $a_i \neq a_j$ for $i \neq j$, then*

$$f(z) = \sum_{i=1}^{n} f(a_i) \prod_{\substack{k=1 \\ k\neq i}}^{n} \frac{z - a_k}{a_i - a_k} + f(a_1, \cdots, a_n, z) \prod_{k=1}^{n} (z - a_k).$$

Proof. If $z = a_i$ for some $i = 1, \cdots, n$, then the theorem is trivially true. Otherwise, denote z by a_{n+1}. Take a closed curve γ which is homotopic to a point with respect to Ω and for which $n(\gamma, a_i) = 1$ for $i = 1, \cdots, n+1$. Pick closed curves γ_i such that $n(\gamma_i, a_j) = \delta_{ij}$, the Kronecker delta, for $i, j = 1, \cdots, n+1$ and for which $\sum_{i=1}^{n+1} \gamma_i$ is homologous to γ with respect to Ω.

Then, since

$$f(a_1, \cdots, a_n, z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)\prod_{k=1}^{n}(\zeta - a_k)},$$

$$f(a_1, \cdots, a_n, a_{n+1}) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\prod_{k=1}^{n+1}(\zeta - a_k)} = \frac{1}{2\pi i} \sum_{i=1}^{n+1} \int_{\gamma_i} \frac{f(\zeta)d\zeta}{\prod_{k=1}^{n+1}(\zeta - a_k)}$$

$$= \sum_{i=1}^{n+1} \frac{f(a_i)}{\prod_{\substack{k=1 \\ k\neq i}}^{n+1}(a_i - a_k)} = \frac{f(z)}{\prod_{k=1}^{n}(z - a_k)} + \sum_{i=1}^{n} \frac{f(a_i)}{(a_i - z)\prod_{\substack{k=1 \\ k\neq i}}^{n}(a_i - a_k)}.$$

The Lagrange interpolation formula follows on solving for $f(z)$.

2.3. INTEGRAL AND DERIVATIVE REPRESENTATIONS. *If $a_i = a$ for $i = 1, \cdots, m$, then, for $n \geqq m$,*

$$f(a_1, \cdots, a_n) = \frac{f^{(m-1)}(a_m, a_{m+1}, \cdots, a_n)}{(m-1)!}.$$

In particular $f(a_1, \cdots, a_m) = f^{(m-1)}(a)/(m-1)!$ and hence

$$n(\gamma, a)f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}}.$$

This will follow from the following lemmas:

LEMMA 1. *If $F(z)$ and $G(z)$ are analytic in Ω and γ is any closed curve in Ω, then*

$$\int_\gamma F'(z)G(z)dz = -\int_\gamma F(z)G'(z)dz.$$

The proof of Lemma 1 is an immediate consequence of 1.0 since $FG' + F'G = (FG)'$.

Now letting $G(z)=1/(z-a)^m$, one has

LEMMA 2. *If a is any complex number, m and n are any integers, $F(z)$ is analytic in Ω, and γ is any closed curve in Ω not passing through a, then*

$$\text{(a)} \qquad \int_\gamma \frac{F'(\zeta)d\zeta}{(\zeta-a)^m} = m\int_\gamma \frac{F(\zeta)d\zeta}{(\zeta-a)^{m+1}}\,.$$

Moreover,

$$\text{(b)} \qquad \int_\gamma \frac{F^{(n)}(\zeta)d\zeta}{(\zeta-a)^m} = m(m+1)\cdots(m+n-1)\int_\gamma \frac{F(\zeta)d\zeta}{(\zeta-a)^{m+n}},$$

$$\text{(c)} \qquad n(\gamma,z)F^{(n)}(z) = \frac{1}{2\pi i}\int_\gamma \frac{F^{(n)}(\zeta)d\zeta}{\zeta-z} = \frac{n!}{2\pi i}\int_\gamma \frac{F(\zeta)d\zeta}{(\zeta-z)^{n+1}},$$

$$\text{(d)} \qquad \frac{d^n}{dz^n}\int_\gamma \frac{F(\zeta)d\zeta}{\zeta-z} = n!\int_\gamma \frac{F(\zeta)d\zeta}{(\zeta-z)^{n+1}}\,.$$

(b) follows inductively from (a). (c) follows from Cauchy's integral formula and (b) with $m=1$. (d) follows from Cauchy's integral formula and (c).

Proof of 2.3: In (d), let $z=a$, $n=m-1$ and then let

$$F(\zeta) = \frac{f(\zeta)}{(\zeta-z)\prod_{i=m+1}^n(\zeta-a_i)}\,.$$

Then, if $n(\gamma,a_i)=n(\gamma,z)=1$ for $i=1,\cdots,n$,

$$\begin{aligned} f(a_1,\cdots,a_m,a_{m+1},\cdots,a_n,z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-a)^m\prod_{i=m+1}^n(\zeta-a_i)} \\ &= \frac{1}{(m-1)!2\pi i}\frac{d^{m-1}}{da^{m-1}}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-a)\prod_{i=m+1}^n(\zeta-a_i)} \\ &= \frac{f^{(m-1)}(a_m,a_{m+1},\cdots,a_n,z)}{(m-1)!}\,. \end{aligned}$$

The rest of the theorem follows by taking $m=n$ and letting $z=a$ in (c).

2.4. TAYLOR'S THEOREM WITH REMAINDER. *If $f(z)$ is analytic in a region Ω and $a\in\Omega$ then, for $n=1,2,\cdots$,*

$$f(z) = f(a)+f'(a)\frac{(z-a)}{1!}+\cdots+f^{(n-1)}(a)\frac{(z-a)^{n-1}}{(n-1)!}+f_n(a,z)(z-a)^n,$$

with $f_n(a,z)$ analytic in Ω and

$$f_n(a,z) = \frac{f^{(n-1)}(a,z)}{(n-1)!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-a)^n},$$

for every closed curve γ in Ω which is homotopic to a point with respect to Ω and for which $n(\gamma, a) = n(\gamma, z) = 1$.

This is a consequence of 2.0 and 2.3.

2.5. Generalized Cauchy Estimate. *If $f(z)$ is analytic in a region Ω, $\{a_k\}$, $k=1, 2, \cdots$, is any sequence in Ω, $\{z : |z-a| \leqq R\}$ is a subset of Ω, $|a-a_k| \neq R$ for any k, and $M = \max_{|z-a|=R} |f(z)|$, then, for $|z-a| < R$ and all n,*

$$|f(a_1, \cdots, a_n, z)| \leqq \frac{MR}{(R - |z-a|) \prod_{k=1}^{n} |R - |a_k - a||}.$$

Moreover, if $\rho < R$ and every limit point of $\{a_k\}$, $k=1, 2, \cdots$, is contained in the disk $\{z : |z-a| < \rho\}$, then

$$|f(a_1, \cdots, a_n, z)| \leqq \frac{M'R}{(R - |z-a|)(R-\rho)^{n-\nu}}$$

for $|z-a| < R$ and all $n \geqq \nu = \max_{|a_k - a| > \rho} k$, where

$$M' = \frac{M}{\prod_{|a_k - a| > \rho} |R - |a_k - a||}.$$

These estimates follow readily from the integral representation of $f(a_1, \cdots, a_n, z)$ using the circle with center a and radius R for γ.

If, in the divided difference theorem, the second estimate holds for $\rho = R/3$, then if $|z-a| < \rho$, $|z-a_k| < 2\rho$ for $k > \nu$ and hence

$$\left| \prod_{k=1}^{n} (z - a_k) f(a_1, \cdots, a_n, z) \right| \leqq \frac{M'R(R-\rho)^{\nu}}{R - |z-a|} \prod_{k=1}^{n} \left| \frac{z - a_k}{R - \rho} \right|$$

for $n > \nu$. Since all except ν terms of the product are less than 1, one has an infinite series representation of $f(z)$. which is valid for $|z-a| < R/3$.

2.6. Unique Factorization Theorem for a Compact Subset of a Region. *If $f(z)$ is analytic in a region Ω and if there is a sequence $\{a_i\}$, $i=1, 2 \cdots$, in Ω for which $f(a_1, \cdots, a_i) = 0$ for all i, then*

(a) *$f(z) = f(a_1, \cdots, a_n, z) \prod_{i=1}^{n} (z - a_i)$ for every n.*

(b) *Moreover, if $\{a_i\}$, $i=1, 2, \cdots$, has a limit point $a \in \Omega$ then $f(z) \equiv 0$.*

(c) *Consequently, if $f(z) \not\equiv 0$ and Δ is any compact subset of Ω, then either $f(z)$ has no zeros in Δ or there exists an integer n and an ordered set $\{b_i\}$, $i=1, \cdots, n$, contained in Δ for which*

$$f(z) = \prod_{i=1}^{n} (z - b_i) f(b_1, \cdots, b_n, z)$$

with $f(b_1, \cdots, b_n, z) \neq 0$ in Δ. In particular, if $f(z) \not\equiv 0$, then every zero of $f(z)$ has a finite order and a neighborhood containing no other zeros of $f(z)$.

Proof. (a) follows from 2.0.

As to the proof of (b), one may as well assume to begin with that $\lim_{n\to\infty} a_n = a$. For, since $\{a_n\}$ has a for a limit point, there is a subsequence $\{a_{n_i}\}$, $i=1,2 \cdots$, which converges to a. But $f(z) = \prod_{i=1}^n (z-a_i)f(a_1, \cdots, a_n, z)$ for $n=1, 2, \cdots$ implies that $f(a_i)=0$ for all i and if $a_i=b$ for k distinct subscripts then $f^{(k-1)}(b)=0$ (*i.e.* the $(k-1)$th divided difference, $f(b, b, \cdots, b)=0$). Hence $f(a_{n_1}, a_{n_2}, \cdots, a_{n_i})=0$ for $i=1, 2, \cdots$.

Now, the argument in the last paragraph of 2.5, shows that $f(z)=0$ for $|z-a|<R/3$ and hence $f^{(k)}(z)=0$ for $|z-a|<R/3$ and all k.

By one of the usual arguments* one may show that $f(z)\equiv 0$ for $z\in\Omega$, completing the proof of (b).

(c) is an immediate consequence of (a) and (b).

2.6. PARTIAL FRACTION REPRESENTATIONS. *If $f(z)$ is meromorphic in Ω and Δ is any compact subset of Ω, then either $f(z)$ is analytic in Δ or there exists a region Ω', an integer n, and an ordered set $\{b_i\}$, $i=1, \cdots, n$, $\subset\Delta\subset\Omega'$ for which*

$$f(z) = \sum_{j=1}^{n} \frac{A_j}{\prod_{k=1}^{j} (z - b_k)} + F(z),$$

with $F(z)$ analytic in Δ. Moreover, if γ is any closed curve in Ω' which is homotopic to a point with respect to Ω' and for which $n(\gamma, b_k)=n(\gamma, z)=1$ for $k=1, \cdots, n$, then

$$A_j = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\prod_{k=1}^{j-1}(\zeta - b_k)d\zeta}{\zeta - z},$$

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}.$$

Proof. One first notes, from the definition of a pole, that the poles are isolated and that, for any Δ, one can find a region Ω' containing Δ and a $g(z)$ analytic in Ω' such that for $z\in\Omega'$, $g(z)=f(z)\prod_{i=1}^n (z-b_i)$ if $z\neq b_i$, $i=1, \cdots, n$.

Then, by the divided difference theorem,

$$g(z) = \sum_{i=1}^{n} g(a_1, \cdots, a_i)\prod_{k=1}^{i-1}(z - a_k) + g(a_1, \cdots, a_n, z)\prod_{k=1}^{n}(z - a_k).$$

Let $a_k=b_{n-k+1}$ for $k=1, \cdots, n$. Then

$$f(z) = \sum_{i=1}^{n} \frac{g(b_n, \cdots, b_{n-i+1})}{\prod_{k=i}^{n}(z - b_{n-k+1})} + g(b_n, \cdots, b_1, z)$$

* See, for example, p. 102 of Ahlfors.

$$= \sum_{i=1}^{n} \frac{g(b_{n-i+1}, \cdots, b_n)}{\prod_{k=1}^{n-i+1} (z - b_k)} + g(b_1, \cdots, b_n, z)$$

$$= \sum_{j=1}^{n} \frac{g(b_j, \cdots, b_n)}{\prod_{k=1}^{j} (z - b_k)} + g(b_1, \cdots, b_n, z).$$

Now, by the divided difference theorem, $g(b_1, \cdots, b_n, z) = F(z)$ is analytic in Δ and

$$g(b_1, \cdots, b_n, z) = \frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)d\zeta}{(\zeta - z) \prod_{k=1}^{n} (\zeta - b_k)} = \frac{1}{2\pi i} \int \frac{f(\zeta)d\zeta}{\zeta - z}.$$

Moreover,

$$A_j = g(b_j, \cdots, b_n) = \frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)d\zeta}{\prod_{k=j}^{n} (\zeta - b_k)} = \frac{1}{2\pi i} \int_\gamma f(\zeta) \prod_{k=1}^{j-1} (\zeta - b_k)d\zeta.$$

Finally, one notes that $\{b_j\}$, $j = 1, \cdots, n$, can be represented by an ordered set of distinct points $\{a_j\}$, $j = 1, \cdots, m$, where each a_j appears in the original set h_j times and $h_1 + \cdots + h_m = n$, and that

$$f(z) = \sum_{k=1}^{m} \sum_{i=1}^{h_k} \frac{B_{k,i}}{(z - a_k)^i} + F(z),$$

the Laurent expansion in Δ.

MATRIX PROOF OF PASCAL'S THEOREM

JAMES J. McMAHON, St. Patrick's College, Maynooth, Ireland

Pascal's theorem states that the points of intersection of the opposite sides of a hexagon inscribed in a conic lie on a line. Many proofs of this celebrated theorem have been given, the most elegant being, perhaps, that of Sylvester [1], but the matrix proof given here has the advantage that it lends itself easily to generalization.

Let A be a real nonsingular symmetric matrix of order $n \times n$, and X and L_i column vectors whose transposes are $X^t = (x_1, \cdots, x_n)$, $L_i^t = (l_{1i}, \cdots, l_{ni})$. Then $L_1L_2^t + L_2L_1^t$ is a symmetric matrix of order $n \times n$ and rank 2. We seek the eigenvectors corresponding to the two nonzero eigenvalues of the system of equations

$$(1) \qquad \lambda AX = (L_1L_2^t + L_2L_1^t)X.$$

Setting $X=aA^{-1}L_1+bA^{-1}L_2$, where a and b are numbers, we find

$$\lambda(aL_1+bL_2) = (aL_2^tA^{-1}L_1+bL_2^tA^{-1}L_2)L_1+(aL_1^tA^{-1}L_1+bL_1^tA^{-1}L_2)L_2,$$

or

$$aL_1^tA^{-1}L_1+bL_1^tA^{-1}L_2=\lambda b, \qquad aL_2^tA^{-1}L_1+bL_2^tA^{-1}L_2=\lambda a.$$

Solving we obtain $a=u_2^2$, $b=\pm u_1u_2$, where

$$u_i^2 = L_i^tA^{-1}L_i. \tag{2}$$

Hence the two linearly independent eigenvectors are

$$u_2L_1 \pm u_1L_2. \tag{3}$$

Let A now be of order 3×3, and consider the conic $X^tAX=0$, in which a hexagon is inscribed. Denote the vertices of the hexagon by p, q, r, s, t, u, and let the equations of the three lines ps, qt, ru be, respectively,

$$L_1^tX=0, \qquad L_2^tX=0, \qquad L_3^tX=0, \tag{4}$$

so that the quadratic forms associated with the pairs of lines ps, qt; qt, ru; ru, ps are $L_1L_2^t+L_2L_1^t$, $L_2L_3^t+L_3L_2^t$, $L_3L_1^t+L_1L_3^t$. For any value of λ

$$X^t(\lambda A - L_1L_2^t - L_2L_1^t)X = 0 \tag{5}$$

is the equation of a conic passing through the points p, s, q, t, and the conic degenerates into a pair of straight lines if λ is an eigenvalue of the equations $\lambda AX=(L_1L_2^t+L_2L_1^t)X$. The corresponding eigenvector is the point of intersection of two lines through the points p, s, q, t. To the eigenvalue $\lambda=0$ corresponds that eigenvector which is the point of intersection of the lines ps, qt. To the two nonzero eigenvalues correspond eigenvectors which are the points of intersection of the two other pairs of lines through the points p, s, q, t. These two eigenvectors are, from (3), $u_2L_1\pm u_1L_2$. Similarly for the pairs of lines through the points q, t, r, u and r, u, p, s we obtain, as eigenvectors corresponding to nonzero eigenvalues,

$$u_3L_2 \pm u_2L_3, \qquad u_1L_3 \pm u_3L_1.$$

Note that $u_i^2=L_i^tA^{-1}L_i=0$ is the condition for $L_i^tX=0$ to touch the conic $X^tAX=0$, and so if the three lines ps, qt, and ru cut the conic, u_1^2, u_2^2, and u_3^2 are all of one sign. Thus it is clear that the set of points

$$u_2L_1 - u_1L_2, \qquad u_3L_2 - u_2L_3, \qquad u_1L_3 - u_3L_1$$

are real, collinear, and lie on the Pascal line of the hexagon.

The generalization, which we now give, of Pascal's theorem to space of three

dimensions differs from that given by Salmon [2], and seems to be more simple and natural. Let

$$L_1^t X = 0, \qquad L_2^t X = 0, \qquad L_3^t X = 0, \qquad L_4^t X = 0 \tag{6}$$

denote four distinct planes, so that the matrix (L_1, L_2, L_3, L_4) is nonsingular and

$$X^t A X = 0 \tag{7}$$

is a quadric, where A is now of order 4×4. Then for any value of λ

$$X^t(\lambda A - L_1 L_2^t - L_2 L_1^t)X = 0 \tag{8}$$

is the equation of a quadric passing through the curves of intersection of the first two planes with the quadric (7). The quadric (8) specializes to a cone when λ is an eigenvalue of the equations

$$\lambda A X = (L_1 L_2^t + L_2 L_1^t)X$$

and the corresponding eigenvector is the vertex of the cone. Six pairs of planes can be obtained from (6) whose common sections are the six edges of the tetrahedron formed by the four planes, and each of the six pairs yields two cones whose vertices, as we see from (3), are the points

12) $u_1L_2 - u_2L_1$,	23) $u_2L_3 - u_3L_2$,	31) $u_3L_1 - u_1L_3$,
41) $u_4L_1 - u_1L_4$,	42) $u_4L_2 - u_2L_4$,	43) $u_4L_3 - u_3L_4$,
12)* $u_1L_2 + u_2L_1$,	23)* $u_2L_3 + u_3L_2$,	31)* $u_3L_1 + u_1L_3$,
41)* $u_4L_1 + u_1L_4$,	42)* $u_4L_2 + u_2L_4$,	43)* $u_4L_3 + u_3L_4$.

Note that the points *ik*) and *ik*)* are the same points as *ki*) and *ki*)*, respectively. The set of points 12), 23), 31), 41), 42), 43) are coplanar, for the matrix

$$\begin{pmatrix} u_2 & -u_1 & 0 & 0 \\ 0 & u_3 & -u_2 & 0 \\ -u_3 & 0 & u_1 & 0 \\ u_4 & 0 & 0 & -u_1 \\ 0 & u_4 & 0 & -u_2 \\ 0 & 0 & u_4 & -u_3 \end{pmatrix} \begin{pmatrix} L_1^t \\ L_2^t \\ L_3^t \\ L_4^t \end{pmatrix},$$

being the product of a matrix of rank three and a nonsingular matrix, is itself of rank three. The plane determined by these six points we call the Pascal plane of the original tetrahedron with respect to the quadric (7). For the same reason the set of points 12)*, 13)*, 14)*, 23), 34), 42) are coplanar. Hence in addition to the Pascal plane we obtain the following four planes:

plane P_1 containing the points 12)*, 13)*, 14)*, 23), 34), 42),
plane P_2 containing the points 23)*, 24)*, 21)*, 34), 41), 13),
plane P_3 containing the points 34)*, 31)*, 32)*, 41), 12), 24),
plane P_4 containing the points 41)*, 42)*, 43)*, 12), 23), 31).

The planes P_1, P_2, P_3, P_4 determine a tetrahedron, and it is readily seen that the Pascal plane cuts the six edges of the tetrahedron in the points 12), 23), 31), 41), 42), 43).

If we take the original tetrahedron formed by the four planes in (6) as a tetrahedron of reference, we can give analytical expression to the above results. In this case the matrix (L_1, L_2, L_3, L_4) becomes the unit matrix, and $u_1^2, u_2^2, u_3^2, u_4^2$ are proportional to the principal minors of order 3×3 in the matrix A. The equation of the Pascal plane is

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0,$$

while the planes P_1, P_2, P_3, P_4 are respectively

$$\begin{aligned} -u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 &= 0,\\ u_1x_1 - u_2x_2 + u_3x_3 + u_4x_4 &= 0,\\ u_1x_1 + u_2x_2 - u_3x_3 + u_4x_4 &= 0,\\ u_1x_1 + u_2x_2 + u_3x_3 - u_4x_4 &= 0, \end{aligned}$$

and determine a tetrahedron whose vertices are

$$\begin{array}{llll} (-1/u_1, & 1/u_2, & 1/u_3, & 1/u_4),\\ (1/u_1, & -1/u_2, & 1/u_3, & 1/u_4),\\ (1/u_1, & 1/u_2, & -1/u_3, & 1/u_4),\\ (1/u_1, & 1/u_2, & 1/u_3, & -1/u_4). \end{array} \tag{9}$$

Provided that each of the four planes (6) intersects the quadric (7), the above planes and points are real.

The form exhibited by the coordinate-sets (9) shows that the following theorem* is true; *The tetrahedron determined by the planes* (6) *belongs to the same desmic system* [**3**] *as the tetrahedron determined by the planes* P_1, P_2, P_3, P_4.

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1. J. Sylvester, Instantaneous demonstration of Pascal's theorem by the method of indeterminate coordinates, Phil. Mag. vol. 37, 1850, p. 212.

2. G. Salmon, Analytic Geometry of Three Dimensions, 4th ed., Dublin, 1882, p. 122.

3. R. Hudson, Kummer's Quartic Surface, Cambridge University, 1905, p. 2.

* The author is indebted to the referee for pointing out this theorem.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

A REMARK ON LAURENT EXPANSIONS

HOWARD OSBORN, University of Illinois

Given a single-valued analytic function f, any Laurent expansion $\sum a_n(z-z_0)^n$ of f can be obtained from any other Laurent expansion $\sum b_n(z-z_0)^n$ of f about the same point by means of an elementary device, and furthermore in many cases the sequence $\{a_n-b_n\}$ is easy to describe. For example, there are two Laurent expansions of $(1+z)^{-1}$ about the origin, one of them an ordinary power series, with $a_n-b_n=(-1)^n$, $n=0, \pm 1, \pm 2, \cdots$.

For convenience take $z_0=0$ and suppose that the annulus A of convergence of $\sum a_n z^n$ surrounds the annulus B of convergence of $\sum b_n z^n$, where both series represent a single-valued analytic function which is meromorphic in the closed annulus C between A and B, and let

$$\sum_{p=1}^{n_k} \frac{a_p^{(k)}}{(z-z_k)^p}$$

be the principal part of f at the pole $z_k \in C$, $k=1, \cdots, N$.

PROPOSITION 1. $a_n-b_n=\sum_{k=1}^{N}\sum_{p=1}^{n_k}\binom{-n-1}{p-1}a_p^{(k)}z_k^{-n-p}$.

Proof. Let α and β be positively oriented circles about the origin, lying in A and B respectively. Using the power series development of $1/z^{n-1}$ at z_k one easily computes the residue

$$\sum_{p=1}^{n_k}\binom{-n-1}{p-1}a_p^{(k)}z_k^{-n-p}$$

of $f(z)/z^{n+1}$ at z_k. Hence

$$a_n-b_n=\frac{1}{2\pi i}\int_{\alpha-\beta}\frac{f(z)}{z^{n+1}}\,dz=\sum_{k=1}^{N}\sum_{p=1}^{n_k}\binom{-n-1}{p-1}a_p^{(k)}z_k^{-n-p}$$

In case all the poles in C are simple the result is $a_n-b_n=\sum_{k=1}^{N}a_1^{(k)}z_k^{-n-1}$, and as a consequence one obtains

PROPOSITION 2. *If C is the common boundary of A and B, a circle of radius r, and if the only singularities of f on C are simple poles at $re^{i\theta_1}, \cdots, re^{i\theta_N}$, where $\theta_1, \cdots, \theta_N$ are commensurate with π, then the sequence $\{r^n(a_n-b_n)\}$ is periodic.*

TWO-SIDED IDEALS OF THE AFFINE NEAR-RING

KENNETH G. WOLFSON, Rutgers University

The set of all affine transformations $N(F, A)$ of a vector space A over a division ring F has been considered in [2]. It is shown there that N is a near-ring with commutative addition but not a ring. In addition, if (F, A) has finite dimension, the set S of constant transformations forms a maximal two-sided ideal and N/S is isomorphic to the simple ring of linear transformations. It is stated also that S is the unique largest ideal, although the proof of this is incomplete. To investigate the situation further we allow (F, A) to be of arbitrary dimension and determine all two-sided ideals and hence all homomorphic images of $N(F, A)$.

Let $T(F, A)$ denote the ring of all linear transformations of (F, A). For each ordinal $\nu \geqq 0$ let $T_\nu(F, A)$ be the set of all those linear transformations of (F, A) whose range has dimension less than $\aleph_\nu$. Set $T_{-1}=0$. By the word ideal we shall understand a two-sided near-ring ideal, a subset of N satisfying conditions a, b, c on page 518 of [2]. The term ring ideal shall refer to a subset of T which is a two-sided ideal of T in the ordinary ring sense. Our results are:

I. *For each* $\nu \geqq -1$, $T_\nu + S$ *is an ideal of* $N(F, A)$, *and every ideal of* N *has this form, where* $[T_\nu + S = \{\sigma + \tau \mid \sigma \in T_\nu, \tau \in S\}]$.

II. $N/(T_\nu+S)$ *is isomorphic to* T/T_ν.

In particular, I implies that the set of ideals of N forms a well-ordered chain with S the minimal element. If (F, A) has finite dimension then $T_\nu = T$ for all $\nu \geqq 0$, so that $T_\nu + S = N$. Hence S is the only proper ideal of N and thus would certainly be the unique largest ideal as stated in [2].

From II it follows that every homomorphic (nonisomorphic) image of N is actually a ring and in fact isomorphic to a dense ring of linear transformations. For $\nu \geqq 0$ it is known that T/T_ν is a dense ring containing no nonzero transformations of finite rank.

Proof of I. Let the constant transformation $x \to k$ be denoted by σ_k so that $x\sigma_k = k$ for all $x \in A$. Then any affine transformation $\tau' = \tau + \sigma_k$ where τ is linear. Now let $Q > 0$ be an ideal of N. Then Q must contain a nonzero transformation which is not linear. For let $\tau \in Q$ be linear and $\neq 0$. Then there exists $k \in A$ such that $k\tau \neq 0$. Then $\sigma_k \in N$ so that $\sigma_k \cdot \tau \in Q$. But $\sigma_k \cdot \tau = \sigma_{k\tau} \neq 0$ and is a constant transformation and certainly nonlinear. Hence Q contains a transformation of the form $\tau + \sigma_k$ where $k \neq 0$, τ is linear ($\tau = 0$ is, of course, possible). Since $NQ \leqq Q$ and $0 \in N$, $0 \cdot (\tau + \sigma_k) = \sigma_k \in Q$. Next we show $Q \geqq S$. Let $\sigma_{k'}$ be any constant transformation. Then there exists a linear transformation α such that $k\alpha = k'$. Since $\sigma_k \in Q$ and $0, \alpha \in N$ we have $(0 + \sigma_k)\alpha - 0 \cdot \alpha = \sigma_k \cdot \alpha = \sigma_{k\alpha} = \sigma_{k'} \in Q$. Hence $Q \geqq S$.

Now if $\tau + \sigma_k \in Q$ then $(\tau + \sigma_k) - \sigma_k = \tau \in Q$ since Q is an additive group. Let J be the set of all linear transformations in Q. We shall show that J is a ring ideal of T. Certainly J is an additive group since Q is one. J is a left ring ideal

since $TJ \leqq NQ \leqq Q$ and the elements of TJ are linear, making $TJ \leqq J$. It remains to show $JT \leqq J$ and it is again sufficient to show $JT \leqq Q$. Let $\beta \in J$ and $\gamma \in T$. Then $\beta \in Q$ and $\gamma \in N$ so that $(0+\beta)\gamma - 0 \cdot \gamma = \beta\gamma \in Q$.

Hence we have that $Q = J + S$ where J is a ring ideal of T. But all ring ideals of T have the form T_ν ([1], p. 198), so that $Q = T_\nu + S$ for some $\nu \geqq -1$.

To complete the proof we must show that $Q = J + S$ is an ideal whenever J is a ring ideal of T. Clearly Q is an additive group. Let $\tau + \sigma_a \in N$ $(\tau \in T)$ and $\alpha + \sigma_b \in Q$ $(\alpha \in J)$. A direct computation yields $(\tau + \sigma_a)(\alpha + \sigma_b) = \tau\alpha + \sigma_{a\alpha} + \sigma_b \in J + S$. Hence $NQ \leqq Q$. To verify condition (c) for Q, let $\tau + \sigma_a$, $\gamma + \sigma_c \in N$ and $\alpha + \sigma_b \in Q$. Then

$$[(\tau + \sigma_a) + (\alpha + \sigma_b)](\gamma + \sigma_c) - (\tau + \sigma_a)(\gamma + \sigma_c) = \alpha\gamma + \sigma_{b\gamma} \in J + S.$$

The last computation can be achieved by applying the previous rule twice.

Proof of II. Consider the mapping $\beta: N = T + S \rightarrow T/T_\nu$ defined by $\tau'\beta = (\tau + \sigma_k)\beta = \tau + T_\nu$. This is obviously a near-ring homomorphism of N onto the ring T/T_ν. The kernel of β is the ideal $T_\nu + S$. Since the homomorphism theorem holds for near-rings the conclusion follows.

References

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ESSENTIAL SIMILARITY: A COUNTEREXAMPLE

A. J. Goldman, National Bureau of Standards

Let G be a group under the operations $x * y$ and $x \# y$. The groups $G(*)$ and $G(\#)$ are called *essentially similar* (*ES*) if there exists c in G such that $x * y \equiv x \# c \# y$. This concept was considered by D. Ellis,* who formulated the following

Conjecture. *If $G(+, *)$ and $G(+, \#)$ are s-fields, then* (*ES*) *subsists.*

We present the following counterexample. Let the additive group $G(+)$ be the direct sum of the additive group of rational numbers with itself; thus G consists of ordered pairs (r_1, r_2) of rational numbers. Let m, n be distinct positive integers which are not perfect squares and define two multiplications on G by

$$(x_1, x_2) * (y_1, y_2) = (x_1y_1 + mx_2y_2, x_1y_2 + x_2y_1),$$
$$(x_1, x_2) \# (y_1, y_2) = (x_1y_1 + xn_2y_2, x_1y_2 + x_2y_1).$$

Then $G(+, *)$ and $G(+, \#)$ are fields since they are isomorphic to the quadratic number fields $R(\sqrt{m})$ and $R(\sqrt{n})$ under the maps $(x_1, x_2) \rightarrow x_1 + x_2\sqrt{m}$ and $(x_1, x_2) \rightarrow x_1 + x_2\sqrt{n}$, respectively.

* D. Ellis, Cross-associativity and essential similarity, this Monthly, vol. 60, 1953, pp. 545–546.

If we set $x=(x_1, x_2)$, $y=(y_1, y_2)$, and $c=(c_1, c_2)$, we find by an easy calculation that c obeys the condition of (ES) if and only if the equations

$$x_1y_1 + mx_2y_2 = x_1(c_1y_1 + nc_2y_2) + nx_2(c_1y_2 + c_2y_1), \tag{1}$$

$$x_1y_2 + x_2y_1 = x_1(c_1y_2 + c_2y_1) + x_2(c_1y_1 + nc_2y_2) \tag{2}$$

hold for all rational x_1, x_2, y_1, y_2 such that $(x_1, x_2) \neq (0, 0)$ and $(y_1, y_2) \neq (0, 0)$. Let $x_1=y_1=1$ and $x_2=y_2=0$ to deduce $c_1=1$ from (1); then set $x_2=y_2=1$ and $x_1=y_1=0$ to deduce $m=n$ from (1). Since we originally assumed $m \neq n$, we have shown that no c obeying the condition of (ES) exists.

I am indebted to the referee for pointing out (a) that nonidentical group multiplications with the same identity element cannot be (ES) and (b) that a counterexample in which multiplications have different identity elements can be obtained from the above by the single change $(x_1, x_2) \# (y_1, y_2) = (x_1y_2+x_2y_1, nx_1y_1+x_2y_2)$.

A DECOMPOSITION THEOREM FOR THE INTEGERS MODULO q*

NEAL ZIERLER, Lincoln Laboratory, Massachusetts Institute of Technology

During an investigation of finite fields which will be reported elsewhere (a preliminary version has appeared as an internal Lincoln Laboratory document), it was observed that not only is it true that a and a^r are of the same order for every element a of a finite field F with $q+1$ elements when r is prime to q but that whenever a and b are elements of F of the same order there exists r prime to q for which $a^r=b$. It is not difficult to see that the second statement is a direct consequence of the following result. Let q be an arbitrary positive integer and let m be any integer. Then there exist a divisor d of q and a number r prime to q such that $m \equiv dr \pmod{q}$. If we observe now that d is uniquely determined by m and q we are led to the following

THEOREM. *Let q be a positive integer and let S be the ring of least positive residues modulo q. Let G be the multiplicative group of regular elements of S. Then the collection $\{dG\}_{d|q}$ is a decomposition of S. That is, $S=\bigcup_{d|q} dG$ and $d_1G \cap d_2G = \phi$ if $d_1 \neq d_2$.*

LEMMA. *Let $q'|q$ and suppose $(r', q')=1$. Then there is an r in G such that $r \equiv r' \pmod{q'}$.*

Proof. Let d denote the product of the distinct prime factors of q which do not divide q'. Then $(d, q')=1$ and so $\{r'+aq'\}$, $a=0, \cdots, d-1$, is a complete residue system modulo d. Hence for some a_0, $r'+a_0q' \equiv 1 \pmod{d}$. If the prime p divides $(q, r'+a_0q')$, then $p \nmid q'$ for otherwise $p | (r', q')$. Thus $p|d$ which implies $p|1$ and so $r=r'+a_0q'$ has the required properties.

* The research in this document was supported jointly by the Army, Navy and Air Force under contract with the Massachusetts Institute of Technology.

Proof of the theorem. For each divisor d of q let A_d denote the set of all those a in S for which $(a, q)=d$. Clearly the collection $\{A_d\}_{d|q}$ is a decomposition of S. Since $dG\subset A_d$ is apparent, the proof is complete if $A_d\subset dG$. Let $a\in A_d$ and let $q'=q/d$. Then $(a/d, q')=1$ and it follows from the lemma that r in G can be found for which $a/d\equiv r \pmod{q'}$. Hence $a\equiv dr \pmod q$ which in S means that $a=dr$.

COROLLARY 1. *Let H be a cyclic group of finite order q and for h in H let $\theta(h)$ denote the order of h. Let S denote the ring of least positive residues modulo q. Let $h\in H$ and let k be any member of S for which $q/(k, q)=\theta(h)$. Then there exists a generator g of H for which $g^k=h$.*

Proof. Let f be any generator of H. Then $f^j=h$ for some j with $\theta(h)=\theta(f^j)=q/(j, q)$ and so $(j, q)=(k, q)$. It follows from the theorem that $k=dr$, $j=dr_1$ where $d|q$ and r, r_1 are in G. Then $g=f^{r_1r^{-1}}$ has the required property.

COROLLARY 2. *Let H be a cyclic group of finite order q and let g and h be two elements of H of the same order. Then there exists r prime to q for which $g^r=h$.*

Proof. Let $d=q/\theta(g)$. By Corollary 1 there exist generators f_1 and f_2 of H such that $f_1^d=g$ and $f_2^d=h$. Then $f_2=f_1^r$ for some r prime to q and so $g^r=h$.

AN n-LINE PROPERTY

J. W. CLAWSON, Ursinus College

Using the notation and method of the article *A chain of circles associated with the n-line*,* the equation of the Miquel circle with center $C_{n!/f}$ passing through $K_{n!/f}$ is

$$z\bar{z}(\mathbf{n!/f}) - z\sum k_1\mathbf{1}^{2n-5}(\mathbf{n!/1f}) - \bar{z}(-1)^n\mathbf{n!/f}\sum k_1\mathbf{n!/1f(n!/1f)}$$
$$= (-1)^n\mathbf{n!/f}\sum k_1k_2\mathbf{n!/12f(n!/12f)}\{\mathbf{1}^{2n-8}-\mathbf{2}^{2n-8}\}\{\mathbf{1}^2+\mathbf{2}^2\}.$$

The equation of the Miquel circle with center at $C_{n!/g}$ may be written down by replacing $\mathbf{f}$ by $\mathbf{g}$ in the above equation. Multiplying the first equation by $(\mathbf{1f})(\mathbf{2f})(\mathbf{3f})\cdots(\mathbf{fn})$ and the second by $(\mathbf{1g})(\mathbf{2g})(\mathbf{3g})\cdots(\mathbf{gn})$ and subtracting, we find the radical axis of these two circles to be

$$z\sum k_1\mathbf{1}^{2n-5}(\mathbf{n!/1}) + \bar{z}(-1)^n\{\mathbf{n!/fg}\}^2\sum k_1\mathbf{1(n!/1)}$$
$$= (-1)^{n-1}\mathbf{n!/fg}\sum k_1k_2\mathbf{n!/12fg(n!/12)}\{\mathbf{1}^{2n-8}-\mathbf{2}^{2n-8}\}\{\mathbf{1}^2+\mathbf{2}^2\}\{\mathbf{1}^2\mathbf{2}^2-\mathbf{f}^2\mathbf{g}^2\}.$$

The radical axis of the pair of Miquel circles with centers at $C_{n!/f}$ and $C_{n!/h}$ may be written down by replacing $\mathbf{g}$ by $\mathbf{h}$ in the above equation. Multiplying the first equation by $\mathbf{g}^2$ and the second by $\mathbf{h}^2$ and subtracting, we find that the radical center of the three circles is

* This MONTHLY, vol. 63, 1956, p. 306.

$$\frac{(-1)^n \mathbf{n}! \sum k_1 k_2 \mathbf{n}!/\mathbf{12}(\mathbf{n}!/\mathbf{12})\{\mathbf{1}^{2n-8} - \mathbf{2}^{2n-8}\}\{\mathbf{1}^2 + \mathbf{2}^2\}}{\sum k_1 \mathbf{1}^{2n-5}(\mathbf{n}!/\mathbf{1})}.$$

Since this is independent of **f**, **g** and **h**, it follows that

THEOREM I. *The n Miquel circles of the $(n-1)$-lines obtained by omitting in turn $l_1, l_2, \cdots, l_n$ from the n-line have a common radical center.*

Let us denote this expression by z_1, the radical center of the "new circles" (p. 312 of the article) by z_2, and the center of the Morley (centric) circle by z_3. If these three points are collinear,

$$(z_1 - z_3)(\bar{z}_2 - \bar{z}_3) = (z_2 - z_3)(\bar{z}_1 - \bar{z}_3).$$

Making use of the fact that $(\mathbf{n}!)(\mathbf{n}!/\mathbf{fg}) = (\mathbf{n}!/\mathbf{f})(\mathbf{n}!/\mathbf{g})(\mathbf{fg})$, it is not hard to show that

$$z_1 - z_3 = \frac{(-1)^n\{\mathbf{n}!\}^2 \sum k_1 \mathbf{1}^3(\mathbf{n}!/\mathbf{1}) \cdot \sum k_1 \mathbf{1}^{2n-9}(\mathbf{n}!/\mathbf{1})}{(\mathbf{n}!) \sum k_1 \mathbf{1}^{2n-5}(\mathbf{n}!/\mathbf{1})},$$

$$\bar{z}_1 - \bar{z}_3 = \frac{-\sum k_1 \mathbf{1}^5(\mathbf{n}!/\mathbf{1}) \cdot \sum k_1 \mathbf{1}^{2n-7}(\mathbf{n}!/\mathbf{1})}{(\mathbf{n}!) \sum k_1 \mathbf{1}(\mathbf{n}!/\mathbf{1})},$$

$$z_2 - z_3 = \frac{(-1)^n\{\mathbf{n}!\}^2 \sum k_1 \mathbf{1}(\mathbf{n}!/\mathbf{1}) \cdot \sum k_1 \mathbf{1}^{2n-9}(\mathbf{n}!/\mathbf{1})}{(\mathbf{n}!) \sum k_1 \mathbf{1}^{2n-7}(\mathbf{n}!/\mathbf{1})},$$

$$\bar{z}_2 - \bar{z}_3 = \frac{-\sum k_1 \mathbf{1}^5(\mathbf{n}!/\mathbf{1}) \cdot \sum k_1 \mathbf{1}^{2n-5}(\mathbf{n}!/\mathbf{1})}{(\mathbf{n}!) \sum k_1 \mathbf{1}^3(\mathbf{n}!/\mathbf{1})}.$$

The required condition follows immediately. Thus

THEOREM II. *The radical center of the n Miquel circles and the radical center of the n new circles are collinear with the center of the Morley (centric) circle of the n-line.*

THE MAPPING WHICH TAKES EACH ELEMENT OF A GROUP ONTO ITS nth POWER

EUGENE SCHENKMAN AND L. I. WADE, Louisiana State University

In an Abelian group the mapping $\bar{n}$ of each element g to g^n is an endomorphism. We consider here the converse question whether a group is Abelian when $\bar{n}$ is an endomorphism.

Let G denote the group with elements a, b, g, *etc.*, and for each integer n let $G(n)$ denote the subgroup of G generated by all nth powers g^n of G; let $G\{n\}$ denote the subgroup of G generated by all elements whose orders divide n.

The results can then be stated as follows:

PROPOSITION. 1) *Let G be a group in which the mapping $\bar{n}$ is an endomorphism, then $G/G\{n^2-n\}$ is Abelian;*

2) *If $\bar{n}$ is an automorphism then $G/G\{n-1\}$ is Abelian;*

3) *If G has no elements whose orders divide n^2-n or if G has no elements whose orders divide $n-1$ when $\bar{n}$ is an automorphism, then G is Abelian.*

It should be noted that if G is the direct product of two groups A and B, where $A(n-1)=(1)$ and $B(n)=(1)$, then $\bar{n}$ leaves A elementwise fixed and maps B into (1). Hence any group of this type admits $\bar{n}$ as an endomorphism, and some such restriction as in 3) is necessary if G is to be Abelian.

The proof of the proposition is as follows. Since $\bar{n}$ is an endomorphism $a^nb^n=(ab)^n$. If a is cancelled on the left and b on the right, then $a^{n-1}b^{n-1}=(ba)^{n-1}$. It follows that $b^{1-n}a^{1-n}=(ba)^{1-n}$ and $\overline{1-n}$ is an endomorphism (*cf.* [1]).

Then $(aa^{n-1}b^n)(a^{1-n}b^{-n}b)=(ab)^n(ab)^{1-n}=ab$, and $1=a^{n-1}b^na^{1-n}b^{-n}$, or $a^{n-1}b^n=b^na^{n-1}$. This means that nth powers commute with $(n-1)$st powers, whence $G(n^2-n)$ is Abelian (*cf.* [2] p. 29 Ex. 4).

Now the product of the two endomorphisms $\bar{n}$ by $\overline{1-n}$ is an endomorphism of G onto the Abelian group $G(n^2-n)$ with kernel $G\{n^2-n\}$. This proves the first statement of the proposition.

Statement 2) follows from the fact that when $\bar{n}$ is an automorphism, every element is an nth power, and therefore the equation $a^{n-1}b^n=b^na^{n-1}$ implies that $G(n-1)$ is in the center of the group. It follows that $\overline{1-n}$ is an endomorphism, mapping G onto the Abelian group $G(n-1)$ with kernel $G\{n-1\}$.

Statement 3) follows immediately from 1) and 2).

We are indebted to the referee for the references to the literature.

References

1. J. W. Young, On the holomorphisms of a group, Trans. Amer. Math. Soc., vol. 3, 1902, pp. 186–191.

2. Hans Zassenhaus, Group Theory (English Edition), New York, 1949.

AN ALGORITHM FOR THE EVALUATION OF FINITE TRIGONOMETRIC SERIES

GERALD GOERTZEL, Nuclear Development Corporation of America, White Plains, N. Y.

The algorithm described below enables one to obtain the simultaneous numerical evaluation of $C=\sum_0^N a_k \cos kx$ and $S=\sum_1^N a_k \sin kx$ for given a_k, $\cos x$, and $\sin x$. Tables for $\sin kx$ and $\cos kx$ are not needed and only about N multiplications and about $2N$ additions or subtractions are required, so the method may be of interest to programmers of digital computers.

The algorithm is defined by

$$\begin{aligned} U_{N+2} &= U_{N+1} = 0; \\ U_k &= a_k + 2\cos x U_{k+1} - U_{k+2}, \qquad k = N, N-1, \cdots, 1\cdot \\ C &= a_0 + U_1 \cos x - U_2, \qquad S = U_1 \sin x. \end{aligned}$$

To establish this result, consider

$$V_k = \sum_{j=k}^{N} a_j \sin (j - k + 1)x; \qquad k = 1, \cdots, N,$$
$$V_{N+1} = V_{N+2} = 0.$$

Then

$$a_k \sin x + 2 \cos xV_{k+1} - V_{k+2}$$
$$= a_k \sin x + \sum_{j=k+1}^{N} a_j[2 \cos x \sin (j - k)x - \sin (j - k - 1)x]$$
$$= a_k \sin x + \sum_{j=k+1}^{N} a_j \sin (j - k + 1)x = V_k,$$

whence $V_k = U_k \sin x$ and, in particular, $S = V_1 = U_1 \sin x$. Furthermore

$$a_0 \sin x + V_1 \cos x - V_2 = a_0 \sin x + \sum_{j=1}^{N} a_j[\cos x \sin jx - \sin (j - 1)x]$$
$$= a_0 \sin x + \sum_{j=1}^{N} a_j \cos jx \sin x = C \sin x,$$

whence $C = a_0 + U_1 \cos x - U_2$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

A DIRECT PROOF FOR THE LEAST SQUARES SOLUTION OF A SET OF CONDITION EQUATIONS

ERWIN SCHMID, Coast and Geodetic Survey, Washington, D. C.

The problem of finding the solution of a set of m independent "condition" equations, linear in the n variables $v_1, \cdots, v_n$, $n > m$

$$(1) \qquad \sum_{j=1}^{n} a_{ij}v_j - a_{i0} = 0, \qquad i = 1, \cdots, m$$

such that $\sum_1^n v_j^2$ be a minimum is generally solved, following Lagrange, by minimizing instead, an equivalent function involving the so-called Lagrangian multipliers.

The following approach seems more direct, and generalizes a basic theorem in analytic geometry to n dimensions.

Multiplying in turn each of equations (1) by one of the m parameters

the required minimum solution since the P^* are a subset of the points of (2) and the point $(p\lambda_j)$ marks the minimum of $\sum v^2$ for all points of (2). The set of parameters k_i^* which make the critical point a P^* are found by substituting (5) in (1), resulting in the usual m normal equations, linear in the m variables k_i^*:

$$
\begin{aligned}
(7)\qquad & \left(\sum_{j=1}^{n} a_{ij}a_{1j}\right)k_1^* + \left(\sum_{j=1}^{n} a_{ij}a_{2j}\right)k_2^* + \cdots \\
& \qquad + \left(\sum_{j=1}^{n} a_{ij}a_{mj}\right)k_m^* - a_{i0} = 0, \qquad i = 1, \cdots m,
\end{aligned}
$$

the solution for which, when substituted back in (5), gives the required least squares solution of the system (1).

With this set of k_i^* it follows from (4) and (6) that the minimum $\sum_1^n v_j^2$ is

$$p^2 = Q^{-1}\left(\sum_{r=1}^{m} k_r a_{r0}\right)^2 = P\sum_{r=1}^{m} k_r a_{r0} = \sum_{r=1}^{m} k_r^* a_{r0}.$$

An n-dimensional geometrical interpretation of this result follows by analogy with the 3-dimensional case for which (1) consists of two linear equations in three unknowns. In the general case the system (1) may be thought of as representing m planes in an n-dimensional Euclidean space with Cartesian coordinates v_j. Equation (2) is any plane containing the intersection of (1). In a Euclidean space $\sum_1^n v_j^2$ is, by definition, the square of the distance from a point in this plane to the origin, and this distance is therefore a minimum for the point (5), an extension to n dimensions of the footpoint of the normal to the plane from the origin. Furthermore, this minimal distance is, by (4), equal to the constant term of the normalized (*i.e.* $\sum_1^n \lambda_j^2 = 1$) linear equation as in the 2 and 3-dimensional cases. The parameters k_i effect a rotation of plane (2) and a corresponding displacement of the footpoint of the normal. When the k_i^* satisfy the normal equations (7) the position of (2) is such that the footpoint lies on the intersection of the planes (1).

THE LINEAR FUNCTIONAL EQUATION

G. S. Young, University of Michigan

In many undergraduate courses such as advanced calculus, it is proved that the only solutions of the functional equation

$$(1)\qquad f(x + y) = f(x) + f(y)$$

which are continuous, or continuous at a point, are the functions $f(x) = mx$, m a constant. To me, the usual proof, of showing that a solution is linear for x rational and then extending by continuity of the irrationals, lacks elegance, and repeats much of what was done—or should have been done—in setting up the real number system. There are more general hypotheses that imply the same conclusion, for example that f is bounded on some interval, or bounded on a set

of positive measure, or measurable. In this note I prove the first of these generalizations by an argument suitable for undergraduates and which is perhaps shorter than the usual one for the continuous case. The proof is at least not well-known, though I can hardly believe it has escaped notice. Green and Gustin have given an excellent discussion of the history of this equation in [1].

THEOREM. *If $F(x)$ is a solution to* (1) *which is bounded over an interval* $[a, b]$, *then it is of the form $F(x)=mx$ for some real number m.*

Proof. We show first that $F(x)$ is bounded over $[0, b-a]$. Suppose that for all y in $[a, b]$, $|F(y)|<M$. If x is in $[0, b-a]$, then $F(x+a)$ is in $[a, b]$, so that from

$$F(x) = F(x + a) - F(a),$$

we get

$$|F(x)| < M + F(a).$$

Accordingly, if $b-a=c$, $F(x)$ is bounded in $[0, c]$. Let $m=f(c)/c$, and let $\phi(x)=F(x)-mx$. Then $\phi(x)$ also satisfies (1). We have $\phi(c)=F(c)-mc=0$. It follows that $\phi(x)$ is periodic, with period c, for

$$\phi(x + c) = \phi(x) + \phi(c) = \phi(x).$$

Further, as the difference of two functions bounded over $[0, c]$, $\phi(x)$ is bounded over $[0, c]$, and from the periodicity, $\phi(x)$ is therefore bounded over the entire x-axis.

Suppose x_0 is a number such that $\phi(x_0)\neq 0$. By an easy induction, we have $\phi(nx_0)=n\phi(x_0)$. We can make $|n\phi(x_0)|$ as large as we please by increasing n, which would contradict the boundedness of $\phi(x)$. Therefore $\phi(x)\equiv 0$, or $F(x)\equiv mx$.

The existence of pathological solutions is something in the reach of quite superior students in advanced calculus, and I have several times had students who were able to report (to me) on Jones' example [2] of a pathological solution to (1) that has a connected graph.

It is easy to modify the argument to prove that the only solutions of

$$f(xy) = f(x) + f(y)$$

bounded on an interval $[1, a]$ are of the form $f(x)=k \log x$. Let $\phi(x)=f(x)-f(a)(\log x)/(\log a)$. Instead of periodicity, show that $\phi(ax)=\phi(x)$, which implies that $\phi(x)$ has the same bound in each interval $[1, a]$, $[a, a^2]$, $\cdots$. But $\phi(x^n)=n\phi(x)$.

References

1. J. W. Green and W. Gustin, Quasi-convex sets, Canad. J. Math., vol. 2, 1950, pp. 489–507.

2. F. B. Jones, Connected and disconnected plane sets and the functional equation $f(x)+f(y)=f(x+y)$, Bull. Amer. Math. Soc., vol. 48, 1942, pp. 115–120.

MULTIPLICATION AND DIVISION BY BINOMIAL FACTORS

R. V. PARKER, Scole School, Diss, England

Professor Miller in his note *A Pascal triangle for the coefficients of a polynomial*, this MONTHLY, vol. 64, 1957, pp. 268–9, gives in effect an algorithm for obtaining the product of binomial factors. The example he gives is essentially the expansion of $(x-2)(x-3)(x-5)(x-7)$. The process includes the case for $(x+1)^n$, expansion of which gives the binomial coefficients, which may be tabulated in the more usual form:

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

TABLE 1

often referred to as Pascal's Triangle, although Pascal himself seems to have given it in the form:

1	1	1	1	1	...
1	2	3	4	...	
1	3	6	...		
1	4	...			
1	...				

TABLE 2

(see *Oeuvres Complètes de Blaise Pascal*, Paris, 1889, vol. 3, p. 244, fig. 52).

The law of formation of Table 1 is that each term is the sum of the term diagonally above to the left and the term immediately above it; *e.g.*, $6=3+3$. We may also put this as:— $6=1.3+1.3$, where the factors 1 and 1 arise from the fact that the coefficients of $(x+1)$ are 1 and 1. If we had to expand $(x+2)^n$, each term of the appropriate table would be equal to the sum of *twice* the term diagonally above it to the left and *once* the term immediately above it:

1						
1	2					
1	4	4				
1	6	12	8			
1	8	24	32	16		
1	10	40	80	80	32	
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.

TABLE 3

which may be expressed as:

1						
1	2.1					
1	2.2	$2^2.1$				
1	2.3	$2^2.3$	$2^3.1$			
1	2.4	$2^2.6$	$2^3.4$	$2^4.1$		
1	2.5	$2^2.10$	$2^3.10$	$2^4.5$	2^5	
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.

TABLE 4

We may here generalize for the expansion of $(ax+b)^n$:

a				
a	ab			
a	$ab(a+1)$	ab^2		
a	$ab(a^2+a+1)$	$ab^2(2a+1)$	ab^3	
a	$ab(a^3+a^2+a+1)$	$ab^2(3a^2+2a+1)$	$ab^3(3a+1)$	ab^4
a	$ab(a^4+a^3+a^2+a+1)$	$ab^2(4a^3+3a^2+2a+1)$	$ab^3(6a^2+3a+1)$	$ab^4(4a+1)$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

TABLE 5

Here every term is the sum of b times the term diagonally above it to the left and a times the term immediately above it.

Applying this schema to Professor Miller's example, we have:

$-3+1$	1	-2			
$-5+1$	1	-5	$+6$		
$-7+1$	1	-10	$+31$	-30	
	1	-17	$+101$	-247	$+210$

TABLE 6

i.e. $(x-2)(x-3)(x-5)(x-7)=x^4-17x^3+101x^2-247x+210$.

The coefficients of the binomial factors may be taken in any order; *e.g.*,

$-2+1$	1	-5			
$-7+1$	1	-7	$+10$		
$-3+1$	1	-14	$+59$	-70	
	1	-17	$+101$	-247	$+210$

TABLE 7

The method can be applied where polynomials are involved, and where the coefficient of x in the binomial factors is not unity:

Expand $(x^2+4x-1)(2x-7)(x+5)$.

$$\begin{array}{r|lllll} -7+2 & 1 & +4 & -1 & & \\ 5+1 & 2 & +1 & -30 & +7 & \\ & 2 & +11 & -25 & -143 & +35 \end{array}$$

$$= 2x^4 + 11x^3 - 25x^2 - 143x + 35.$$

If it is necessary to place a trinomial to the left of the vertical line, the mental work involved is slightly more onerous: $(x^2-4x+7)(2x^2+3x+4)$.

$$\begin{array}{r|lllll} 4+3+2 & 1 & -4 & +7 & & \\ & 2 & -5 & +6 & +5 & +28 \end{array}$$

$$= 2x^4 - 5x^3 + 6x^2 + 5x + 28.$$

The reverse process would of course be division by binomial factors. As an example, in the case of, say, $x^4+4x^3+6x^2+4x+1$ to be divided repeatedly by $x+1$, we have:

1	4	6	4	1
1	3	3	1	
1	2	1		
1	1			
1				

TABLE 8

Here each term is found by adding the term diagonally below it to the left to the term immediately below it; *e.g.* $6=3+3=1.3+1.3$, where again the factors 1 and 1 derive from the coefficients of $x+1$. Generally, if we wish to divide a polynomial expression by the binomial $ax+b$, we may write the resultant polynomial beneath the polynomial to be divided, and each term of the dividend will be equal to the sum of b times the term of the quotient diagonally below it to the left and a times the term of the quotient immediately below it.

Divide $6x^4-29x^3+40x^2-7x-12$ by $3x-4$.

$$\begin{array}{r|lllll} & 6 & -29 & +40 & -7 & -12 \\ -4+3 & 2 & -7 & +4 & +3 & \end{array}$$

$$= 2x^3 - 7x^2 + 4x + 3.$$

Thus we finally arrive at the method for turning a polynomial into factorial powers, which we define as:

$$x^{(n)} = x(x-1)\cdots(x-n+1)$$

This involves division by $(x-1)$, $(x-2)$, $\cdots$ and so on, and we may set down the work briefly as below:

Find the sum of $x^4-3x^3+5x^2+x-3$.

$$\begin{array}{r|lllll} & 1 & -3 & +5 & +1 & (-3 \\ -1+1 & 1 & -2 & +3 & (+4 & \\ -2+1 & 1 & +0 & (+3 & & \\ -3+1 & 1 & (+3 & & & \end{array}$$

Whence

$$\sum = \frac{(x+1)^{(5)}}{5} + \frac{3(x+1)^{(4)}}{4} + \frac{3(x+1)^{(3)}}{3} + \frac{4(x+1)^{(2)}}{2} - 3x$$

$$= \frac{x}{20}(4x^4 - 5x^3 + 10x^2 + 45x - 34), \text{ on reduction.}$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1296. *Proposed by Beckham Martin, Owens-Illinois Glass Co., Toledo, Ohio*

Problem E 1242 [1957, 433] says that the circle orthogonal to the circles (A'), (B'), (C') inscribed in the squares constructed exteriorly (or interiorly) on the sides of a triangle ABC is concentric with the nine point circle of triangle $A'B'C'$. Show, more generally, that all circles cutting the circles (A'), (B'), (C') under angles α, β, γ, respectively, where

$$\cos\alpha \sin A = \cos\beta \sin B = \cos\gamma \sin C,$$

are concentric with the nine point circle of triangle $A'B'C'$.

E 1297. *Proposed by D. C. B. Marsh, Colorado School of Mines*

In Solution II of Problem E 1243 [1957, 434] occurs the following conjecture for proof or disproof: When an integer and its reversal are unequal, their product is never a square except when both are squares.

Show that for any $n>2$ there is a nonsymmetric, nonsquare, n digit integer whose product with its reversal is a square.

E 1298. *Proposed by H. D. Grossman, New York, N. Y.*

It is not difficult to show that the longest linear section of a triangle is the longest side of the triangle. Is the greatest planar section of a tetrahedron the largest face of the tetrahedron?

E 1299. *Proposed by P. L. Chessin, Westinghouse Electric Corporation*

Find the limit of $1+x(1-x[1+x(1-x[1+ \cdots])])$, for $|x|<1$.

E 1300. *Proposed by D. A. Robinson, University of Wisconsin*

Let $D_1, D_2, D_3, \cdots$ be the determinants

$$|1|, \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{vmatrix}, \cdots,$$

respectively. Find the value of D_n for any positive integer n.

SOLUTIONS

A Diophantine System

E 1266 [1957, 365]. *Proposed by D. C. B. Marsh, Colorado School of Mines*

Solve $a^3-b^3-c^3=3abc$, $a^2=2(b+c)$ simultaneously in positive integers.

I. *Solution by E. W. Marchand, Eastman Kodak Company.* The first equation may be written $(a-b-c)[a^2+(b-c)^2+ab+bc+ca]=0$. Since the second factor cannot vanish for positive a, b, c, we must have $a=b+c=a^2/2$, whence the only solution in positive integers is $a=2$, $b=c=1$.

II. *Solution by Raymond Huck, Marietta College.* It is seen from the cubic relation that $c<a$, $b<a$. It follows from the quadratic relation that $2(b+c)=a^2<4a$, that $a<4$, and that a is even. Hence the only solution is $a=2, b=c=1$.

III. *Solution by H. M. Gehman, University of Buffalo.* Let us first solve without imposing the restriction that a, b, c be positive integers, but only requiring them to be real. If the second equation is solved for c, and the result substituted in the first equation, a small amount of manipulation gives

$$a(a-2)[(a^2+4a)^2+3(a^2-4b)^2]=0.$$

Hence either $a=0$ and $b=-c$, or $a=2$ and $b+c=2$, or $a=-4$ and $b=c=4$. The only solution in positive integers is $a=2$, $b=c=1$.

Also solved by Leon Bankoff, Merrill Barnebey, Robert Bart, D. M. Bloom, Margaret R. Blue, W. J. Blundon, D. A. Breault, D. M. Brown, P. L. Chessin, Eleanor G. Dawley, Brothers Day, Monte Dernham, Warren Dickinson, N. Ersec, Hazel E. Evans, J. F. Faley, N. J. Fine,

G. R. Flowers, Calvin Foreman, L. D. Goldberg, Michael Goldberg, F. L. Griffin, C. A. Grimm, A. B. Harper, Jr., A. R. Hyde, Edgar Karst, A. F. Kaupe, Jr., M. S. Klamkin, Frank Kocher, Sam Kravitz, Sidney Kravitz, M. A. Laframboise, J. T. Liddle, Joe Lipman, Bernhard Marzetta, J. B. Muskat, Herbert Nadler, T. D. Nagle, Bart Park, Paul Payette, J. L. Pietenpol, C. F. Pinzka, Montfort Plebstnoch, L. A. Ringenberg, Jeff Ritterman, B. S. Sackman, R. A. Sebastian, R. R. Seeber, Jr., R. E. Shafer, Nancy Siljander, Conrado Silva, Arnold Singer, E. P. Starke, R. F. Steinhart, R. S. Underwood, Chih-yi Wang, C. H. Wells, Jr., R. H. Wilson, Jr., the proposer, and one anonymous contributor. Late solution by Seymour Kass.

The proposer remarked that the system $a^3-b^3-c^3=kabc$, $a^2=m(b+c)$, where k, m are positive integers with $k>3m/2$, has no simultaneous solution in positive integers a, b, c. If $k=3$ and $m\geqq 2$, then $a=m$ and b and c are any two positive integers whose sum is m.

A Divergent Series

E 1267 [1957, 365]. *Proposed by Ivan Niven, University of Oregon*

The divergence of the harmonic series $\sum 1/n$ is often established by comparison with the obviously divergent series $\sum f(n)$ where $f(n)=2^{-k}$, the integer k being defined by the inequality $2^k\geqq n>2^{k-1}$. Establish the convergence or divergence of the series $\sum(1/n-f(n))$.

Solution by C. F. Pinzka, Xavier University, Cincinnati, Ohio. Since $\sum_{n=1}^{2^r} f(n)=r/2+1$ and $\sum_{n=1}^{2^r} 1/n>\int_1^{2^r+1}dx/x>r\ln 2$, $\sum_{n=1}^{2^r}(1/n-f(n))>r(\ln 2-1/2)-1$ and the series diverges.

Also solved by Julian Braun, T. S. Chihara, W. H. Colbert, Jr., N. J. Fine, D. S. Greenstein, F. L. Griffin, R. H. Hou, A. R. Hyde, N. D. Kazarinoff, J. F. Kennison, M. S. Klamkin, Joe Lipman, D. C. B. Marsh, Bernhard Marzetta, J. B. Muskat, Paul Payette, L. A. Ringenberg, Jeff Ritterman, Michael Rosen, R. E. Shafer, G. B. Thomas, Jr., and the proposer.

A Special Circulant

E 1268 [1957, 365]. *Proposed by A. J. Goldman and O. S. Wolf, Princeton University*

Evaluate the determinant D_n which has $(1, \cdots, n)$ as first row, $(2, \cdots, n, 1)$ as second row, *etc.*

Solution by A. E. Danese, Eastman Kodak Company. It is just as easy to evaluate the determinant which has $(a, a+d, \cdots, a+(n-1)d)$ as first row, $(a+d, a+2d, \cdots, a)$ as second row, *etc.* Take $n>2$. By adding all subsequent rows to the first row, we obtain $na+n(n-1)d/2$ as a factor. Removing this and performing the column operations c_r-c_n, $r=1, \cdots, n-1$, we obtain d^{n-1} as a factor. Now we can readily reduce the order of the determinant by one. In the reduced determinant we perform the column operations $c_r+(n-r)c_1$, $r=2, \cdots, n-1$, to obtain a determinant having only zeros below the secondary diagonal, only ones in the first column, and all remaining elements equal to n. This is readily evaluated so as to yield for the final result

$$(-1)^{n(n-1)/2}(nd)^{n-1}[a+(n-1)d/2],$$

which proves to be true also for $n=1$ or 2, and which for $a=d=1$ becomes

$$(-1)^{n(n-1)/2}\, n^{n-1}\,(n+1)/2.$$

These results occur in Muir, *A Treatise on the Theory of Determinants*, New York, 1933, pp. 442–444.

Also solved by Robert Bart, A. P. Boblétt, Julian Braun, K. A. Bush, F. M. Djoup, Jr., and A. G. Konheim (jointly), A. L. Epstein, N. Ersec, N. J. Fine, G. R. Flowers, J. F. Foley, Calvin Foreman, H. M. Gehman, Lee Goldberg, G. A. Harris, Jr., R. H. Hou, A. R. Hyde, P. G. Kirmser, M. S. Klamkin, M. A. Laframboise, Joe Lipman, R. L. London, Marshall Luban, E. W. Marchand, D. C. B. Marsh, Bernhard Marzetta, W. B. Morgan, W. L. Murdock, J. B. Muskat, C. O. Oakley, T. J. Pignani, C. F. Pinzka, E. J. F. Primrose, L. A. Ringenberg, Jeff Ritterman, D. A. Robinson, R. E. Shafer, D. D. Strebe, Ling-Erl Ting, Chih-yi Wang, Alan Wayne, David Zeitlin, and the proposers.

Editorial Note. The generalized problem has also been located in G. Kowalewski, *Determinantentheorie*, Chelsea, 1948, p. 109. Both the proposed problem and the generalization are solved in Problem 2774 [1920, 235].

A Roulette Problem

E 1269 [1957, 365]. *Proposed by Frank Kocher, Pennsylvania State University*

Prove that the area under one arch of the curve generated by a vertex of a regular polygon rolling on a straight line is equal to the area of the polygon plus twice the area of its circumscribed circle.

Solution by Michael Goldberg, Washington, D. C. The area under the arch can be divided by straight lines from the cusps to the stationary points into $n-1$ sectors and $n-2$ triangles. The lengths r_i of the straight lines are sides or diagonals of the polygon. Hence the triangles correspond to the $n-2$ triangles into which the polygon is cut by diagonals from a vertex.

The area of each sector is $\pi r_i^2/n$, where r_i is the length of a radius. The sum of the areas of the sectors is $(\pi/n)\Sigma r_i^2$. If unit masses are placed at the vertices of the polygon whose circumcircle is of radius R, then the total moment of inertia of these masses about the center is nR^2. Therefore the total moment of inertia about a vertex is nR^2+nR^2, since the center of gravity is translated the distance R. But this is the same as Σr_i^2. Hence $(\pi/n)\Sigma r_i^2=(\pi/n)2nR^2=2\pi R^2$.

When n becomes infinite, the familiar area of $3\pi R^2$ under the cycloid is obtained.

Also solved by Robert Bart, R. C. Brown, W. B. Carver, N. J. Fine, A. R. Hyde, L. M. Lewandowski, Joe Lipman, D. C. B. Marsh, Paul Payette, L. A. Ringenberg, Jeff Ritterman, F. G. Schmitt, Jr., Chih-yi Wang, and the proposer.

Regular Simplexes

E 1270 [1957, 365]. *Proposed by Leo Moser, University of Alberta*

What is the smallest positive even integer n such that in both n and $n+1$ dimensions the regular simplex of edge 1 will have a rational number as its content? (Dedicated to Professor H. S. M. Coxeter.)

Solution by W. J. Blundon, Memorial University of Newfoundland. The content of the regular simplex of edge 1 in E_n is given by $\{(n+1)/2^n\}^{1/2}/n!$. Since n is to be even, we must have $n+1=x^2$ and $(n+2)/2=y^2$, where x and y are positive integers. This yields the Pellian equation $x^2-2y^2=-1$, successive solu-

tions of which are (1, 1), (7, 5), (41, 29), (239, 169), $\cdots$. Since $x>1$, min $x=7$, so that min $n=48$. The next smallest values of n are clearly $41^2-1=1680$ and $239^2-1=57120$.

Also solved by N. J. Fine, Calvin Foreman, Michael Goldberg, N. W. Johnson, Paul Payette, and the proposer.

Editorial Note. For the content of a regular simplex see D. M. Y. Sommerville, *An Introduction to the Geometry of n Dimensions*, p. 126, or H. S. M. Coxeter, *Regular Polytopes*, p. 295.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4768. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let O be the center of the circumcircle (O) of a regular convex polygon $A_1A_2A_3 \cdots A_{25}A_{26}$ of 26 sides, and let O_1 and O_2 be symmetric to O with respect to the lines $A_{25}A_1$ and A_2A_6. Prove that O_1O_2 is equal to a side of an equilateral triangle inscribed in (O).

4769. *Proposed by H. Schwerdtfeger, University of Melbourne, Australia*

(1) Find all analytic functions $f(z)=u+iv$ of the complex variable $z=x+iy$, whose real part appears in the form $u=g(x)+h(y)$, where $g(x)$ and $h(y)$ are two real continuous functions of the real variables x and y whose first and second derivatives are continuous.

(2) The same except that $u=g(x)\cdot h(y)$.

4770. *Proposed by J. R. Holdsworth and J. R. Smart, China Lake, California*

Determine whether the following series is divergent or not:

$$S=\sum_{n=1}^{\infty}\sin^n n.$$

4771. *Proposed by Leonard Carlitz, Duke University*

Muir (*Contributions to the History of Determinants* 1900–1920, London and Glasgow) reproduces the assertion of Vogt that the determinant

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2^2 & \cdots & 2^n & 2^{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ n & n^2 & \cdots & n^n & n^{n+1} \\ \dfrac{n}{2} & \dfrac{n^2}{3} & \cdots & \dfrac{n^n}{n+1} & \dfrac{n^{n+1}}{n+2} \end{vmatrix}$$

vanishes for $n \geqq 2$ and even. Prove the truth of this assertion. Also show that $D_n \neq 0$ for n odd.

4772. *Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.*

It is easy to show that there exist consecutive prime pairs such that their difference is arbitrarily large. Do there exist consecutive prime triplets P_1, P_2, P_3 such that min (P_2-P_1, P_3-P_2) is arbitrarily large?

SOLUTIONS

Minimum Least Common Multiple

3834 [1937, 394]. *Proposed by Paul Erdös.*

Let $a_1 < a_2 < \cdots < a_n \leqq 2n$ be a sequence of positive integers. Then

$$\min\,[a_i, a_j] < 6([n/2] + 1),$$

where $[a_i, a_j]$ denotes the least common multiple of a_i and a_j. This is the best possible estimate.

Solution by C. R. Phelps, Rutgers University. The stated conclusion is invalid; for, when $n=3$, the sequence 3, 4, 5 has min $[a_i, a_j]=12=6([n/2]+1)$, and when $n=4$, the sequence 5, 6, 7, 8 has min $[a_i, a_j]=24>6([n/2]+1)$. However, the following corrected conclusion is valid:

$$\min\,[a_i, a_j] \leqq 6([n/2] + 1), \qquad n \neq 4.$$

Consider first, for any n, the special sequence in which $a_i=n+i$. The first even number in this sequence is $2([n/2]+1)$, and the number $3([n/2]+1)$ is in the sequence also, for $n=3$ and $n\geqq 5$. Hence min $[a_i, a_j]\leqq 6([n/2]+1)$ for this special sequence, except when $n=4$, as the conclusion is evident for $n=2$.

For any other sequence $a_1<a_2<\cdots<a_n\leqq 2n$, for any a_i there exists an integer $k_i\geqq 1$ such that $n<k_ia_i\leqq 2n$. If $k_ia_i=k_ja_j$ for some i, j, then this $[a_i, a_j]$ divides k_ia_i, and is thus $\leqq 2n$. Otherwise the k_ia_i, $i=1, \cdots, n$, are all distinct and exactly the special sequence in some order, so that for some p, q, k_pa_p and k_qa_q are precisely $2([n/2]+1)$ and $3([n/2]+1)$, whence $[a_p, a_q]$ divides $6([n/2]+1)$. Thus, in any case, min $[a_i, a_j]\leqq 6([n/2]+1)$.

We show also that $6([n/2]+1)$ is the exact minimum for the special sequence, for each n, and hence the best possible value. Suppose, in the special sequence, that $[a_i, a_j]=M<6([n/2]+1)$, with $a_i<a_j$. Then $a_j \geqq n+3$, so that $a_j>2([n/2]+1)$, with $M=ma_j$ for some integer m. Hence $m=2$, and $M=ka_i=2a_j$, so that $k\geqq 3$. If $k=3$, then $a_i<2([n/2]+1)$ with a_i even, an impossibility. If $k\geqq 4$, then $a_i<(3/2)([n/2]+1)$, so that $a_i<n+1$, also impossible. Hence $M<6([n/2]+1)$ is impossible.

Disjoint Permutations

4647 [1955, 447; 1957, 277]. *Proposed by James Munkres, Los Alamos Scientific Laboratory*

A permutation of the integers $1, \cdots, n$ is called an n-chain; two n-chains are disjoint if any two integers which are adjacent in one chain are not adjacent in the other. (The first integer is considered adjacent to the last for this purpose.) Does there exist, for each n, a collection containing $[(n-1)/2]$ mutually disjoint n-chains?

Editorial Note. III. Both Roberto Frucht and Haim Hanani have pointed out that the problem is classical (as your Editor should have remembered) being given by Lucas, (*Récréations Mathématiques*, Paris, 1891, t. II, pp. 162–168) under the title, *Les Rondes Enfantines*. The same discussion is given in Kraitchik, *Mathematical Recreations*, New York, 1942, pp. 227–229.

Lucas' solution may be summarized as follows. Write down the number n and, to its right, the numbers 1 and 2. Then put 3 in the 5th place, 4 in the 7th place, *etc.* When the last odd place has been filled put the remaining numbers (in order) in the even places returning from right to left. For example with $n=17$:

17, 1, 2, 16, 3, 15, 4, 14, 5, 13, 6, 12, 7, 11, 8, 10, 9.

The $[(n-1)/2]$ other required permutations are obtained, each from the preceding, by holding n fixed, replacing $(n-1)$ by 1, and increasing each of the other numbers by unity. For the case above we continue:

17, 2, 3, 1, 4, 16, 5, 15, 6, 14, 7, 13, 8, 12, 9, 11, 10;
17, 3, 4, 2, 5, 1, 6, 16, 7, 15, 8, 14, 9, 13, 10, 12, 11;

and so on until eight lines are complete.

That this procedure meets the requirements is almost obvious from the simple pattern. Lucas gives a geometrical device from which the proof is evident upon noticing that, because of the slopes of the lines, no two points can be joined by more than one line in all the $[(n-1)/2]$ designated paths.

Also solved by J. J. Harkema.

Binomial Coefficients

4723 [1957, 116]. *Proposed by Joachim Lambek, McGill University*

Given a nonnegative integer n and a prime p, obtain an expression for the number of binomial coefficients $\binom{n}{r}$ which are not divisible by p. (This generalizes problem I, 7 of the Putnam Competition Examination for 1956. See this MONTHLY, vol. 64, 1957, p. 24.)

Editorial Note. Write n in the scale of p, and let n_i be the digits which appear. Then the re-

quired number is $\prod(n_i+1)$. This is Theorem 2 of the note by N. J. Fine, *Binomial coefficients modulo a prime*, this MONTHLY, vol. 54, 1947, pp. 589–592.

Also solved by Robert Breusch, L. Carlitz, A. L. Davis, Harley Flanders, Emil Grosswald, A. P. Hillman and C. T. Long and H. Schneider, J. H. Hodges, Joe Lipman, D. C. B. Marsh, J. B. Roberts, Leopold Sauve, L. K. Williams, Chih-yi Wang, and the proposer. Late solution by F. W. Perkins.

Normal Matrices

4725 [1957, 116]. *Proposed by Olga Taussky, National Bureau of Standards*

A square $n\times n$ matrix $A=(a_{ik})$ with complex numbers as elements is called normal if $AA^*=A^*A$, where A^* is the transposed and complex conjugate of A. A matrix A is called nilpotent if for some integer $r\geqq 1$ the matrix A^r is the zero matrix 0. Prove (by rational methods only) that a normal and nilpotent matrix is the zero matrix.

I. *Solution by P. P. Saworotnow, The Catholic University of America.* It is easy to see that $A=0$ if and only if $AA^*=0$. Indeed, let $AA^*=(b_{ij})$, then $b_{ii}=\sum_{k=1}^{n}|a_{ik}|^2$ and so $b_{ii}=0$ only if each $a_{ik}=0$, $k=1,\cdots,n$.

It follows that if $A\neq 0$ and $A=A^*$ then $A^2\neq 0$ and $A^{2^m}\neq 0$ for each integer m and hence $A^r\neq 0$ for every integer r.

Now let A be normal and suppose $A^r=0$, then $(AA^*)^r=A^r(A^*)^r=0$. But then $AA^*=0$ which in turn implies $A=0$.

II. *Solution by Howard Eves, University of Maine.* A proof may be based on the well known fact that a complex normal matrix is unitarily similar to a diagonal matrix. Thus there is a nonsingular matrix P and a diagonal matrix D such that $D=P^{-1}AP$. Then $D^r=P^{-1}A^rP=0$, and, since D is diagonal, $D=0$. Therefore $A=PDP^{-1}=0$.

Also solved by Kurt Bing, Robert Breusch, T. A. Brown, L. Carlitz, P. M. Cohn, Harley Flanders, D. S. Greenstein, Ronald Jacobowitz, D. S. Kahn, R. L. McFarland, Albert Madansky, D. C. B. Marsh, M. H. Pearl, R. J. Pegis, H. Schneider, Melvin Schwartz, H. Schwerdtfeger, D. P. Squier, Hirofumi Uzawa, Chih-yi Wang, and the proposer.

Concurrent Lines and a Conic

4726 [1957, 117]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the parallels to the asymptotes of a conic (C), drawn through an arbitrary point P of its plane, intersect (C) in P_1 and P_2, if the perpendiculars to PP_1 and PP_2 at P_1 and P_2 intersect in a point O, and if the polar of P with respect to (C) intersects the conic in M_1 and M_2, then the perpendicular bisector of segment M_1M_2 passes through O.

Solution by Joe Lipman, University of Toronto. Let the lines PP_1 and PP_2 intersect the line M_1M_2 in K_1 and K_2, and intersect the ideal line in I_1 and I_2 respectively. Then the points P, P_1, K_1, I_1 are a harmonic range, whence P_1 is the midpoint of PK_1. Similarly P_2 is the midpoint of PK_2. Therefore O is the circumcenter of triangle PK_1K_2 and the perpendicular from O upon K_1K_2 bisects it. It remains only to show that the midpoint of K_1K_2 coincides with the midpoint of M_1M_2.

Let P be (0, 0), and let

$$(C) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \tag{1}$$

$$M_1M_2 \equiv gx + fy + c = 0, \tag{2}$$

$$(PP_1, PP_2) \equiv ax^2 + 2hxy + by^2 = 0. \tag{3}$$

Now the coordinates of the midpoint of M_1M_2 are found by taking $-\frac{1}{2}$ times the ratio of the coefficient of the linear term to that of the quadratic term in the equation resulting when y and x, respectively, are eliminated between (2) and (1), or, equivalently, between (2) and

$$\begin{aligned} &ax^2 + 2hxy + by^2 + 2gx + 2fy + c - 2(gx + fy + c) \\ &\qquad = ax^2 + 2hxy + by^2 - c = 0. \end{aligned} \tag{4}$$

The midpoint of K_1K_2 is found by a similar treatment of (2) and (3). Since (3) and (4) differ only in the absolute term, the results of both treatments will be the same.

Also solved by W. B. Carver, R. Deaux, Josef Langr, O. J. Ramler, P. Somanadham, Chih-y Wang, and the proposer.

Divergent Integral and Convergent Series

4727 [1957, 117]. *Proposed by G. R. MacLane, the Rice Institute*

Find a function $f(x)$, upper semi-continuous and nonnegative on $[0, \infty)$, bounded on each finite interval $(0, T)$, such that $\int_0^\infty f(x)dx = \infty$ and $\sum_{n=1}^\infty f(nh) < \infty$ for every $h>0$. (*Cf.* problem 4670, [1957, 119].)

Solution by the Proposer. We construct a sequence of closed nowhere dense sets E_n, $E_n \subset (n, n+1)$, $n \geqq 1$, such that $m(E_n) > 0$ and such that: if $rh \in \bigcup_1^n E_k$, $sh \in E_{n+1}$ for some $h>0$ and integers r and s, then $h < 1/(n+1)$. From such a sequence an f of the required sort is easily constructed:

$$f(x) = \begin{cases} 0, & x \notin \bigcup_1^\infty E_n \\ 1/m(E_n), & x \in E_n,\ n \geqq 1. \end{cases}$$

Set $aE_n = \{x \mid x/a \in E_n\}$. The sequence E_n is chosen inductively. Let E_1 be any closed nowhere dense set of positive measure in (1, 2). Assuming that $E_1, \cdots, E_n$ have been determined, set

$$F_n = \bigcup_{r,s} \left(\bigcup_{k=1}^n \frac{s}{r} E_k \right),$$

where $\bigcup_{r,s}$ is the union for all rationals s/r with $r \leqq (n+1)^2$. Now F_n is clearly nowhere dense; hence $(n+1, n+2)$ contains an open interval disjoint from F_n. Choose E_{n+1} to be any closed *etc.*, subset of this interval. The required property follows, for if $rh \in \bigcup_1^n E_k$ and $h \geqq 1/(n+1)$, then $n+1 \geqq rh \geqq r/(n+1)$, or $r \leqq (n+1)^2$. Thus $sh \in \bigcup_1^n (s/r)E_k \subset F_n$, and hence $sh \notin E_{n+1} \subset F_n'$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. I: Theory of Statistics. Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956. ix+208 pp. $6.00.*

This volume, along with its four companion volumes, joins the *Proceedings* of the two earlier Berkeley symposia as a source of fascinating results, stimulating conjectures, and novel formulations. Warm thanks should be expressed to Jerzy Neyman, organizer and editor of the symposium, and to his associates, for their efforts in arranging and publishing these *Proceedings.*

Sixteen papers appear in this book, representing the work of seventeen authors. The following outline crudely classifies these papers and very briefly describes their contents:†

Nonparametric analysis

Z. W. Birnbaum. Upper confidence limit for Pr $\{Y<X\}$.

B. L. van der Waerden. Two-sample test based on ranks transformed by the inverse normal cumulative function‡.

J. L. Hodges, Jr. and E. L. Lehmann. Stochastic approximation (Robbins-Monro) methods; choice of the sequence of coefficients to be used.

Aryeh Dvoretsky. Stochastic approximation methods in a very general framework.§

Parametric analysis

Joseph Berkson. Comparisons of first and second moments for six estimators in quantal bio-assay.

L. LeCam. Asymptotic estimation and testing theory, extending the work of Neyman and Wald.

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* By a special arrangement, members of the Mathematical Association of America may purchase any or all of the five volumes of the *Proceedings* at a 25% discount. Orders must be sent to the Statistical Laboratory, University of California, Berkeley 4, California. Reference must be made to membership in the Association and checks, made out to the University of California Press for the correct amount, must accompany orders.

† A more detailed version of this review (including references to relevant material published later) will be sent, on request to the reviewer, until the store of copies is exhausted.

‡ Tables for this test have since appeared. (B. L. van der Waerden and E. Nievergelt, Tafeln zum Vergleich Zweier Stichproben Mittels X-test und Zeichentest, Berlin, 1956.)

§ For a subsequent simplification, see J. Wolfowitz, On stochastic approximation methods, Ann. Math. Statist., vol. 27, 1956, pp. 1151–1156. For an expository article, see Cyrus Derman, Stochastic approximation, Ann. Math. Statist., vol. 27, 1956, pp. 879–886.

Herman Chernoff and Herman Rubin. Estimation of the unknown point of discontinuity of a probability density function having only one such point.

Charles Stein. The usual estimator for the mean of a multivariate normal distribution with three or more components, and with known covariance structure, is inadmissible with respect to the natural quadratic loss function.

Samuel Karlin. Detailed and extensive results on two-decision theory for the general exponential and still more general Pólya type distributions.

Sylvain Ehrenfeld. Complete class theorems for experimental designs in the linear hypothesis context.

G. Elfving. Allocation theory for linear hypothesis designs when repeated observations under the same conditions cannot be made arbitrarily often.

Intermediate between parametric and nonparametric analysis or not readily classified in these terms.

Wassily Hoeffding. Effects on statistical techniques of nonfulfillment of assumptions, *e.g.*, normality. Distinguishability of two sets of distributions.

Charles Stein. A connection between parametric and nonparametric theory relating to the existence of a one-dimensional problem for each state of nature that is asymptotically as difficult as the original problem.

Herbert Robbins. Bayes estimation when information about the prior distribution is accumulated in a sequence of observations.

Ulf Grenander and Murray Rosenblatt. Estimation of the spectral density of a stationary stochastic process.

Murray Rosenblatt. Comparison of Markov and least-squares estimators for estimating the regression parameters of a stochastic process in which a stationary residual process is superimposed on a mean function linear in the unknown regression parameters.*

The physical format of this volume is first-rate in all respects and I noticed very few typographical errors. The book suffers from two probably inevitable difficulties in any symposium publication. First, because of the fixed presentation date, about a quarter of the papers are of a progress-report nature and would probably not have been published until later were it not for the existence of the symposium. (This characteristic has merits as well as drawbacks.) Second, the absence of normal refereeing has resulted in lack of clarity in some of the papers.

WILLIAM KRUSKAL
University of Chicago

* The approach of Grenander and Rosenblatt to time series is expounded in their recent book, Statistical Analysis of Stationary Time Series, New York, 1957.

Fundamental Mathematics. By Leslie H. Miller. Henry Holt, New York, 1957. 323 pp. $3.50.

This book is a text and a workbook to be used for a remedial course on the freshman level in colleges. I like the introduction of algebraic notations and concepts in the arithmetic portion of the book, because it gives the student a better understanding of the arithmetic and prepares him for the formal algebra part of the book. More emphasis is placed on the comprehension of the fundamental concepts than in the other remedial workbooks with which I am familiar. With Chapters 10, 11, 12 on "Quadratic Equations," "Common Logarithms," and "Preview of Trigonometry," the book can be used for a three- or four-hour course per week for one semester. It can be used to advantage in Teachers Colleges for those going into elementary work.

I believe Professor Miller has taken care of the very heterogeneous group that one finds in a remedial class in college. Most of his material is of a very elementary nature, but there are still enough challenging ideas to keep the better of the poor students from becoming bored. It is very unfortunate that we have students in college who need an elementary remedial course; but since this is the situation, I know of no better book for the review than this workbook by Professor Miller.

JOHN C. KNIPP
University of Pittsburgh

Numerical Methods. By R. A. Buckingham. Pitman, New York, 1957. xii+597 pp. $15.00.

This is the most comprehensive book on computational techniques the reviewer has yet seen. The book is oriented toward the computer with a desk calculator rather than a high-speed electronic calculator, but can be of real value to both. In spite of its rather high price it should be in the library of every computing installation at least, and preferably on the desks of those who are called analysts in the high-speed digital computing industry.

The first chapter is an introduction to computation. The next five chapters are on polynomial fits to functions by the applications of the various differencing techniques. These fits are used for interpolation, integration and differentiation of tabulated functions.

The real worth of this book makes itself felt almost immediately. Not only does the author have comprehensive discussions of practically every technique known, with detailed references to the literature, but also he transmits a kind of "feel" for the subject by noting when each formula is particularly powerful or useful and the various pitfalls a computer is likely to fall into with each formula. In the reviewer's opinion it is this, not strictly mathematical, advice that makes this book outstanding in its field.

Chapters 7 and 8 are concerned with both approximate and recurrence methods of solving ordinary differential equations. Various types of equations and both single and two-point boundary conditions are discussed. Again there are extensive references to the literature and many helpful hints and warnings on each method discussed.

The next five chapters are on various algebraic problems such as finding the zeros of polynomials, least-squares fits, and simultaneous equations. These chapters contain comments on essentially every known useful technique including a good discussion of relaxation methods.

Chapter 14 contains discussions of matric and vector techiques of solving ordinary differential equations and certain integral equations.

Chapter 15 discusses polynomial fits by difference techniques to multivariable data. Many formulas are derived and the techniques explained for deriving other special-purpose formulas.

The last chapter contains a discussion of the partial differential equations readily susceptible to numerical solution and expounds these techniques with some emphasis on relaxation methods. These later chapters too are well referenced and full of helpful hints.

The book closes with four appendices on

(1) Relations Between Powers and Factorials.
(2) Summary of Difference Formulas, with Remainders.
(3) Lagrangian Formulas for Differentiation and Integration.
(4) Orthogonal Polynomials.

There is then an "Index of Useful Formulas in Text," which the reviewer has found very convenient, and finally a fairly complete index.

The book has with each important technique carefully worked out examples with comments keyed into them. At the end of each chapter is a collection of problems.

This book is easy to read and understand. It is very good as a reference work and should have a wide appreciative audience. The book seems a bit formidable as a text for a formal course in numerical analysis, though it could be used with considerable picking and choosing by the instructor.

JOHN E. MAXFIELD
Naval Ordnance Test Station,
China Lake, California

I Am a Mathematician. By Norbert Wiener. Doubleday, Garden City, N. Y., 1956. 380 pp. $5.00.

Professor Norbert Wiener is a pioneer in many fields, In *I Am a Mathematician* he has done it again. In autobiographical form the story begins in 1919 when Wiener came to Massachusetts Institute of Technology as an instructor in the department of mathematics, and contains a chronological account of his work, his travels, his scientific contacts, and his personal experiences during the next thirty-five years.

Although the book is rich in anecdote, the major emphasis in the story is on Wiener's intellectual development and his relation to the forward movement of scientific ideas during the period. He has taken considerable pains to discuss scientific concepts in everyday language and to illustrate their meanings by examples which are within the experience of the average citizen. Thus he gives a readable account of his early work on such abstract topics as Brownian motion, harmonic analysis, vector spaces, and quantum mechanics, relating each to its historical setting and to other fields.

Many factors influenced the development of his career. Most important was the fact that from the first he was convinced of the necessity to the mathematician of maintaining contact with the physical interpretations of purely mathematical problems. In many instances he found that his physical sense determined the images he formed and the tools by which he sought to solve problems. His frequent visits abroad to study, to lecture, and to attend scientific meetings brought him the thrill of meeting young contemporaries and the inspiration of knowing the older mathematicians whom he regarded as "links connecting us with the great past of mathematics," and he gives many vivid sentence descriptions of these people.

Of particular interest in the technical account of his later work is his analysis of the high-speed digital computer, and of the parallelism between computing machines and the human brain and nervous system. He includes also an interesting chapter giving his views on the social responsibility of a scientist. A scholar, he concludes, is not justified in withholding his research. His ideas belong to the times rather than to himself. He should, insofar as he is able, make public the social and economic dangers associated with his inventions.

As the record of his career unfolds, we see how the individual pieces of work he has done have built themselves into an organized pattern, coming inevitably to a climax in Cybernetics.

Although the book is primarily the story of ideas, it is no less the story of the man whose consecration and scientific discipline were such that he was able to produce those ideas. Despite his genius, his success was not achieved without hard work. In his early years at M.I.T., he taught twenty hours a week and still found time to study and to create mathematics. He had to overcome indifference and a certain amount of antipathy born of jealousy and social incompatability. Work in an active field of mathematics is at best a highly competitive business, so he was often working under pressure. He was hampered also by physical and emotional handicaps, and he had to experience the usual difficulties which come to a father of two children. Nevertheless he remained consecrated to the pursuit of truth. Nothing for him was greater than the thrill of creation. The task of the mathematician, he says, is "to use a rigid and demanding medium to express a new and significant vision of some aspect of the universe." He is therefore no less a creative artist than the painter or the musician, even though the strict discipline necessary to appreciate the aesthetic qualities of mathematical work means that only a few are able to recognize a

powerful theory or a clever and elegant proof.

In the opinion of this reviewer, Professor Wiener's book is both meaningful and readable for the educated reading public, and of especial interest to students and teachers of mathematics and other sciences. In particular, it should provide useful collateral reading for a "general education" course in mathematics.

JEANNE AGNEW
Oklahoma State University

BRIEF MENTION

Insights into Modern Mathematics. Twenty-third Yearbook. The National Council of Teachers of Mathematics, Washington, D. C., 1957. viii+440 pp.

The National Council of Teachers of Mathematics should indeed be proud in their achievement. This volume provides a careful treatment of representative topics in modern mathematics. The expositions are clear and have been developed on the assumption that the reader has about two years of college mathematics. Yet there has been no attempt to "water down" the mathematics of a more advanced level. This is indeed a book which every teacher of mathematics, either high school or college, should welcome. It is this reviewer's sincere hope that the effect of this volume, both upon college and high school teaching, will be profound. This does not mean that the subject matter of the twenty-third yearbook should be taught in these classes, but rather that it will help provide the teacher with the background necessary to make intelligent choices. There is considerable new mathematics which has been discovered recently, but the biggest change in high school is in *emphasis*. The reader can gain insight into the contents of the book by noting its thirteen chapters and the competence of the authors thereof:

I. Introduction, by Carroll V. Newsom.
II. The Concept of Number, by Ivan Niven.
III. Operating with Sets, by E. J. McShane.
IV. Deductive Methods in Mathematics, by Carl B. Allendoerfer.
V. Algebra, by Saunders MacLane.
VI. Geometric Vector Analysis and the Concept of Vector Space, by Walter Prenowitz.
VII. Limits, by John F. Randolph.
VIII. Functions, by Rudolph E. Langer.
IX. Origins and Development of Concepts of Geometry, by S. H. Gould.
X. Point Set Topology, by R. H. Bing.
XI. The Theory of Probability, by Herbert Robbins.
XII. Computing Machines and Automatic Decisions, by Charles B. Tompkins.
XIII. Implications for the Mathematics Curriculum, by Bruce E. Meserve.

Congratulations to the National Council of Mathematics on the publication of this, the most eximious of their yearbooks.

Engineering Analysis (*A Survey of Numerical Procedures*). By Stephen H. Crandall. McGraw-Hill, New York, 1957. x+417 pp. $9.50.

It is always a pleasure to welcome another of the Engineering Societies Monographs, which the McGraw-Hill Book Company publishes under the auspices of the American Society of Civil Engineers, American Institute of Mining and Metallurgical Engineers, the American Society of Mechanical Engineers, and the American Institute of Electrical Engineers. Among other problems treated from an engineering viewpoint in this book are the eigenvalue problems for both continuous and discrete systems. The use of finite difference methods and relaxation procedures may seem a bit startling to a few of the old guard, but is certainly compatible with current engineering practice. A mathematician who wishes to keep abreast of engineering students of today, could well spend some profitable hours with this monograph.

Pure Mathematics (*A First Course*). By J. K. Backhouse and S. P. T. Houldsworth. Longmans, Green, New York, 1957. xi+472 pp. $2.60.

It is always interesting to compare our current textbooks with the modern books from England. The current American books seem to be in much more of a state of flux than those of our English cousins, if this book is a typical example.

Basic Mathematics for Radio and Electronics. By F. M. Colebrook and J. W. Head. Philosophical Library, New York, 1957. 359 pp. $6.00.

Mathematics for Science and Engineering. By Philip L. Alger. McGraw-Hill, New York, 1957. xi+360 pp. Text edition, $5.50; Trade edition, $6.95.

This revision of Steinmetz's *Engineering Mathematics* should be an interesting book to teach, particularly for students who need some brush-up between calculus and their modern engineering mathematics. This reviewer recommends it for your consideration.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

NRC COMMITTEE ON APPLICATIONS OF MATHEMATICS

In 1954 the Division of Mathematics of the National Research Council established a standing Committee on Applications of Mathematics with the following functions:

(a) To facilitate cooperation among organizations concerned with various aspects of mathematics in applied settings.
(b) To call attention to the emergence of new areas in which significant applications of mathematics may be possible.
(c) To serve as a focus for the continuing scrutiny of problems concerned with training and research in mathematics as related to its applications.
(d) To take whatever steps are deemed appropriate to enhance the effectiveness of mathematics in its applications.

Dr. A. S. Householder, Oak Ridge National Laboratory, is now Chairman of this Committee. This Committee has been concerned with problems of training at the high school level. While doubtless many factors contribute in tending to repel even superior students from courses in mathematics, one of the important factors is lack of information about the careers that are open to mathematicians, and a scant and often distorted conception of mathematicians and their activities. The Committee felt that a good contribution toward making mathematics interesting might be by attempting to make mathematicians interesting. It seemed that this could be done best by presenting sketches, or profiles in the New Yorker sense, of living mathematicians.

The National Science Foundation has provided funds for interviews and for writing the profiles. It is hoped that eventually these profiles will be printed in a brochure to be made available at no cost to high school students, as well as their teachers and counsellors. It is planned to include in this brochure a statement regarding mathematical requirements of various occupations.

The Chairman of this Committee welcomes suggestions for future projects.

1958 COLLEGE-INDUSTRY CONFERENCE OF ASEE

"New Concepts in the Education and Development of Technical Manpower" is the theme of the 1958 College-Industry Conference of the American Society for Engineering Education to be held at the University of Michigan, January 30–31.

Four speakers are scheduled to present the new concepts arising from current research: Paul Reinert, President of St. Louis University, *New Concepts in Education from the President's Committee on Education Beyond High School;* Andrew A. Kucher, Vice President of Engineering and Research for the Ford Motor Company, *New Concepts in Engineering;* Donald C. Pelz, University of Michigan Survey Research Center, *New Concepts Useful in Motivating Engineering and Technical Personnel;* Lee Danielson or John W. Riegel, U-M Bureau of Industrial Relations, *New Management Concepts Obtained Through Interviews with Employed Engineers and Technicians.*

Four additional speakers will describe new programs designed by their companies to utilize new concepts of professional development: W. R. Collings, President of Dow-Corning Corporation; E. E. Sivacek, Chief Engineer, King-Seeley Corporation; A. L. Bain, Superintendent of Engineering Training for Western Electric Company; L. V. French, Director of Engineering Personnel for Whirlpool-Seeger Corporation. These reports hold many implications for education as well as for employers of engineers.

Registration will be held from 8:30 to 10:00 A.M., January 30 and the conference will adjourn at 2:00 P.M., January 31. For additional information contact the College of Engineering, 255 West Engineering Building, University of Michigan, Ann Arbor, Michigan.

PERSONAL ITEMS

Professor D. J. Struik, Massachusetts Institute of Technology, has been nominated to an Extraordinary Professorship at the National University of Mexico and elected an Honorary Member of the Mexican Mathematical Society.

American University: Mrs. Lucille Reifman, Associate, George Washington Univer-

sity, has been appointed Instructor; Dr. W. F. Shenton, Chairman, Department of Mathematics, has retired; Professor J. H. Smith has been appointed Chairman.

Boston University: Dr. Philip Maker has been appointed Assistant Professor; Mr. Chacko Abraham, Bombay University, has been appointed Instructor; Assistant Professor Donald Blackett has been promoted to Associate Professor; Professor E. B. Mode has retired with the title Professor Emeritus; Professor G. E. Noether is on leave as a Fulbright Lecturer at Tübingen; Professor Francis Scheid is on leave at the National Bureau of Standards.

Canisius College: Dr. T. D. Pozniak, St. Joseph's College, has been appointed Assistant Professor; Mr. R. L. Uschold has been promoted to Assistant Professor.

Cornell University: Dr. R. G. Heyneman, Research Assistant, University of California, Berkeley, and Dr. Constantine Kassimatis, Lehigh University, have been appointed Instructors.

DePaul University: Mr. E. G. McNiel, Instructor in Business Mathematics at the University, has been appointed Instructor in Mathematics; Dr. W. F. Darsow, Dr. L. M. Wiener, and Dr. G. L. Weiss have been promoted to Assistant Professors; Dr. Mary Weiss has obtained a postdoctoral research fellowship at the University of Chicago for 1957–58.

Eastern Michigan College: Mr. James H. Northey has been appointed Assistant Professor; Mr. P. D. Drees has been appointed Instructor.

Gonzaga University: Mr. B. E. Brown, Field Engineer, General Electric Company, and Associate Professor Ruth E. Dillavou, Western Montana College of Education, have been appointed Lecturers.

Lehigh University: Dr. J. W. Wall, Jr., recently at the Institute for Advanced Study, has been appointed Assistant Professor; Mrs. Marguerite Gravez, Interpreter, French Embassy, has been appointed Instructor; Mr. E. K. Dorff, Mr. John Nassar, and Mr. F. C. Oglesby, Graduate Assistants at the University, have been promoted to Instructors.

Marquette University: Dr. Frank Wagner, Midwest Research Corporation, Kansas City, Missouri, has been appointed Assistant Professor; Mr. William Weideman, Graduate Student, University of Michigan, and Miss Sheila Conheady, Graduate Student at the University, have been appointed Instructors; Mr. Ernest Gloyd has been appointed Assistant Instructor; Rev. L. J. Heider, Assistant Professor, has been promoted to Associate Professor; Dr. Earl Swokowski and Mr. John Kelley have been promoted to Assistant Professors.

Miami University: Mr. R. F. DeMar, Graduate Assistant, University of Wisconsin, has been appointed Instructor; Mr. C. C. Crell and Mr. R. W. Emmert have been promoted to Assistant Professors; Assistant Professor T. C. Holyoke is on leave on a Postdoctoral National Science Foundation Fellowship at the University of California.

Montana State University: Assistant Professor W. R. Ballard, Air Force Institute of Technology, and Mr. H. E. Reinhardt, Teaching Fellow, University of Michigan, have been appointed Assistant Professors; Dr. F. H. Young, Senior Engineer, Autonetics, Downey, California, has been appointed Associate Professor; Assistant Professor Wolfgang Schmidt is on leave at the University of Vienna.

New York University, Institute of Mathematical Sciences: Assistant Professor J. T. Schwartz, Yale University, has been appointed Associate Professor; Associate Professors Harold Grad, P. D. Lax, and Louis Nirenberg have been promoted to Professors; Research Associate Professor J. B. Keller has been promoted to Research Professor; Dr. Cathleen S. Morawetz has been promoted to Associate Professor; Dr. D. S. Carter, Los Alamos Scientific Laboratories, New Mexico, Mr. S. R. Foguel, Yale University, Professor M. P. Gaffney, Jr., Northwestern University, Dr. S. R. Goldner, Stellenbosch University, Union of South Africa, Dr. R. K. Juberg, University of Minnesota, Dr. T. E. Nieminen, University of Helsinki, Professor Robert Osserman, Stanford University,

Dr. R. N. Pederson, University of Minnesota, Dr. Maurice Roseau, Poitiers University, France, and Dr. Leonard Topper, U. S. Atomic Energy Commission, have been appointed Temporary Members for the academic year 1957–58.

New York University, Washington Square College: Research Associate Professor Joseph Keller has been promoted to Research Professor; Research Assistant Professor Samuel Karp has been promoted to Research Associate Professor; Assistant Professor Warren Hirsch has been promoted to Associate Professor.

Northwestern University: Dr. R. R. Goldberg, Mathematician, Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania, Mr. R. J. Driscoll, Research Assistant at the University, and Dr. Bruno Harris, Postdoctoral NSF Fellow, Yale University, have been appointed Instructors; Dr. M. P. Drazin, Research Fellow, Trinity College, Cambridge, England, has been appointed Visiting Lecturer; Dr. Edoardo Vesentini, Pisa University and the Politechnico di Milano, has been appointed Lecturer; Assistant Professors Meyer Dwass and Alex Rosenberg have been promoted to Associate Professors; Professor W. T. Reid is on leave for the academic year 1957–58 as Visiting Professor at the University of California; Assistant Professor Esther Seiden was on leave at the University of Wisconsin for the fall semester of 1957–58; Associate Professor M. A. Rosenlicht is on leave on a Guggenheim award.

Oklahoma State University: Assistant Professor J. W. Hamblen has been promoted to Associate Professor and Director of the Computing Center; Assistant Professor R. D. Morrison has been promoted to Associate Professor.

Oregon State College: Professor Cornelius Lanczos, Dublin Institute for Advanced Studies, has been appointed Visiting Professor; Professor Gustav Lochs, University of Innsbruck, has been appointed Visiting Associate Professor; Dr. Helmut Groemer, Postdoctoral Student, University of Vienna, has been appointed Instructor; Mr. R. R. Reynolds has been promoted to Assistant Professor; Professor Emeritus W. E. Milne is Visiting Professor at U. S. Naval Postgraduate School, Monterey, California, during 1957–58.

Pennsylvania State University: A new program for the degree of Master of Engineering has been inaugurated. This program permits people with an undergraduate engineering background to earn this degree off campus with no residence requirements. Professor B. H. Bissinger, Chairman of the Department of Mathematics of Lebanon Valley College, has been appointed to the staff of the University to teach mathematics in this program and is teaching in Harrisburg during 1957–58.

Purdue University: Associate Professor John Warfield, University of Illinois, and Dr. Annette Sinclair, Research Fellow, Harvard University, have been appointed Associate Professors; Dr. George Pedrick, University of Kansas, Dr. Lincoln Turner, Research Assistant at the University, and Dr. Norman Alling, Lecturer, Columbia University, have been appointed Assistant Professors; Mr. James Abbott, Teaching Assistant, University of Illinois, Mr. Irvin Lynn, University of Cincinnati, and Mr. Joseph Stefanac, Captain, U. S. Navy, have been appointed Instructors; Mr. W. R. Fuller, Mr. Frank Kozin, Dr. Judah Rosenblatt, and Dr. A. M. Yaqub have been promoted to Assistant Professors; Assistant Professors Melvin Henriksen, C. R. Hicks, G. T. Miller, and Stanley Reiter have been promoted to Associate Professors; Assistant Professor P. W. Overman has retired; Professor Arthur Rosenthal has retired with the title Professor Emeritus.

Randolph-Macon Woman's College: A Conference, Science in Perspective, was held at the College on November 12–14, 1957. The purposes of the conference were to interpret some of the major problems and issues before natural scientists today and to stimulate the interest of both secondary school and college students in the natural sciences. The following mathematicians participated in the program. Professor A. W. Tucker, Princeton University, spoke on *Natural Science and the Social Sciences* in the Symposium on Relationships between Natural Science and Other Fields; In the Sym-

posium on Current Issues and Developments, Dr. Louis Robinson, Manager of the Mathematics and Applications Department of the I.B.M. Corporation, spoke on *Applications of the IBM-704 Data-Processing Machine to the Computing Problems in Tracking the Earth Satellite;* Dean Mina Rees, Hunter College, gave an address entitled *Problems for Machines* in the Symposium on New Horizons in Science.

Rutgers University: Professor O. E. Villamayer, University of La Plata, Argentina, has been appointed Visiting Professor; Assistant Professor Harry Gonshor, Pennsylvania State University, has been appointed Assistant Professor; Mr. D. G. Malm, Teaching Assistant, Brown University, has been appointed Instructor; Dr. E. R. Gentile, Assistant, University of La Plata, has been appointed Lecturer; Assistant Professor V. L. Shapiro has been promoted to Associate Professor; Dr. R. E. Heaton and Dr. Abe Shenitzer have been promoted to Assistant Professors.

Stanford University: Dr. Karel deLeeuw, University of Wisconsin, has been appointed Assistant Professor; Assistant Professor G. E. Latta has been promoted to Associate Professor.

Syracuse University: Visiting Associate Professor Wolfgang Jurkat, Ohio State University, has been appointed Visiting Associate Professor; Assistant Professors Kathryn Morgan and O. O. Pardee have been promoted to Associate Professors.

Texas Christian University: Mrs. Florence Messinger has been appointed Instructor; Professor J. I. Tracy has retired.

University of Akron: Dr. Karl Johannes, Wisconsin State College, Whitewater, has been appointed Associate Professor; Miss Catharine Howard has been appointed Instructor; Assistant Professor Ernest Tabler has been promoted to Associate Professor; Associate Professor Louis Ross was Visiting Associate Professor at Western Reserve University during the summer of 1957. The University is now offering a Master of Science Degree in Engineering requiring 18 credit hours of graduate work in mathematics and physics. The University received a gift of an aerojet-general nucleonics Model 201 Nuclear Reactor from the General Tire and Rubber Company.

University of Alaska: Professor F. D. Parker, on leave from Clarkson College of Technology, is Professor and Head of the Department of Mathematics; Mr. W. A. Peterson, Boeing Aircraft Company, and Mrs. Ingrid Owren have been appointed Instructors; Mr. J. O. Distad has been promoted to Assistant Professor.

University of California, Berkeley: Professor Herie de Volgelaere, Notre Dame University, and Professor Istvan Fary, University of Montreal, have been appointed Visiting Associate Professors; Professors Leo Breiman and Steven Orey, University of Minnesota, have been appointed Visiting Assistant Professors; Dr. Jacob Feldman, National Science Foundation Fellow, Columbia University, Professor R. H. Lehman, and Dr. Maurice Sion, Member, Institute for Advanced Study, have been appointed Assistant Professors; Dr. A. H. Lightstone and Dr. Marianne Smith have been appointed Lecturers for 1957–58; Professors Harley Flanders and J. L. Kelley are on sabbatical leave in Cambridge, England; Miss Eva Kallin is spending the year in Cambridge, England, on an American Association of University Women Fellowship; Professor Albert Tarski has been appointed honorary member of the Dutch Mathematical Society; Professor Bernard Friedman, New York University, has been appointed Professor.

University of Delaware: Dr. C. C. Braunschweiger, Purdue University, has been appointed Assistant Professor; Dr. W. A. Thompson, Jr., Air Defense Board, has been appointed Associate Professor; Miss Edith A. McDougle has retired with the title Instructor Emeritus.

University of Detroit: Mr. J. B. Eckstein, Teacher, Harper Woods High School, Michigan, and Dr. Violet B. Haas, University of Connecticut, have been appointed Assistant Professors; Mr. R. G. Kane, Mr. G. E. Meike, and Miss Maryjo M. Nichols, have been appointed Instructors; Associate Professors G. E. Markle, E. D. McCarthy, and Emily C. Pixley have been promoted to Professors; Mr. J. G. Sowul has received a National

Science Foundation Faculty Fellowship and is studying at Wayne State University.

University of Kansas: Associate Professor Frank Gamblen, University of Western Australia, Nedlands, has been appointed Visiting Associate Professor; Dr. U. W. Hochstrasser, Assistant Professor at American University and Mathematician, Computation Laboratory, National Bureau of Standards, has been appointed Associate Professor and Director of the Computation Center; Dr. J. C. Lillo, Graduate Student, Princeton University, has been appointed Assistant Professor; Associate Professor George Springer has been promoted to Professor; Assistant Professor W. F. Donoghue, Jr., has been promoted to Associate Professor; Professor Nachman Aronszajn is on sabbatical leave in Cambridge, England, and Belgium; Professor Robert Schatten has received a research grant from the National Science Foundation and is at the University of California, Berkeley; Associate Professor W. F. Donoghue, Jr., presented a paper in Helsinki, Finland, at the International Colloquium on Function Theory, sponsored by the International Union of Mathematicians.

University of Maryland: Dr. G. J. Rieger, Privatdozent, University of Giessen, Germany, Professor John Horvath, University de los Andes, and Dr. J. A. Hummel, Postdoctoral Fellow, Stanford University, have been appointed Assistant Professors; Mrs. Shuh-yin Mar has been appointed Instructor; Dr. J. R. Mayor, American Association for the Advancement of Science, has been appointed Part-time Professor; Associate Professor G. S. S. Ludford is on sabbatical leave and is studying at Harvard University.

The Advisory Board on Education of the National Academy of Sciences has announced the first experimental use of color television in the teaching of a graduate course, Foundations of Analysis, offered by the University. The closed-circuit color T.V. facility at Walter Reed Army Medical Center is employed in teaching this course to a group of inservice high-school teachers of mathematics and science in the Washington, D. C. area. Professor R. A. Good of the University is lecturer for the course and Dr. J. R. Mayor, Director of Education, AAAS, is serving as consultant. A grant from the Fund for the Advancement of Education of the Ford Foundation has made possible the production of color kinescopes which will be used in the comparison of various techniques. These kinescopes will later be made available to other suitably equipped institutions for further evaluation tests.

University of Mississippi: Associate Professors N. A. Childress and R. D. Sheffield have been promoted to Professors; Professor Benjamin Ernest Mitchell has retired.

University of Missouri: Dr. D. F. Dawson, University of Texas, has been appointed Assistant Professor; Professor G. M. Ewing is on leave for the year 1957–58 as a mathematician at Fort Sill, Oklahoma.

University of Montreal: Dr. Alexis Zinger has been appointed Assistant Professor; Assistant Professors Geoffrey Fox and Jean Maranda have been promoted to Associate Professors; Professor Maurice L'Abbé has been appointed Chairman of the Department of Mathematics.

University of Rochester: Professor W. F. Eberlein, University of Wisconsin, has been appointed Professor; Mr. William Browder, Princeton University, Mr. P. J. Cohen, National Science Foundation Fellow, University of Chicago, Dr. Arshag Hajian, Research Assistant, Yale University, and Mr. D. E. Schroer, Acting Instructor, University of California, Berkeley, have been appointed Instructors; Associate Professors Dorothy L. Bernstein and Walter Rudin have been promoted to Professors; Professor Bernstein has received a two-year grant from the National Science Foundation and is now at the Institute for Numerical Analysis, University of California at Los Angeles; Dr. R. W. MacDowell is at Cornell University on a Science Faculty Fellowship from the National Science Foundation.

University of Virginia: Dr. R. T. Ives, University of Washington, has been appointed Instructor; Assistant Professor E. C. Paige, University of Illinois, has been appointed Assistant Professor.

University of Western Ontario: Mr. Douglas Clarke, Officer, R.C.A.F., and Mr. W. H. Adamson have been appointed Instructors.

Yale University: Dr. Leonard Gross, Research Assistant, University of Chicago, Dr. Walter Koppelman, Coder-Programmer, Institute of Mathematical Sciences, New York University, Dr. P. J. Message, Cambridge University, England, and Dr. A. B. Simon, Tulane University, have been appointed Instructors; Dr. George Seligman has been promoted to Assistant Professor.

Dr. W. R. Abel, University of Missouri, has been appointed Instructor at the University of Nebraska.

Dr. J. W. Addison, who has spent the past year at the Institute for Advanced Study and the Mathematical Institute of the Polish Academy of Sciences, has been appointed Assistant Professor at the University of Michigan.

Mr. C. E. Antle, Aerophysics Engineer, Convair, Fort Worth, Texas, has been appointed Instructor at Missouri School of Mines and Metallurgy.

Assistant Professor H. A. Arnold, University of California, Davis, has been promoted to Associate Professor.

Dr. D. D. Aufenkamp, Systems Laboratories Corporation, Sherman Oaks, California, has a position as Associate Research Scientist at Lockheed Missile Systems Division, Palo Alto, California.

Associate Professor Ray Authement, McNeese State College, has been appointed Associate Professor at Southwestern Louisiana Institute.

Associate Professor G. B. Ax, Virginia Military Institute, has been promoted to Professor.

Dr. W. L. Baily, Jr., Massachusetts Institute of Technology, has been appointed Assistant Professor at the University of Chicago.

Associate Professor Grace E. Bates, Mount Holyoke College, has been promoted to Professor and Chairman of the Department of Mathematics.

Mr. Jacques Bazinet, Graduate Student, University of Montreal, has been appointed Professor at the University of Sherbrooke, Province of Quebec, Canada.

Associate Professor R. F. Bell, Eastern Washington College of Education, has been promoted to Professor and Chairman of the Department of Mathematics.

Mr. W. C. Bennewitz, Graduate Assistant, University of Illinois, has been appointed Instructor at the University of Southern California.

Assistant Professor Donald C. Benson, Carnegie Institute of Technology, has been appointed Assistant Professor at the University of California, Davis.

Mrs. Shirley A. Blackett, Northeastern University, has been promoted to Assistant Professor.

Mr. D. J. Boyce, Graduate Assistant, Oklahoma State University, has been appointed Instructor, Central State College.

Associate Professor B. W. Brewer, Oregon State College, has been appointed Professor at Agricultural and Mechanical College of Texas.

Mr. C. A. Bridger, Director, Bureau of Vital Statistics, Division of Health, Jefferson City, Missouri, has been appointed Chief, Bureau of Statistics, State Department of Public Health, Springfield, Illinois.

Dr. F. A. Butter, Jr., Research Physicist, Hughes Aircraft Company, Culver City, California, has a position as an engineering specialist at Northrop Aircraft, Inc., Hawthorne, California.

Mr. A. J. Carlan, Research Physicist, American Optical Company, Southbridge, Massachusetts, is now employed as Physicist at Hoffman SemiConductor Division, Evanston, Illinois.

Miss L. Virginia Carlton, Visiting Lecturer, Northwestern University, has been appointed Chairman, Department of Mathematics, Centenary College.

Assistant Professor R. C. Carson, Lehigh University, has been appointed Assistant Professor at Western Reserve University.

Professor Abraham Charnes, Purdue University, has been appointed Research Professor, Northwestern University.

Miss Virginia B. Christian, Instructor, Eastern Illinois State College, has been appointed Instructor at South Dakota State College.

Assistant Professor R. A. Clark, Case Institute of Technology, has been promoted to Associate Professor.

Dr. M. L. Coffman, Senior Nuclear Engineer, Convair, Ft. Worth, Texas, has been appointed Associate Professor at Abilene Christian College.

Mr. Gerald Derman, Teaching Assistant, Rutgers University, is now a teacher at New Brunswick High School, New Jersey.

Dr. Jim Douglas, Jr., Assistant Research Engineer, Humble Oil and Refining Company, Houston, Texas, has been appointed Assistant Professor at Rice Institute.

Mr. R. E. Ekstrom, Mathematician, U. S. Naval Ordnance Plant, Indianapolis, Indiana, has been appointed Research Associate, Department of Engineering Mechanics, University of Florida.

Assistant Professor William Golomski, Marquette University, is now employed by the Oscar Meyer Company, Madison, Wisconsin.

Assistant Professor W. H. Greub, University of Maryland, has been appointed Assistant Professor at Johns Hopkins University.

Dr. J. K. Hale, Sandia Corporation, Albuquerque, New Mexico, has accepted a position as a member of the Mathematical Research Department, Remington Rand Univac, St. Paul, Minnesota.

Mr. R. W. Harruff, Student, Kent State University, is now Research Engineer at North American Aviation, Downey, California.

Associate Professor G. P. Henderson, University of Western Ontario, has joined the research staff of Imperial Oil, Ltd., Toronto, Ontario, Canada.

Mr. William Hoyt, Northwestern University, is an ONR Fellow at Johns Hopkins University.

Assistant Professor W. E. Jenner, Northwestern University, has been appointed Associate Professor at Bucknell University.

Professor Ralph Johanson, Boston University, has a position as Mathematician with the I.B.M. Corporation.

Mr. H. C. Kerr, Blinn College, is employed as Research Engineer with Convair, Daingerfield, Texas.

Mr. D. R. King, Teaching Assistant, Rutgers University, is now employed by the Radio Corporation of America, Camden, New Jersey.

Assistant Professor G. F. Leger, Syracuse University, has been appointed Assistant Professor at the University of Pittsburgh.

Research Associate Professor S. C. Lowell, New York University, has been appointed Professor and Director of the Graduate Program in Applied Mathematics and Physics, Adelphi College.

Mr. P. T. Mielke, Senior Group Engineer, Boeing Airplane Company, Seattle, Washington, has been appointed Associate Professor at Wabash College.

Dr. C. N. Mills, Visiting Lecturer, Florida State University, has been appointed Professor at Sioux Falls College.

Dr. H. W. Milnes, Senior Mathematician, Data Processing Group, General Motors Corporation, Detroit, Michigan, is now Senior Research Scientist, Science Group of the Corporation.

Dr. Josephine Mitchell, Westinghouse Research Laboratory, has been appointed Associate Professor at the University of Pittsburgh.

Dr. A. E. Nussbaum, Rensselaer Polytechnic Institute, has been promoted to Assistant Professor.

Mr. J. M. Osborn, Jr., Yale University, has been appointed Assistant Professor, Georgia Institute of Technology.

Dr. M. H. Pearl, University of Rochester, has a position with the National Bureau of Standards, Commerce Department, Washington, D. C.

Assistant Professor O. P. Sanders, University of Arkansas, has been appointed Professor at Louisiana Polytechnic Institute.

Dr. Albert Schild, Temple University, has been promoted to Associate Professor.

Professor J. P. Scholz, Head, Department of Mathematics, Western College for Women, has been appointed Professor at New Mexico Institute of Mining and Technology.

Mr. Aaron Siegel, Teaching Assistant, Rutgers University, has been appointed Instructor at the College of South Jersey of the University.

Professor Jack Silber, Roosevelt University, has returned to the University after spending four months as Consultant to the Operations Analysis Office at the Air Force Missile Test Center.

Assistant Professor G. L. Spencer, University of Maryland, has been appointed Associate Professor at Williams College.

Mr. Frederick R. White, University of Buffalo, has a position as Mathematician, Sylvania Electric Products, Mountain View, California.

Dr. J. B. Wilson, Assistant, University of Florida, has been appointed Assistant Professor at North Carolina State College.

Mr. A. W. Yonda, Temple University, has been appointed Associate Scientist at AVCO Manufacturing Company, Lawrence, Massachusetts.

Mr. Joseph Zilber, Visiting Lecturer, Northwestern University, is now Editorial Consultant, *Mathematical Reviews*, American Mathematical Society, Providence, Rhode Island.

Assistant Professor Stanley Bolks, Purdue University, died on January 26, 1957.

Associate Professor Mildred Crawford, Eastern Michigan College, died in November, 1956.

Professor Emeritus C. B. Upton, Teachers College, Columbia University, died on September 25, 1957.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

TELEVISION AND FILMS

It is proposed to collect in one issue of the MONTHLY as many articles as possible on the use of television and of films in the teaching of mathematics. Articles intended for this special issue should be submitted to the Editor, Professor R. D. James, Department of Mathematics, University of British Columbia, Vancouver 8, Canada, as soon as possible and in any event, by March 15, 1958. Since space is limited, such articles should be concise and should not include unnecessarily elaborate statistical data.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 160 persons have been elected to membership by the Board of Governors on applications duly certified.

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A SPECIAL MEETING OF THE IOWA SECTION

The first meeting of the Iowa Section of the Mathematical Association of America with a sizeable group of high school teachers of mathematics was held at the State University of Iowa, Iowa City, Iowa, on October 10, 1957, in connection with the Twenty-seventh Annual Conference of Teachers of Mathematics. Provost H. H. Davis of the

State University of Iowa (a former teacher of high school mathematics) welcomed the participants with *Some observations concerning mathematics.* Professors M. F. Smiley and H. V. Price presided at the morning and afternoon sessions, respectively. There were 98 persons in attendance of which 22 were members of the Iowa Section.

The program consisted of four invited lectures. At the morning session, Professor Henry Van Engen, Iowa State Teachers College, spoke on *Proposals of the Commission on Mathematics of the College Entrance Examination Board* and Professor H. P. Evans, University of Wisconsin, spoke on *The need for better coordination between high school and college mathematics.* The speakers of the afternoon session were Professor Max Beberman, University of Illinois, *The University of Illinois School Mathematics Project,* and Mr. Frank Allen, Lyons Township High School and Junior College, LaGrange, Illinois, *The work of the Secondary School Curriculum Committee of the National Council of Teachers of Mathematics.* These addresses elicited lively discussion by both college and high school teachers of mathematics.

M. F. SMILEY, *Governor*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, May 13, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS

KENTUCKY, University of Kentucky, Lexington, April 19, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Randolph-Macon Woman's College, Lynchburg, Virginia, April 26, 1958

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April 19, 1958.

MICHIGAN, University of Michigan, Ann Arbor, March 22, 1958.

MINNESOTA

MISSOURI, University of Missouri, Columbia, April 26, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA, San Francisco State College, January 18, 1958.

OHIO, Denison University, Granville, April 26, 1958.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April, 1958.

UPPER NEW YORK STATE, Ecole Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

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CONTENTS

FEBRUARY 1958

The AMERICAN MATHEMATICAL MONTHLY

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EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. JAMES, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

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THE CASE OF THE FORGETFUL BURGLAR

CAXTON FOSTER AND ANATOL RAPOPORT, Mental Health Research Institute, University of Michigan

While there has been considerable attention paid to random walk problems in one or more dimensions, there has as yet been little investigation of walks in which each previously occupied point becomes an absorbing site, which, if occupied again, ends the walk. Such a walk is a model for the case of the forgetful burglar who, working one side of a street, feels compelled to pass no more than one house by without plying his trade. Thus coming out of a particular house he may enter the one next door or skip that and enter the succeeding one. He becomes so absorbed in his burgling that he forgets from which direction he entered a particular house and which houses on the street he has already entered. If he is so unfortunate as to re-enter a house, the already aroused occupants detain him to his permanent disadvantage. The question of interest is "How long, on the average, can he expect to keep this up?" And if many such burglars operate in this fashion, "What is the distribution of the lengths of their paths?"

This problem is of course generalizable to burglars operating in two, three, or even more dimensions and skipping none, one, two, or more houses with either random distribution of choices or with some sort of bias in the choice. One could also include an arbitrary "decay" time for the arousal of the home owners. The general solution of this type of problem would be of interest not only to forgetful burglars but also to itinerant medicine shows, standard random-type drunkards with superstitions against retracing steps, some neutrons in chain reactions, impulses in a neural network, and to sociologists studying sociograms. Indeed, it was from an attempt to build a model for sociograms that interest in this problem originally arose.

The special case of the one-dimensional walk with only single steps is trivial, because the first reversal of direction terminates the walk. The authors have not succeeded in finding a general solution for problems of this type. But the special case of the one-dimensional walk in which steps may be either singles or doubles has been solved and may have heuristic value for a future generalization.

Consider, then, an infinite one-dimensional grid of sites. Starting from an arbitrary site, the walker takes with equal probability one or two steps in either direction. When a site previously occupied is re-entered, the walk terminates. It is required to calculate the frequency distribution of walk lengths.

1. Definitions. A *chain* is a path that has not yet terminated, that is, one in which no site has been entered more than once.

A *closed hook* is a chain which starts to the left (right) and ends at the site immediately to the right (left) of the starting site. We will call a closed hook which starts to the left, a left closed hook and one that starts to the right, a right

closed hook. We will call the starting site "0," the site immediately to the right "+1," the site immediately to the left as "−1." We will sometimes abbreviate "closed hook" to "hook."

A *trip* is a chain without reversal of direction. Obviously any chain is composed of trips of alternating directions.

A *coil* is a chain which proceeds steadily in one direction except for possible retrograde single steps, and ends on an extreme site of the chain.

An *open hook* is a chain which starts to the right (left) and ends at +1 (−1), and has at least one reversal of direction.

Note that an additional double step in the same direction changes an open hook into a closed hook. Right and left open hooks are analogous to right and left closed hooks, if the starting and ending sites are interchanged.

Examples of a closed hook, a coil, and an open hook are shown in Figure 1. In the proofs that follow it will be assumed that each path starts to the left. Complementary lemmas follow by symmetry.

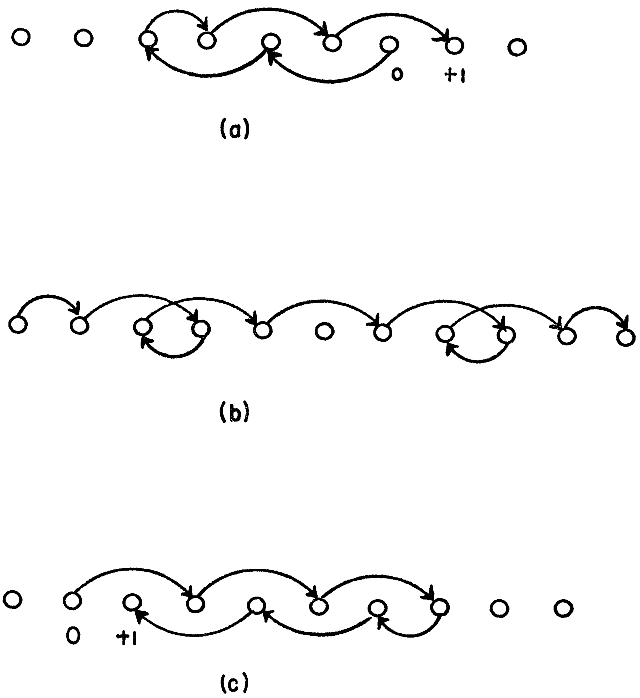

FIG. 1. (a) closed hook; (b) coil; (c) open hook.

LEMMA 1. *A two-step left closed hook starts with a single step. Conversely, if a closed hook starts with a single step, it is a two-step closed hook.*

Proof. If the first step is a double to the left, the hook cannot be completed

on the second step, because a single to the right lands on -1 and a double to the right terminates the path. Conversely if the first step is a single, a reversal must follow immediately; otherwise two sites $(0, -1)$ will have to be skipped to complete the hook, which is impossible. The reversal must be a double step, since otherwise the path is terminated. But the double step completes the hook on the second step, hence the lemma.

LEMMA 2. *In a closed hook exactly one reversal of direction occurs.*

Proof. At least one reversal is obviously necessary. We must show that no more than one can occur. If the trip to the right following the first reversal carries the path beyond 0, then $+1$ must be occupied, otherwise two consecutive sites $(0, +1)$ must be skipped, which is impossible. But then either $+1$ ends the hook and the lemma is proved, or $+1$ must be occupied again, a contradiction, since a hook is not a terminated path. If the first trip to the right does not carry past 0, then $+1$ must be reached after a subsequent trip to the right. But this is impossible since a double reversal following a trip to the left establishes a barrier of two consecutive occupied sites which no subsequent trip to the right can cross.

LEMMA 3. *All the steps of a left hook of h steps are completely determined.*

Proof. By Lemma 1, this is true for $h=2$. If $h>2$, the first step must be double by Lemma 1, and so must every subsequent step to the left except possibly the last one; otherwise some consecutive pair of sites will have to be skipped on the trip to the right, which is impossible. If the last (kth) step to the left is double, the first step to the right must be single followed by k double steps, making $h=2k+1$ in all. If the kth step to the left is single, the first step to the right must be double, followed by $k-1$ double steps, making $h=2k$ in all. Thus all steps are determined for $h=2k+1$ (odd) and for $h=2k$ (even). But this exhausts all cases.

COROLLARY 1. *At the completion of a left closed hook of h steps, h consecutive sites to the left of $+1$ are occupied.*

COROLLARY 2. *After completion of a left closed hook, the chain can continue only to the right.*

LEMMA 4. *If two consecutive reversals occur in a chain (that is reversals separating three consecutive trips), the trip following the first of these reversals either carries beyond $+1$, or consists of one single step.*

Proof. Suppose, on the contrary, that the first of these reversals is followed by more than one step but does not carry the trip beyond $+1$. Then at the end of this trip there are more than one occupied sites to the left, and the second reversal is impossible.

LEMMA 5. *All the steps of an open hook are determined.*

Proof. An open hook is equivalent to a closed hook if the starting and ending sites are interchanged.

COROLLARY. *An open hook has exactly one reversal.*

These results can be summarized in the principal theorem on the structure of chains.

THEOREM. *Every chain is composed of a closed hook of h steps, a coil of c steps, and an open hook of y steps.* $(h, c, y \geqq 0, h \neq 1, y \neq 1)$.

Proof. Suppose the first h steps complete a left hook. Then by Corollary 2 the chain must continue to the right. If reversals occur, then the first and second reversals must be of the type occurring in a coil; so also will be the third and fourth, the fifth and sixth, *etc.*, by Lemma 4.

Thus if the last reversal is to the right, we have a closed hook followed by a coil. If the last reversal is to the left, the number of steps in the trip to the left cannot be specified. But then each step completes some open hook since it must enter a site immediately to the right of another, which can be taken to be the origin of a right open hook. Then we have a closed hook, a coil, and an open hook. If $+1$ is not reached following the first reversal, we have a coil to the left and a left open hook or a left open hook alone. This exhausts all possibilities.

2. It remains to calculate the probability of occurrence of a chain consisting of a closed hook of h steps, a coil of c steps, and an open hook of y steps. We shall do so for the case where the chain starts with a left hook. The resulting probabilities will then be doubled to get the final result.

Since by Lemma 3, all the steps of a left hook are determined, we have for the probability of occurrence of a left hook of h steps

$$P_h(h) = (1/4)^h. \tag{1}$$

Let the coil which follows the hook have c steps in all and ρ double reversals. Then 3ρ steps are specified completely, since a double reversal, given a direction, consists of a double, a single, and a double in specified directions. The remaining $c-3\rho$ steps are specified as to direction only, their sizes being arbitrary. These ρ reverses can be thought of as determining ρ partitions and hence $\rho+1$ bins, into which the remaining $c-3\rho$ steps may be placed in any way whatsoever. Therefore there are

$$\frac{[(c-3\rho)+(\rho+1)-1]!}{(c-3\rho)![(\rho+1)-1]!} = \frac{(c-2\rho)!}{(c-3\rho)!\rho!} \tag{2}$$

ways of making a coil with c steps and ρ double reversals. The probability of any one of these is

$$P_1(c, \rho) = (1/4)^{3\rho}\cdot(1/2)^{c-3\rho} = (1/2)^{c+3\rho}. \tag{3}$$

Since ρ may range from 0 to $[c/3]$, the probability of making a coil of c steps with any number of reversals is

$$P_c(c) = \sum_{\rho=0}^{[c/3]} (1/2)^{c+3\rho} \frac{(c-2\rho)!}{(c-3\rho)!\rho!}. \tag{4}$$

If a hook is followed by a coil containing x steps between them, then the probability of making a "hooked coil" of x steps is

$$P_{hc}(x) = \sum_{h=2}^{x} P_c(x-h)P_h(h) + P_c(x), \tag{5}$$

since the steps may be divided between the hook and the coil in any fashion except for a single step hook.

If following the hooked coil the path enters an open hook, the probability of making an open hook of y steps is (as for the closed hook)

$$P_y(y) = (1/4)^y. \tag{6}$$

Since the hooked coil and the open hook may divide the total number of n between them in any fashion except for a single-step open hook, the probability of making a hook-coil-open-hook, which by our theorem is indeed the most general chain, is

$$P_{hcy}(n) = P_{hc}(n) + \sum_{x=2}^{n} P_{hc}(n-x)P_y(x). \tag{7}$$

Allowing now for the choice of initial direction, we have the probability of a chain of at least n steps

$$P_{ch}(n) = 2P_{hcy}(n), \tag{8}$$

and the probability of making a path of exactly n steps, *i.e.*, a path terminating on the nth step, is

$$P_p(n) = P_{ch}(n-1) - P_{ch}(n). \tag{9}$$

The final result can be stated as an iterated function. If we define

$$F_0(n) = \sum_{i=0}^{[n/3]} (1/2)^{n+3i} \frac{(n-2i)!}{(n-3i)!i!}, \tag{10}$$

and, by iteration,

$$F_{k+1}(n) = F_k(n) + \sum_{x=2}^{n} F_k(n-x)(1/4)^x, \tag{11}$$

then the probability that a path does not terminate by the nth step is $2F_2(n)$ and the probability that a path terminates in exactly n steps is

$$P_p(n) = 2[F_2(n-1) - F_2(n)]. \tag{12}$$

Table 1 gives the first few values of $P_{ch}(n)$ and $P_p(n)$.

TABLE 1

n	1	2	3	4	5	6	7
$P_{ch}(n)$	1	12/16	30/64	70/256	160/1024	360/4096	802/16384
$P_p(n)$	0	4/16	18/64	50/256	120/1024	280/4096	638/16384

FIGURES INSCRIBED IN CONVEX SETS

H. G. EGGLESTON, Trumpington, Cambridge, England

We shall say that a regular polygon or polyhedron is inscribed in a convex set* X if every vertex of the polygon or polyhedron is a point of the frontier of X. It is known that it is possible to inscribe a square in any plane convex set (see [1], [3], [5], [7]) and that if a point p is given interior to a three-dimensional convex set X then it is possible to inscribe a square in X with p as its centre ([2]).

On the other hand it is known that about any convex set in n dimensions it is possible to circumscribe an n-dimensional cube, *i.e.* there are support hyperplanes of the convex set which form the faces of a cube ([4], [6]). The dual of this result in two dimensions or three dimensions asserts that it is possible to inscribe a square or a regular octahedron (as the case may be) in a central convex set.

The object of this note is to give examples which show that these results are, from certain points of view, the most that can be asserted. Examples are given with the following properties.

(A) A plane set of constant width† in which it is impossible to inscribe a regular n-gon with $n > 4$.

(B) A three-dimensional convex set such that for every $n > 4$ there is a point P interior to the set such that it is not possible to inscribe a regular n-gon in the set with P as circumcentre.

(C) A central three-dimensional convex set in which it is impossible to inscribe a cube.

* By "convex set" is always meant "bounded closed convex set." The containing space is real Euclidean space.

† A convex set is of constant width if the distance apart of two parallel support lines is the same whatever the direction of the lines.

Example (A). Let Γ be a Reuleaux triangle§ with vertices A, B, C and of width λ. Any n-gon with $n \geqq 5$ inscribed in Γ must have two vertices on the same arc AB, BC, or CA. Suppose that the n-gon $P_1P_2 \cdots P_n$ is such that P_1 and P_2 lie on BC.

Since BC is an arc of a circle whose centre is A the perpendicular bisector of the segment P_1P_2 passes through A. If n is odd it also passes through the vertex $P_{(n+3)/2}$ which, since it lies on the frontier of Γ lies either at A or on the subarc P_1P_2 of the arc BC. In the second case the polygon is not regular since the segment P_1P_2 would be of greater length than P_2P_3. In the first case, if the polygon is regular there are two vertices of it on each arc AB and AC of Γ. For either $P_1 = B$ and $P_2 = C$ or the length of segment P_1P_2 is less than λ. If the second alternative holds, the length of segments $P_{(n+3)/2}P_{(n+5)/2}$ and $P_{(n+1)/2}P_{(n+3)/2}$ are both less than λ; thus $P_{(n+5)/2}$ lies on arc AB and $P_{(n+1)/2}$ lies on arc AC. But then by the same argument as above the points B and C are vertices of the polygon. Now every vertex of a regular polygon with an odd number of vertices lies on the perpendicular bisector of a side of the polygon. But every side of our polygon is a chord of one of the arcs AB, BC, CA of Γ and this implies that the only possible positions for the vertices of the polygon are A, B, C. This is a contradiction with the fact that the n-gon has more than four vertices. Thus no regular n-gon with n odd, $n \geqq 5$ can be inscribed in Γ.

Next consider the case when n is even, $n \geqq 8$. Again a regular n-gon $P_1 \cdots P_n$ with three vertices on one of the arcs AB, BC, CA say P_1, P_2, P_3 on the arc BC is such that the centre of the n-gon (which lies on the perpendicular bisector of P_1P_2 and on the perpendicular bisector of P_2P_3) must lie at A and cannot therefore be contained in the Reuleaux triangle. Thus no regular n-gon with n even and $n \geqq 8$ can be inscribed in Γ.

Finally consider the case $n = 6$. As above $P_1 \cdots P_6$ has at most two vertices on each arc AB, BC, CA and thus it must have exactly two vertices on each of these arcs, say P_1P_2 on BC, P_3P_4 on CA and P_5P_6 on AB. Let l denote the perpendicular bisector of P_1P_2. Then if P_1P_2 lie on BC, l passes through A. Further $P_1 \cdots P_6$ is symmetrical about l. If we reflect our figure in l, P_3 becomes P_6, P_4 becomes P_5. Hence the arc AB is the reflection of the arc AC in l, and l must bisect the arc BC. Similarly the perpendicular bisector of P_3P_4 passes through B. Thus the centre of the polygon coincides with the circumcentre of Γ and the regular hexagon that can be inscribed in Γ is unique.

We construct a set $\Gamma(x)$ by modifying Γ as follows. On the arc AC of Γ take a point A_1, near to A, at a distance of x from A and let A_2 denote the midpoint of the arc AB. Let D be a point of intersection of the two circles which have radius λ and centres A_1, A_2 and suppose that D is such that the circular arcs of radius λ, A_1A_2, A_2B, BD, DC, CA_1 bound a pentagon of constant width λ. (The circular arc A_1A_2 is part of the circle centre D and radius λ.) We denote this pentagon by $\Gamma(x)$.

§ A Reuleaux triangle is the common part of three closed circular discs of radius r, the three centres of which form an equilateral triangle of side r.

As x decreases to zero, $A_1 \to A$, $D \to C$ and the pentagon A_1A_2BDC tends to Γ. It follows that for any n, $n \geqq 7$ or $n=5$ there exists a positive number δ_n such that for $0 \leqq x \leqq \delta_n$ it is impossible to inscribe a regular n-gon in $\Gamma(x)$. Further the only regular hexagons that can be inscribed in $\Gamma(x)$ must tend to the unique regular hexagon that can be inscribed in Γ, as $x \to 0$. Thus such a hexagon, if it exists, will have two vertices on arc A_1C, one each on arcs BA_2 and A_2A_1 and two on arc BD.

We show firstly that if x is small $\Gamma(x)$ contains no regular inscribed hexagon. Suppose that such a hexagon exists with vertices $P_1 \cdots P_6$ where P_1, P_2 lie on arc BD, P_3P_4 lie on arc CA_1, P_5 lies on arc A_1A_2 and P_6 lies on arc A_2B. The perpendicular bisector of segment P_3P_4 passes through B, and the hexagon is symmetrical about it. Thus if we reflect our figure in this line, P_6 reflects onto P_1 and thus arc BA is the reflection of arc BD. Hence arc A_2A_1 reflects into an arc which is entirely interior to $\Gamma(x)$ except for its end points. But arc A_2A_1 contains P_5 which is not one of its end points and which reflects onto P_2, a frontier point of $\Gamma(x)$. This contradiction shows that $\Gamma(x)$ contains no regular inscribed hexagon.

Take an integer N so large that a regular n-gon with $n \geqq N$ which is inscribed in a set of constant width has sides of length at most $\lambda/8$. In defining $\Gamma(x)$ choose x so small that the distance of A_1 from C is greater than $3\lambda/4$ and such that no regular n-gons can be inscribed in $\Gamma(x)$, where $n=5, \cdots, N$. If a regular n-gon with $n>N$ could be inscribed in $\Gamma(x)$ there would be at least three vertices on arc A_1C (for otherwise either all vertices would lie on arcs BA_2 and BD or there would be two consecutive vertices at a distance apart greater than $\lambda/8$; neither of these alternatives is possible). But then if n is even, B is the centre of the polygon which is therefore not contained in $\Gamma(x)$, and if n is odd B is a vertex of the polygon which is equidistant from the three vertices on A_1C. This is impossible.

Thus finally $\Gamma(x)$ is a set of constant width in which it is impossible to inscribe any regular polygon with $n \geqq 5$.

Example (B). Let Y be a plane convex set satisfying the conditions of example (A) and let its circumcentre be k and let X be a right cylinder on Y as base. Let P_n be a sequence of points converging to k and contained in the interior of Y. For every fixed $n \geqq 5$ there is an integer m such that there is no regular n-gon inscribable in X with P_m as circumcentre.

Suppose that this is not the case. Then by the Blaschke selection theorem we can select a subsequence P_{n_i} such that each P_{n_i} is the circumcentre of a regular n-gon, say T_{n_i}, inscribed in X and convergent to a set T. T is either a frontier point of X or a regular n-gon whose circumcentre is k. But in the second case since $T \subset X$ we must have T contained in Y. Thus there would be a regular n-gon inscribed in Y. We know that this is impossible. In the first case the dimensions of T_n tend to zero as i tends to infinity. But each T_m must contain a vertex not on Y and thus at a fixed positive distance say τ from k. Hence as

$P_{n_i} \to k$ the linear dimension of T_{n_i} is bounded below and the first alternative cannot occur either.

Thus for every $n > 4$ there is a point P interior to X such that no n-gon can be inscribed in X whose circumcentre coincides with P.

Example (C). Of the eight vertices of a cube the set of six points obtained by omitting two diametrically opposite vertices is called a V set. We need the following lemma.

LEMMA. *If the points of a V set belong to the frontier of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{1}$$

where no two of the positive real numbers a, b, c are equal, then they are of the form $(\pm d, \pm d, \pm d)$ where any choice of sign is possible and where d satisfies

$$d^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 1.$$

Let the center of the V set be (f, g, h); then the points of this set are of the form $(f+k_i, g+l_i, h+m_i)$ and $(f-k_i, g-l_i, h-m_i)$, $i=1, 2, 3$. Further the points of V do not lie in a plane through (f, g, h). Thus

$$\begin{vmatrix} k_1 & l_1 & m_1 \\ k_2 & l_2 & m_2 \\ k_3 & l_3 & m_3 \end{vmatrix} \neq 0.$$

But since the points of the V set lie on the frontier of the ellipsoid, simple algebraic manipulations lead to

$$f\frac{k_1}{a^2} + g\frac{l_1}{b^2} + h\frac{m_1}{c^2} = 0,$$

$$f\frac{k_2}{a^2} + g\frac{l_2}{b^2} + h\frac{m_2}{c^2} = 0,$$

$$f\frac{k_3}{a^2} + g\frac{l_3}{b^2} + h\frac{m_3}{c^2} = 0.$$

Hence $f = g = h = 0$.

Next suppose for definiteness that $a > b > c > 0$ and let V_1 be the V set formed from the six points $(d, d, -d)$, $(-d, -d, d)$, $(-d, d, -d)$, $(d, -d, d)$, $(-d, d, d)$, $(d, -d, -d)$. Then any ellipsoid which has the origin as centre and in which V_1 is inscribed is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \lambda(yz + xz + xy) = 1 - \lambda d^2, \tag{2}$$

where $\lambda d^2 < 1$. The points $(a(1-\lambda d^2)^{1/2}, 0, 0)$, $(0, 0, c(1-\lambda d^2)^{1/2})$ satisfy (2). Thus if the lengths of the semimajor and semiminor axes of (2) are a' and c', respectively, then $a' \geq a(1-\lambda d^2)^{1/2}$, $c' \leq c(1-\lambda d^2)^{1/2}$. Moreover, equality holds only if the tangent plane to (2) at $(a(1-\lambda d^2)^{1/2}, 0, 0)$ is parallel to the plane $x=0$ and this is so if and only if $\lambda = 0$. Thus if $\lambda \neq 0$ then $a'/c' > a/c$. Hence no ellipsoid of the form (2) is similar to the ellipsoid (1) unless it is identical with it. Since any two V sets are similar, it follows that no V set can be inscribed in (1) except the V set V_1 or a V set obtained from V_1 by a similarity transformation which transforms (1) into itself. Such a transformation transforms the set of points $(\pm d, \pm d, \pm d)$ into itself. This proves the lemma.

We now denote the points on or inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

by E and obtain the required set by a modification of E. Let $\pi(x)$ be the plane parallel to the tangent plane to E at (d, d, d) which passes through (x, x, x). When $x \neq 0$ denote the closed half space bounded by $\pi(x)$ and containing the origin by $L(x)$. Write $E(x) = E \cap L(x) \cap L(-x)$.

Let x approach d by values less than d, say in the sequence $\{x_i\}$. If there is a cube C_i inscribed in $E(x_i)$ then as $i \to \infty$, C_i must approach the cube with vertices $(\pm d, \pm d, \pm d)$ since this is the only cube inscribable in E. Thus for i sufficiently large C_i has one vertex each on $\pi(x)$ and $\pi(-x)$ and the remaining six on E, *i.e.* there is a V set of six vertices of C_i on E. By the lemma this implies that C_i is the cube with vertices $(\pm d, \pm d, \pm d)$. But this is impossible since the set $E(x_i)$ does not contain the point (d, d, d) or the point $(-d, -d, -d)$.

Thus finally if i is large, no cube can be inscribed in $E(x_i)$. Since $E(x_i)$ is central we have established the third example.

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BRINGING CALCULUS UP-TO-DATE

M. E. MUNROE, University of Illinois

1. Introduction. There are many conversations, committee meetings, *etc.*, today about the modernization of the undergraduate calculus course; but all too often the attack on the problem falls short of being comprehensive. Calculus has been in cold storage for over fifty years now. These have been highly productive years in mathematics, and the result is that more changes are in order than most people would like to admit.

The relevant branches of modern mathematics would seem to be real function theory and differential geometry, and a survey of these fields suggests at least the five points listed below. It is significant that point (i) can be gleaned from a study of real function theory and is already coming into vogue in calculus, while points (ii)–(v) come largely from differential geometry and are too generally overlooked. Calculus is too often erroneously classified as a part of real function theory only.

(i) It is essential to distinguish between a function f and its values $f(a)$.

(ii) The coordinate variables x and y of analytic geometry are themselves mappings; x maps points into their abscissas; y maps points into their ordinates.

(iii) There is an important difference in point of view toward coordinate variables in analytic and differential geometry. In analytic geometry x and y are defined over the entire plane, and $y=f(x)$ is a conditional equation describing a locus. In differential geometry the locus is preassigned; x and y are restricted to this locus, and $y=f(x)$ is an identity. A calculus problem starts as analytic geometry and finishes as differential geometry. Thus, somewhere in the middle the symbols change meaning. (More details on this point in Section 2.)

(iv) Modern differential geometry has finally produced a definition of the differential that is quite satisfactory for purposes of integration and differentiation theory. In this definition the concept of differential has been divorced from that of approximate increment. (Details in Section 3.)

(v) In the modern, advanced study of integration of differential forms the theory of the integral is lifted from real function theory, and the algebra of the differential forms is shown to fit into this framework. When this abstract theory is specialized to the simplified, concrete cases discussed in calculus, modern differential theory becomes an effective (and simpler) tool for certain developments in the theory of the integral. (See Sections 5 and 6.)

2. Functions and variables. To begin with, define the word *function* to mean a mapping of numbers into numbers. A later generalization will include

mappings of n-tuples of numbers into numbers. This is the traditional use of the word function in calculus.

One also encounters spaces whose elements are called points. These take the form of lines, planes, curves, surfaces, 3-space, space-time, *etc.* An essential part of the machinery is the mapping of these points into numbers. Let these mappings be called *variables*. Linguistically incongruous as it may seem, this is traditionally the most common use of the word variable in calculus. This definition follows the principle that language is established by usage, not by logic. It retains the word and gives a more enlightened description of what it has been used to mean. It should be noted that probability theory has already established this usage with the phrase *random variable.*

The coordinate variables x and y are variables in the sense just defined. In plane analytic geometry each of these symbols stands for a mapping of the entire plane onto the real number system. If x has this meaning and f is a function, then $f(x)$ is used to denote a composite mapping:

$$\text{point} \xrightarrow{x} \text{number} \xrightarrow{f} \text{another number}.$$

The form

$$y = f(x) \tag{1}$$

is now a *conditional equation*. Here there are two symbols, y and $f(x)$, each standing for a mapping; but in analytic geometry (1) is not regarded as an assertion that these are two names for the same mapping. Rather, (1) appears (or should appear) only as a noun clause in the phrase, "the locus of $y=f(x)$." This locus is, of course, the set of all points p in the common domain of x and y for which $y(p)=f[x(p)]$. Note that this last "$=$" means what it should.

Suppose, now, that (1) has been given in proper context and its locus has been found and named C. The next step in the logical analysis of a calculus problem is to restrict the mappings x and y to C. Note that this could not be done before because the superstructure described above was used to define C. It would help if the restricted mappings were given new names; say, start with X and Y and boil them down to x and y. However, this is probably asking for too much of a change in set habits. In any case, with x and y restricted to C, (1) becomes an *identity*—an assertion that two mappings are the same. At this stage one has a special case of the type structure studied in modern differential geometry and so can turn to the literature of that discipline for further enlightenment. What follows is an informal summary of the specialization of this work to calculus. For a more complete discussion of the general theory see, for example, Chevalley, *Theory of Lie Groups*, Princeton, 1946.

3. Differentials. A *one-dimensional manifold* is a structure consisting of a point set C and a set of *coordinate variables* defined thereon. Precise postulates can be given, but a very loose description will suffice for the present discussion. The set C is a smooth curve, and each of the coordinate variables is continuous

and locally one-to-one. Since each is locally one-to-one, each pair u, v is related at least locally by an identity of the form $v=f(u)$, and it is convenient to assume that each such connecting function f has three derivatives. Note that a locus in the plane with the rectangular coordinate variables restricted to it will generally form such a structure.

Consider three classes of variables on C.

X: all variables on C,
Y: those related to a coordinate by a differentiable function,
Z: those related to a coordinate by a thrice-differentiable function.

Let D be a mapping of Y into X, and let Du denote the map of u by D. If p is a point on the curve, denote the value of Du at p by $(Du)_p$. Such a mapping D is called a *derivative operator* at p provided

$$\text{(a)}\quad [D(au+bv)]_p = a(Du)_p + b(Dv)_p,$$

$$\text{(b)}\quad [D(uv)]_p = u(p)(Dv)_p + v(p)(Du)_p.$$

Here u and v are variables and a and b real numbers.

Regarding a constant a as a variable, observe that

$$(Da)_p = 0, \tag{2}$$

because by (a), $[D(au)]_p=a(Du)_p$, and by (b), $[D(au)]_p=a(Du)_p+u(p)(Da)_p$; and if u is chosen so that $u(p)\neq 0$, then (2) follows. Observe also that setting $u=v$ in (b) yields

$$[D(u^2)]_p = 2u(p)(Du)_p. \tag{3}$$

Now, let u be a coordinate, and let $v=f(u)$. Assume f has three derivatives; then there is a differentiable function g such that

$$v = v(p) + f'[u(p)][u - u(p)] + g(u)[u - u(p)]^2. \tag{4}$$

Recalling (2) and (3), operate on (4) with D at p to obtain

$$(Dv)_p = f'[u(p)](Du)_p. \tag{5}$$

Each of the other terms on the right vanishes either because of (2) or because it contains the factor $u(p)-u(p)$. Note that in this step D must be defined on Y because it must operate on $g(u)$. However, (5) is proved only for $v\in Z$ because this is required in order to set up (4).

Let D_u be the operator such that $D_uu=1$ at each point on the curve. By (5) this determines D_u on v where $v=f(u)$, and indeed

$$D_uv = f'(u). \tag{6}$$

The variable D_uv is called the *derivative of v with respect to u.* The notation f' denotes a purely function theoretic concept; f' is defined from f in terms of limits of difference quotients. On the other hand, D_u is an operator defined by purely

algebraic conditions. Thus, (6) is not a definition; it is a theorem relating these two basically different notions.

Consider now D_1 and D_2, two derivative operators at p. If for some variable u, $(D_1u)_p = a(D_2u)_p$, then for $v = f(u)$,

$$(D_1v)_p = f'[u(p)](D_1u)_p = af'[u(p)](D_2u)_p = a(D_2v)_p.$$

That is, at p any two derivative operators are constant multiples one of the other. So, the set of derivative operators at p has the algebraic structure of a straight line. Identify it with T_p, the tangent line to the curve at p, and define the *tangent bundle* for the given curve as the set of all ordered pairs (p, D) where p is a point of the curve and D is a derivative operator at p.

Finally, the *differential* is defined as follows. For u a variable on the curve, du is a variable on the tangent bundle. Its value at (p, D) is denoted by $du_p(D)$, and du is defined by

$$du_p(D) = (Du)_p. \tag{7}$$

Substitute (6) into (5) to obtain

$$(Dv)_p = (D_uv)_p(Du)_p. \tag{8}$$

By (7), $(Dv)_p$ and $(Du)_p$ are values of appropriate differentials; so (8) may be written

$$dv_p(D) = (D_uv)_p du_p(D).$$

This is for an arbitrary p on the curve and D on T_p; so

$$dv = D_uv du \tag{9}$$

over the tangent bundle. An appropriate name for (9) is "Fundamental Theorem on Differentials." At any rate, in the modern theory it is a theorem, not a revolving definition that changes meaning with every change of "independent" variable.

Multidimensional cases follow the same pattern. Let C be an n-dimensional manifold—intuitively, a uniformly n-dimensional, smooth set with variables attached. The basic form relating variables will be $v = f(u_1, \cdots, u_n)$. Derivative operators are defined by the same postulates as before, and by the n-dimensional generalization of (4) it is shown that they form a family of n-dimensional vector spaces. A derivative operator is determined if specified on n variables, and the operators $\partial/\partial u_i$ defined by

$$\frac{\partial u_j}{\partial u_i} = \begin{cases} 1 & \text{for } j = i, \\ 0 & \text{for } j \neq i \end{cases}$$

form a basis in the tangent bundle. Differentials are defined again by (7), and the fundamental theorem reads

$$dv = \sum_{i=1}^{n} \frac{\partial v}{\partial u_i} du_i. \tag{10}$$

4. Some immediate advantages. The ultimate advantage of saying things correctly rather than incorrectly need not be discussed. Let it be understood, then, that the following list—which is quite probably nonrepresentative—merely points out a few of the common difficulties that can be cleared up by the modern approach.

(a) *Differentials identified with variables.* A variable v on C generates a differential dv on the tangent bundle, and this differential retains its identity whether $v=f(u)$ or $v=g(w)$. Note that df and dg are not defined. Understanding of this will help to clear up many notation obscurities.

(b) *Second differentials.* No one seems to know exactly what these ought to be. Function theorists sometimes want $d^2v=D_u^2v du^2$; at other times (when changes of variable are in order) they want $d^2v=D_u^2vdu^2+D_uvd^2u$. In working, say, with parametric equations the student is apt to use the first of these forms as though it had the invariance properties of the second. It seems advisable to leave these out of calculus completely, and the development outlined above does this very nicely. There the differential of dv is not defined because the domain of dv is the tangent bundle while the differential is defined only for a variable whose domain is C.

(c) *Chain rules.* Given $f(x, y, u, v)=g(x, y, u, v)=0$, it is common practice to find, say, $\partial u/\partial x$ in terms of other partial derivatives by writing out various expressions modeled after (10), making substitutions and equating coefficients of an appropriate differential. This is an effective technique for deriving chain rules, and in the modern theory it is quite acceptable. However, if (10) is a definition rather than a theorem, it must be proved invariant under coordinate changes. One must often know that the differential form is invariant under the very chain rule he is seeking to find!

(d) *Increments and approximations.* The undergraduate is afraid to use the differential—in such manipulations as (c), for example—because of the approximation bugaboo introduced with the now outmoded definition. In the modern theory this fear is never introduced, and the approximate increment problem is properly classified—under Taylor's theorem.

5. Line and surface integrals. Let u and v be variables on a curve C. Partition C by points $p_0, p_1, \cdots, p_n$. For each i, let q_i be the point on the tangent line T_{p_i} whose distance from p_i is the length of the arc p_ip_{i+1} on C. Intuitively, "unroll" each increment of arc onto a tangent line. As noted in Section 3, the set of derivative operators at p_i forms a "line," to be identified with the tangent line T_{p_i}. To obtain the usual geometric interpretation, specify that if s is the arc length variable on C, then D_s is identified with the unit tangent vector. In any

case, each point q_i constructed above is identified with a derivative operator at p_i; so the notation $dv_{p_i}(q_i)$ has an obvious meaning. Now, form sums

$$\sum_{i=0}^{n-1} u(p_i)dv_{p_i}(q_i). \tag{11}$$

Take an appropriate limit of these, and call the result

$$\int_C udv. \tag{12}$$

The important feature to this development of the line integral is that the points q_i on the tangent lines are determined by an intrinsic notion (arc length) on C and have no connection with the variables u and v. Thus, the substitution theorem, $\int_C udv = \int_C uD_w v dw$, is an immediate consequence of the fundamental theorem on differentials (9). The approximating sums (11) are invariant under the substitution; hence so is the integral. The familiar $\int_a^b f(x)dx$ is now a special case of (12)—in which C is an interval on the x-axis. In this development the differential is clearly the same thing in differential and integral calculus.

In a double integral—carefully to be distinguished from an iterated integral —*exterior multiplication* of differentials must be introduced. This is an operation denoted by $*$ and characterized by the rule

$$du * dv = - dv * du. \tag{13}$$

This stems from the fact that $du * dv$ is to be a signed measure on oriented plane regions. Setting $u = v$ in (13) yields the corollary $du * du = 0$. Using these rules of multiplication (plus a distributive law) and substituting from the two-dimensional case of (10), one has

$$\begin{aligned} du * dv &= \left(\frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy\right) * \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y}\, dy\right) \\ &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dx * dx + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\, dx * dy \\ &\quad + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} dy * dx + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\, dy * dy \\ &= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) dx * dy \\ &= \frac{\partial(u, v)}{\partial(x, y)}\, dx * dy. \end{aligned} \tag{14}$$

In general, a pair (du_p, dv_p) forms a set of affine coordinates in the tangent plane T_p. If the pair is orthogonal (and some pairs always are), then multipliers

a and b can be chosen so that (adu_p, bdv_p) forms a set of rectangular coordinates in T_p. In general, a and b depend on p; for example, on the tangent planes to the unit sphere $(d\phi, \sin\phi\, d\theta)$ generates rectangular coordinates. In any case, if (adu_p, bdv_p) is a rectangular set on T_p, and if A is a region in T_p positively oriented with respect to this coordinate system, then define $ab(du * dv)_p(A)$ to be the area of A. Since the Jacobian of a rotation is unity, (14) shows that this is consistent for all such rectangular sets. Values of other exterior product operators are then determined by (14).

Now, the substitution rule for multiple integrals consists merely of making the substitution (14) behind the integral sign. A definition of the surface integral patterned after that given above for the line integral makes this remarkably easy to justify.

Take a smooth surface S and partition it in an arbitrary fashion—not necessarily following coordinate lines—into subsets S_i. At some point p_i in S_i take a tangent plane. Here a minor complication appears; one cannot "unroll" the S_i as he did the arcs and preserve measure. However, define surface area—there are many tricks for accomplishing this—and construct on each tangent plane a set having the same area as S_i. Call this plane set A_i. The shape of A_i is immaterial because (see definition above) $(du * dv)_{p_i}(A_i)$ depends only on the area and orientation of A_i, not on its shape. Now, form sums

$$\sum_{i=1}^{n} w(p_i)(du * dv)_{p_i}(A_i). \tag{15}$$

Take the usual limit, and call it

$$\iint_S wdu * dv.$$

Since the A_i depend in no way on u and v, the sums (15) are invariant under the substitution (14); and there emerges the familiar result

$$\iint_S wdu * dv = \iint_S w\frac{\partial(u, v)}{\partial(x, y)}dx * dy.$$

6. Theorems of Green, Gauss, and Stokes. This view of the multiple integral brings an elegant unification to the theory relating an integral over a manifold to one over the boundary. There is a master formula:

$$\int \cdots \int_B udv_1 * \cdots * dv_k = \iint \cdots \int_M du * dv_1 * \cdots * dv_k \tag{16}$$

where M is an oriented $(k+1)$-dimensional manifold and B is its boundary. This is proved in the usual way by reducing the integral on the right to an iterated integral and performing one integration.

Now, if A is a region in the plane and C its boundary, then (16) yields

$$\int_C (udx + vdy) = \int\int_A (du * dx + dv * dy).$$

However,

$$du * dx = \left[\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right] * dx = -\frac{\partial u}{\partial y} dx * dy,$$

$$dv * dy = \left[\frac{dv}{\partial x} dx + \frac{\partial v}{\partial y} dy\right] * dy = \frac{\partial v}{\partial x} dx * dy;$$

so the result is Green's theorem. A similar maneuver with the form

$$\int_C (udx + vdy + wdz)$$

where C is an appropriate space curve yields the classical theorem of Stokes. The divergence theorem is also a special case of (16) as is easily shown by direct computation. Minus signs do not appear in the divergence theorem because in all this work the permutations of differentials are cyclic permutations, and a cyclic permutation of three things is an even permutation.

When presented in this light, all these theorems appear as generalizations of the fundamental theorem of calculus. For, this latter may be written $u(q) - u(p) = \int_p^q du$, and this is a specialization of (16) in which the difference on the left is regarded as an integral over the zero-dimensional boundary consisting of two points. Subtraction results because the boundary is considered oriented.

7. Notation. Roughly, the structure studied in calculus seems to consist of mappings (variables) "up" from point sets into the real number system together with other mappings (functions) "across the top." It is well to preserve this stratification in the notation, and the usual use of f and g for functions and the last letters of the alphabet for variables is all to the good. However, there are other distinctions that common notational practice fails to make.

The derivative of a function is another function whose values are appropriate limits of difference quotients. Note that a function is not differentiated "with respect to" anything in particular. Thus, an appropriate notation for the derivative of f is f', and it would be well if the prime symbol were reserved for functions only.

Derivatives of variables are generated by the derivative operators introduced above, and the basic form is $D_x y$. The differential form dy/dx is not merely an alternative to this. It means $dy \div dx$, but it follows at once from the fundamental theorem on differentials that $dy/dx = D_x y$.

Hybrid notation such as y' and df/dx abounds in the literature, but it is not really well-defined. One step in the modernization of calculus is to break some well-formed (too well-formed) habits in this respect. Frequently, the confusion stems from writing $y = f(x)$ and then confusing the symbols y and f. As for habits,

however, it is sad to relate that the author of a recent textbook made essentially the comment just made here about confusing y and f and then proceeded to do exactly that three pages later.

Similar distinctions need to be made in the multidimensional cases. If f maps n-tuples of numbers into numbers, its arguments are distinguished by position only. For example, $f(a, b)=a-b$ and $f(b, a)=b-a$ define the same function f. Thus, f_i is appropriate notation for the partial derivative of f with respect to its ith argument. As noted above, $\partial/\partial x$ is a derivative operator in manifold theory and should be reserved for application to variables.

8. **Existence theorems.** Little has been said so far about the role of real function theory in a program of modernizing calculus. In a sense this is a question quite independent of the ones raised so far here, but certainly the following question is pertinent to the present discussion. The algebraic theory of derivative operators and differentials outlined above yields formulas and techniques in a very elegant fashion, but it certainly is not self contained. Is any additional real function theory required as a background to tighten up the logic in this development?

In integral calculus one must develop the notion of integral of a function alongside that of integral of a variable (outlined above). The two are fairly easily related, and the usual existence proof appears in the function theory.

In differential calculus the existence questions are answered *a priori* in the postulates for a manifold. That is, differentiability conditions are imposed by fiat on the connecting functions in such a way that the algebraically defined derivative operators do represent differentiations in the function theoretic sense. However, to keep from operating in a vacuum, one must show that certain things are manifolds.

For purposes of calculus one gets an adequate supply of manifolds by taking the rectangular coordinates in $(n+k)$-space and considering the locus of n equations in them. The implicit function theorem (a standard item in advanced calculus) gives conditions under which k of these variables become local coordinates on the locus, and the Brouwer theorem on invariance of domain (now appearing in the better advanced calculus texts) guarantees that the locus has the proper dimension.* To verify the hypotheses of these theorems, one must know the differentiability properties of the elementary functions. These may be established in the usual way through theorems on differentiability of sums, products, composites, *etc.*

It is probably fair to say, then, that the introduction of the modern theory of the differential changes very little the number or the nature of the real function theory existence proofs required for a logically sound development of calculus. This, of course, is subject to the stipulation that calculus is restricted to

* Added in proof: A recent note by Yamabe, this MONTHLY, vol. 64, 1957, p. 725, gives precisely the result needed.

the well-behaved cases. A general attack on the topological problems of manifold theory is much more difficult.

9. Two facets to the problem. Most programs for the improvement of calculus amount to the injection into the course of more and better real function theory. It is the purpose of the present paper not to discourage this, but to suggest that a more imperative project is the injection of better—if not more—differential geometry. This latter is more imperative because the garden variety calculus contains more downright errors in differential geometry than in real function theory. On the other hand, it is more difficult for at least two reasons. First, to bring the differential geometry up to date will change the appearance of a calculus text; it will modify definitions, terminology, notation, procedures. Second, while most calculus teachers are well grounded in real function theory, for many of them graduate study came before some of the important developments in differential geometry.

Despite these difficulties an effort should be made. In failing to bring calculus up to date we are transmitting to the next generation not the information available to our contemporaries, but that available to our grandfathers.

ON THE COEFFICIENTS OF RECIPROCAL POWER SERIES

JOHN LAMPERTI, California Institute of Technology

1. Introduction. Suppose that the two sequences of numbers $\{u_n\}$ and $\{f_n\}$ satisfy the relations

$$u_n = \sum_{i=0}^{n-1} u_i f_{n-i} + \delta_{0n}, \qquad n = 0, 1, \cdots, \tag{1}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Upon taking the generating functions of the sequences,

$$U(z) = \sum_{n=0}^{\infty} u_n z^n \quad \text{and} \quad F(z) = \sum_{n=1}^{\infty} f_n z^n,$$

relation (1) may be re-expressed as

$$U(z) = U(z)F(z) + 1, \tag{2}$$

where the power series and their multiplication are to be interpreted in a purely formal way. If $U(z)$ and $F(z)$ satisfy (2), then we shall say that $F(z)$ is the re-

ciprocal series determined by $U(z)$ (despite that in the usual sense $U(z)$ and $1-F(z)$ are reciprocals).

It is clear from (1) that either of the sequences $\{u_n\}$ or $\{f_n\}$ determines the other uniquely. It is also easy to see that if every $f_n \geqq 0$, then also each $u_n \geqq 0$ (but not conversely). Theodor Kaluza in [3] has studied the problem of deciding which sequences $\{u_n\}$ determine sequences $\{f_n\}$ such that each $f_n \geqq 0$, and it is also to this problem that the present paper is devoted. In Section 2 a necessary and sufficient condition for $f_n \geqq 0$ (all n) is derived. Although it is cumbersome, this condition has several interesting consequences, and these are discussed in Section 3.

Equation (1) has an important probability interpretation. If E is a "recurrent event" in the sense described in [1] (Ch. 12), then the quantities u_n, equal to the probability that E occurs at time n, and f_n, equal to the probability that E occurs for the first time at time n, satisfy (1). This situation supplies a motivation for most of the results to follow, but will not be made use of explicitly. In the probability interpretation it is of course necessary that $\sum f_n \leqq 1$, while in this paper the f_n need not be bounded.

Before proceeding to the basic theorem, examples may be mentioned with $f_n \geqq 0$, where $U(z)$ is known explicitly and is other than a rational function. One such is

$$U(z) = (1 - az^2)^{-1/2}, \qquad F(z) = 1 - (1 - az^2)^{1/2}, \qquad a > 0.$$

These functions occur in a gambling problem ([1], p. 246). Another example* is

$$U(z) = (\tan z)/z, \qquad F(z) = 1 - z \cot z = - \sum_{k=1}^{\infty} (-4)^k B_{2k} z^{2k}/(2k)!$$

where B_{2k} are Bernouilli numbers; still another is given in [3], Theorem 2.

2. A condition for $f_n \geqq 0$.

THEOREM 1. *The sequence $\{f_n\}$ obtained from* (1) *satisfies $f_n \geqq 0$ for all n if and only if there exists a matrix $P = [p_{ij}]$ with $p_{ij} \geqq 0$ and only a finite number of nonzero terms in each row, such that*

$$u_n = p_{11}^{(n)}, \quad \textit{where} \quad P^n = [p_{ij}^{(n)}]. \tag{3}$$

Proof. Let $P = [p_{ij}]$ be any matrix with a finite number of nonzero terms in each row. Then multiplication is well-defined even if the matrix is infinite, and

$$p_{11}^{(n)} = \sum_{i_1} \cdots \sum_{i_{n-1}} p_{1i_1} p_{i_1 i_2} \cdots p_{i_{n-1}1}. \tag{4}$$

Define

$$r_n = \sum_{i_1 \neq 1} \cdots \sum_{i_{n-1} \neq 1} p_{1i_1} p_{i_1 i_2} \cdots p_{i_{n-1}1}. \tag{5}$$

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* This example was suggested by the referee.

Now let the terms making up the sum in (4) be partitioned into classes E_j, where $p_{1i_1} \cdots p_{i_{n-1}1} \in E_j$ if $i_1 \neq 1, \cdots, i_{j-1} \neq 1$ and $i_j = 1$, and $p_{1i_1} \cdots p_{i_{n-1}1} \in E_n$ if $i_1, \cdots, i_{n-1} \neq 1$. Then if $j = n$,

$$\sum_{E_n} p_{1i_1} \cdots p_{i_{n-1}1} = r_n$$

by definition, while if $j < n$,

$$\begin{aligned} &\sum_{E_j} p_{1i_1} \cdots p_{i_{n-1}1} \\ &= \left\{ \sum_{i_1 \neq 1} \cdots \sum_{i_{j-1} \neq 1} p_{1i_1} \cdots p_{i_{j-1}1} \right\} \left\{ \sum_{k_1} \cdots \sum_{k_{n-j-1}} p_{1k_1} \cdots p_{k_{n-j-1}1} \right\} = r_j p_{11}^{(n-j)}. \end{aligned} \tag{6}$$

Summing (6) over j yields, for $n > 0$,

$$p_{11}^{(n)} = \sum_{j=1}^{n} \sum_{Ej} p_{1i_1} \cdots p_{i_{n-1}1} = \sum_{j=1}^{n-1} r_j p_{11}^{(n-j)} + r_n,$$

or, letting $p_{11}^{(0)} = 1$,

$$p_{11}^{(n)} = \sum_{j=0}^{n-1} r_{n-j} p_{11}^{(j)} + \delta_{0n}. \tag{7}$$

It follows from (7), since solutions to (1) are unique, that if $p_{11}^{(n)} = u_n$, then $r_n = f_n$. If, therefore, all $p_{ij} \geqq 0$, it is obvious from (5) that $f_n \geqq 0$.

To prove the necessity of the condition, it is only required to exhibit a suitable matrix P. Suppose that $\{u_n\}$ does determine $f_n \geqq 0$, and let

$$P = \begin{bmatrix} f_1 & 1 & 0 & 0 & 0 \cdots \\ f_2 & 0 & 1 & 0 & 0 \cdots \\ f_3 & 0 & 0 & 1 & 0 \cdots \\ f_4 & 0 & 0 & 0 & 1 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \cdots \end{bmatrix}.$$

It is easy to see from (5) that $r_n = f_n$; therefore from (7) it follows that $p_{11}^{(n)} = u_n$. This completes the proof.

3. Consequences. Although Theorems 2 and 3 are special cases of Theorem 4, they precede it for reasons of aesthetics.

THEOREM 2. *Suppose $U(z)$ and $V(z)$ are formal power series with $u_0 = v_0 = 1$ such that their reciprocal power series have nonnegative coefficients. Then $U(z) * V(z) = \sum_{n=0}^{\infty} u_n v_n z^n$ has this property also.*

Proof. Let P and Q be matrices with nonnegative entries such that $u_n = p_{11}^{(n)}$ and $v_n = q_{11}^{(n)}$ (Theorem 1); let $R = P \otimes Q$ be the Kronecker product of P and Q (see for instance [2], pp. 176–179). Then since $R^n = P^n \otimes Q^n$,

$$r_{11}^{(n)} = p_{11}^{(n)} q_{11}^{(n)} = u_n v_n.$$

But R also has nonnegative entries and finitely many nonzero terms in each row. Another application of Theorem 1 therefore completes the proof.

THEOREM 3. *Let $U(z)$ and $V(z)$ be as in Theorem 2. Then $W(z) = \sum_{n=0}^{\infty} w_n z^n$, where $w_n = \sum_{i=0}^{n} \binom{n}{i} u_i v_{n-i}$ and $w_0 = 1$, also determines a reciprocal series with nonnegative coefficients.*

Proof. Choose P and Q as in the proof of Theorem 2. Let I_1 and I_2 be identity matrices with the same number (possibly infinite) of rows and columns as P and Q, respectively. Let $A = P \otimes I_2$, $B = I_1 \otimes Q$, and $C = A + B$. Then A and B commute, so that

$$C^n = \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i} = \sum_{i=0}^{n} \binom{n}{i} (P^i \otimes I_2)(I_1 \otimes Q^{n-i}) = \sum_{i=0}^{n} \binom{n}{i} P^i \otimes Q^{n-i},$$

which implies $c_{11}^{(n)} = w_n$. Since C satisfies the conditions of Theorem 1, this proves Theorem 3.

A more general result, which in a sense gives a connection between the Hadamard product $(*)$ of Theorem 2 and the "product" discussed in Theorem 3, is the following:

THEOREM 4. *Let $U(z)$ and $V(z)$ be as above and let $W(z) = \sum_{n=0}^{\infty} w_n z^n$, where $w_0 = 1$ and*

$$w_n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} \alpha^i \beta^j \gamma^k u_{i+j} v_{i+k},$$

with α, β, and γ nonnegative real numbers. Then $W(z)$ determines a reciprocal series with nonnegative coefficients.

Proof. By means of the matrix

$$D = \alpha(P \otimes Q) + \beta(P \otimes I_2) + \gamma(I_1 \otimes Q),$$

the proof may be carried out similarly to that of Theorem 3. Note that 0^n is to be interpreted as δ_{0n}, so that putting $\alpha = 1$, $\beta = \gamma = 0$ gives Theorem 2, while $\alpha = 0$, $\beta = \gamma = 1$ yields Theorem 3.

Also readily obtainable is a necessary condition:

THEOREM 5. *If $U(z)$ has a reciprocal series $F(z)$ with nonnegative coefficients, then $u_{n+m} \geqq u_n u_m$ for all n and m.*

Proof. Let the matrix P satisfy $p_{11}^{(n)} = u_n$ as guaranteed by Theorem 1. Since $p_{11}^{(n)} \geqq 0$,

$$u_{n+m} = p_{11}^{(n+m)} = \sum_i p_{1i}^{(n)} p_{i1}^{(m)} \geqq p_{11}^{(n)} p_{11}^{(m)} = u_n u_m.$$

The class of formal power series with reciprocals having nonnegative coefficients is a semigroup under Hadamard multiplication ($*$) by Theorem 2. From Theorem 5 it is easy to describe the units of this semigroup:

COROLLARY. *$U(z)$ and $U^*(z) = \sum_{n=0}^{\infty} (1/u_n)z^n$ both have reciprocal series with nonnegative coefficients if and only if $U(z) = \sum_{n=0}^{\infty} (az)^n$ for some $a > 0$.*

Proof. The "if" part is obvious. To show necessity, first note that $U^*(z)$ is defined if $u_n \neq 0$ for all n, which by Theorem 5 occurs when $u_1 \neq 0$. Also by Theorem 5 the coefficients of U and U^* must satisfy respectively

$$u_n \geqq u_1^n \quad \text{and} \quad \frac{1}{u_n} \geqq \left(\frac{1}{u_1}\right)^n,$$

and so $u_n = u_1^n$ with $u_1 = a > 0$.

To conclude, we shall state a theorem which is proved in [3], and use it to obtain a result on nonformal series.

THEOREM (Kaluza). *Suppose that $u_0 = 1$, $u_1 > 0$ and that*

$$\begin{vmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{vmatrix} \geqq 0, \qquad n \geqq 1.$$

Then if $\{f_n\}$ and $\{u_n\}$ are related by (1), (*i.e., are the coefficients of reciprocal power series*), *$f_n \geqq 0$ for all n.*

COROLLARY. *If $U(z) = \sum_{n=0}^{\infty} u_n z^n$ has a zero inside its* (*assumed nonnull*) *convergence circle and u_n are all real with $u_0 \neq 0$, then for some $k \geqq 1$, $u_k^2 > u_{k-1}u_{k+1}$.*

The idea of the proof of the corollary is that if the conclusion failed, the reciprocal series $F(z)$ to $(1/u_0)U(z)$ would have $f_n \geqq 0$. In this case, by (1), $f_n \leqq u_n/u_0$ so that the radius of convergence of F is not less than that of U. Then $(1/u_0)U(z) = 1/(1-F(z))$ cannot vanish inside its convergence circle. This same idea, together with our Theorem 2, yields

THEOREM 6. *Let $W(z) = U(z) * V(z)$, where u_n and v_n are all real and $u_0 \neq 0 \neq v_0$. Then if $W(z)$ has a zero inside its convergence circle, for some $n \geqq 1$ either*

$$u_n^2 > u_{n-1}u_{n+1} \text{ or } v_n^2 > v_{n-1}v_{n+1}.$$

This theorem is also true if the Hadamard product $U * V$ is replaced by the more general "product" of Theorem 4.

References

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INVERSES OF VANDERMONDE MATRICES

N. MACON AND A. SPITZBART,* General Electric Company, Evendale

1. Introduction. In a recent paper [1], one finds explicit formulas for the derivatives of a polynomial $y=f(x)$ of degree n in terms of its values $y_i=f(x_i)$ at $n+1$ points defined by $x_i=x_0+ih$, $(i=0, \cdots, n)$. These results are used here to invert the Vandermonde matrix

$$V(x_1, \cdots, x_n) = \begin{bmatrix} 1 & 1 & \cdots 1 \\ x_1 & x_2 & \cdots x_n \\ \cdot & \cdot & \cdots\cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots x_n^{n-1} \end{bmatrix},$$

where the x_i are distinct, different from zero, but otherwise arbitrary.

The paper is in three parts. In the first we outline the results from [1] required later. In the second, these formulas are applied to the special Vandermonde matrix

$$V_{n+1}(x_0, h) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_0+h & \cdots & x_0+nh \\ \cdot & \cdot & \cdots & \cdot \\ x_0^n & (x_0+h)^n & & (x_0+nh)^n \end{bmatrix};$$

and the elements of $V_{n+1}^{-1}(x_0, h)$ are obtained in terms of Stirling numbers. Finally, the methods of [1] are extended in such a way that the elements of $V^{-1}(x_1, \cdots, x_n)$ can be expressed in terms of the elementary symmetric functions of the x's. This last result is offered as an alternative to the derivation one would obtain from the classical formula for the values of Vandermonde determinants with missing powers ([2], p. 99).

2. Preliminary results. Let $y=f(x)$ be a polynomial of degree n, and $y_i=f(x_i)$, $(i=0, 1, \cdots, n)$, where $x_i=x_0+ih$. It was shown in [1] that if we write

$$h^k f^{(k)}(x) = \sum_{i=0}^{n} A_{mi}^k y_i, \tag{1}$$

then

$$A_{mi}^k = \sum_{j=k}^{n} \frac{(-1)^{i+j}\binom{j}{i} C_{mj}^k}{j!}, \tag{2}$$

where

* Now at Alabama Polytechnic Institute and University of Wisconsin-Milwaukee, respectively.

$$\binom{j}{i} = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > j, \end{cases}$$

$$C_{mj}^{k} = \sum_{r=k}^{j} p_k(r)\ S_j^r m^{r-k}, \tag{3}$$

m is any real number, and $x = x_0 + mh$. In the above, $p_k(r)$ denotes the factorial polynomial of degree k, and the S_k^j are the Stirling numbers of the first kind. Thus, we have $p_k(x) = \sum_{j=1}^{k} S_k^j x^j$. A wide variety of classical numerical differentiation formulas are special cases of (1).

It was shown further that these results enable one to invert the Vandermonde matrix

$$M(m) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -m & 1-m & \cdots & n-m \\ \cdot & \cdot & \cdots & \cdot \\ (-m)^n & (1-m)^n & \cdots & (n-m)^n \end{bmatrix}.$$

If we write $M^{-1}(m) = \{a_{\lambda,\mu}^{m}\}$, $(\lambda, \mu = 1, \cdots, n+1)$, then

$$a_{\lambda,\mu}^{m} = \frac{A_{m,\lambda-1}^{\mu-1}}{(\mu-1)!}, \tag{4}$$

where $A_{m,\lambda-1}^{\mu-1}$ is given by (2).

3. Vandermonde Matrices for equally spaced points. It is easy to show that

$$\begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & h & 0 \cdots 0 \\ \cdot & \cdot & \cdots\cdots \\ 0 & 0 & 0 \cdots h^n \end{bmatrix} M(m) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -mh & -mh+h & \cdots & -mh+nh \\ \cdot & \cdot & \cdots & \cdot \\ (-mh)^n & (-mh+h)^n & \cdots & (-mh+nh)^n \end{bmatrix}$$

If we write $x_0 = -mh$, it follows that

$$V_{n+1}(x_0, h) \equiv \begin{bmatrix} 1 & 1 & & 1 \\ x_0 & x_0+h & \cdots & x_0+nh \\ \cdot & \cdot & \cdots & \cdot \\ x_0^n & (x_0+h)^n & \cdots & (x_0+nh)^n \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & h & 0 \cdots 0 \\ \cdot & \cdot & \cdots\cdots \\ 0 & 0 & 0 \cdots h^n \end{bmatrix} M(-x_0/h),$$

and so

$$V_{n+1}^{-1}(x_0, h) = M^{-1}(-x_0/h)\begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & h^{-1} & 0 \cdots 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0 \cdots h^{-n} \end{bmatrix}.$$

Thus, by (4) the element in the λth row and μth column of $V_{n+1}^{-1}(x_0, h)$ is precisely

$$\nu_{\lambda\mu} = \frac{1}{h^{\mu-1}(\mu-1)!} A^{\mu-1}_{-x_0/h,\lambda-1}.$$

It is interesting to note that even though $V_{n+1}^{-1}(1, 1)$ can be obtained directly from the above, a slight modification enables one to express the elements of this inverse very compactly. For simplicity of notation, we apply the method to obtain $V_n^{-1}(1, 1)$.

Let us write

$$M(0) = \begin{bmatrix} 1 & Q_{1n} \\ R_{n1} & P_{nn} \end{bmatrix}, \qquad M^{-1}(0) = \begin{bmatrix} b_{11} & S_{1n} \\ T_{n1} & U_{nn} \end{bmatrix},$$

where $Q_{1n} = (1, \cdots, 1)$, R_{n1} is a column vector of zeros, and

$$P_{nn} = \begin{bmatrix} 1 & 2 & \cdots & n \\ 1 & 4 & \cdots & n^2 \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ 1 & 2^n & \cdots & n^n \end{bmatrix}.$$

It suffices to invert the matrix P_{nn}, since its columns are scalar multiples of those of $V_n(1, 1)$. Now

$$M(0)M^{-1}(0) = \begin{bmatrix} b_{11} + Q_{1n}T_{n1} & S_{1n} + Q_{1n}U_{nn} \\ R_{n1}b_{11} + P_{nn}T_{n1} & R_{n1}S_{1n} + P_{nn}U_{nn} \end{bmatrix} = I,$$

and so $I_n = R_{n1}S_{1n} + P_{nn}U_{nn} = P_{nn}U_{nn}$. Hence $U_{nn} = P_{nn}^{-1}$. Denote the elements of U_{nn} by μ_{ik}. Since b_{11} contains only a single element, it follows that $\mu_{ik} = a^0_{i+1,k+1}$, and in turn, by (4), $\mu_{ik} = A^k_{0i}/k!$. We have from (2) that

$$A^k_{0,i} = \sum_{j=k}^{n} \frac{(-1)^{j+i}\binom{j}{i} C^k_{0j}}{j!} \qquad (i, k = 1, \cdots, n).$$

From (3) $C^k_{0j} = p_k(k)S^k_j = k!S^k_j$. Finally, we have

$$A_{0i}^{k} = \sum_{j=k}^{n} \frac{(-1)^{j+i}\binom{j}{i} k! S_j^k}{j!},$$

and so, by (4),

$$\mu_{ik} = \sum_{j=k}^{n} \frac{(-1)^{j+i}\binom{j}{i} S_j^k}{j!}. \tag{5}$$

Since the kth column of V_{nn} (1, 1) is k^{-1} times the kth column of P_{nn}, ($k=1, \cdots, n$), it follows that V_n^{-1} $(1, 1)=\{i\mu_{ik}\}$, ($i, k=1, \cdots, n$), where μ_{ik} is given by (5). As an illustration, the following inverses are given:

$$P_{44}^{-1} = \begin{bmatrix} 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ -3 & \frac{19}{4} & -2 & \frac{1}{4} \\ \frac{4}{3} & -\frac{7}{3} & \frac{7}{6} & -\frac{1}{6} \\ -\frac{1}{4} & \frac{11}{24} & -\frac{1}{4} & \frac{1}{24} \end{bmatrix}, \quad V_4^{-1}(1, 1) = \begin{bmatrix} 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ -6 & \frac{19}{2} & -4 & \frac{1}{2} \\ 4 & -7 & \frac{7}{2} & -\frac{1}{2} \\ -1 & \frac{11}{6} & -1 & \frac{1}{6} \end{bmatrix}.$$

These inverses may be checked by multiplication with the corresponding direct matrix. P_{44}^{-1} was obtained directly, by use of (5).

4. The inverse of the general Vandermonde matrix. Let $x_0=0, x_1, \cdots, x_n$ be the given distinct numbers. The polynomial $y=y(x)$ of degree n assuming $n+1$ arbitrary values $y_i=y(x_i)$, $i=0, \cdots, n$, can be written, by Lagrange's interpolation formula, as

$$y = \sum_{i=0}^{n} A_{x,i} y_i. \tag{6}$$

If we write the kth derivative of $y(x)$ as

$$y^{(k)} = \sum_{i=0}^{n} A_{xi}^{k} y_i \qquad (k = 1, \cdots, n), \tag{7}$$

where

$$A_{x,i}^{k} = \frac{d^k}{dx^k} A_{x\,i}.$$

and set $x=x_0=0$, we have

(8)
$$y_0^{(k)} = \sum_{i=0}^{n} A_{0,i}^{k} y_i.$$

In the following, we obtain explicit expressions for the $A_{0,i}^k$, and then show that the $A_{0,i}^k$ satisfy systems of linear equations having the same coefficient matrix. Thus, we obtain the inverse of this matrix and, in turn, the inverse of $V(x_1, \cdots, x_n)$ in terms of the $A_{0,i}^k$.

The functions A_{xi}, as given by the Lagrange interpolation formula, are

$$\begin{aligned} A_{xi} &= \frac{1}{p_n{}'_{+1}(x_i)} \cdot \frac{p_{n+1}(x)}{(x - x_i)} \\ &= \frac{1}{p_n{}'_{+1}(x_i)} (x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n); \end{aligned}$$

and so

$$\begin{aligned} A_{xi} = \frac{1}{p_n{}'_{+1}(x_i)} [x^n &- \overset{i}{\sigma}_{1,n-1} x^{n-1} + \overset{i}{\sigma}_{2,n-1} x^{n-2} - \cdots \\ &+ (-1)^{n-2} \overset{i}{\sigma}_{n-2,n-1} x^2 + (-1)^{n-1} \overset{i}{\sigma}_{n-1,n-1} x], \end{aligned}$$

where $\sigma_{j,n-1}^i$ is the sum of all products of j of the numbers $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ without permutations or repetitions ($\sigma_{0,n-1}^i \equiv 1$). If we differentiate the above k times, set $x=0$, and notice that

$$p'_{n+1}(x_i) = \prod_{j=0}^{n}{}' (x_i - x_j),$$

where the dash indicates that $i \neq j$, we obtain*

(9)
$$A_{0i}^{k} = \frac{(-1)^{n-k} \cdot k!}{\prod_{j=0}^{n}{}' (x_i - x_j)} \cdot \overset{i}{\sigma}_{n-k,n-1}.$$

In order to exhibit the linear systems mentioned above, we expand $y(x)$ about $x_0=0$, and substitute $x=x_i$ to get

$$y_i = \sum_{\mu=0}^{n} \frac{1}{\mu!} y_0^{(\mu)} x_i^{\mu}.$$

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* More generally, if we do not substitute $x=0$ after the above differentiation, and substitute the result in (7), we obtain

$$y^{(k)}(x) = \sum_{i=0}^{n} y_i \sum_{j=k}^{n} \frac{(-1)^{n-j} \overset{i}{\sigma}_{n-j,n-1}}{p_n{}'_{+1}(x_i)} j(j-1) \cdots (j-k+1) x^{j-k},$$

which is a formula for an arbitrary derivative of $y(x)$ at an arbitrary x.

Inserting this into (8) and rearranging, we get

$$y_0^{(k)} = \sum_{\mu=0}^{n} \frac{1}{\mu!} y_0^{(\mu)} \sum_{i=0}^{n} x_i^{\mu} A_{0i}^{k} \qquad (k = 1, \cdots, n).$$

Since these are identities in $y_0^{(k)}$, we must have

$$\sum_{i=0}^{n} x_i^{\mu} A_{0i}^{k} = \delta_{\mu k} k! \qquad (\mu = 0, 1, \cdots, n),$$

where $\delta_{\mu k}$ is the Kronecker delta. Since $x_0 = 0$, it follows that

$$\sum_{i=1}^{n} x_i^{\mu} A_{0i}^{k} = \delta_{\mu k} \cdot k! \qquad (\mu = 1, \cdots, n).$$

For each k $(k = 1, \cdots, n)$, this is a system of n linear equations in the unknowns $A_{01}^k, A_{02}^k, \cdots, A_{0n}^k$. These n systems can be combined into the matrix equation

$$\{x_k^i\} \cdot \{A_{0i}^k\} = \begin{bmatrix} 1! & 0 & 0 \cdots 0 \\ 0 & 2! & 0 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots n! \end{bmatrix}$$

where A_{0i}^k is the element of $\{A_{0i}^k\}$ in the ith row and kth column, and similarly for the element x_k^i of $\{x_k^i\}$. (In x_k^i the i denotes an actual exponent.) Denoting the right member of the above equation by $D(n)$, we may write $\{A_{0i}^k\} = \{x_k^i\}^{-1} \cdot D(n)$. Thus, the element $b_{\lambda\mu}$ in the λth row and μth column of $\{x_k^i\}^{-1}$ $(\lambda, \mu = 1, \cdots, n)$ is given by

$$b_{\lambda\mu} = \frac{1}{\mu!} A_{0\lambda}^{\mu}, \tag{10}$$

where the $A_{0\lambda}^{\mu}$ may be obtained from (9).

Finally, if $V^{-1}(x_1, \cdots, x_n) \equiv \{v_{\lambda\mu}\}$, a method similar to that used in Section 3 yields $v_{\lambda\mu} = x_\lambda b_{\lambda\mu}$, $(\lambda, \mu = 1, \cdots, n)$, which, together with (10), gives an explicit representation for the inverse of the general Vandermonde matrix.

References

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ON THE CURRICULUM FOR PROSPECTIVE HIGH SCHOOL TEACHERS

G. PÓLYA, Stanford University

Within the framework of a National Science Foundation Institute held at Stanford University in the present academic year 1957–58 I am conducting two classes for high school teachers:

(1) *Seminar in Problem Solving*: 2 hours in the Fall and Winter Quarters, 3 hours in the Spring Quarter.

(2) *From Elementary Mathematics to the Calculus to Scientific Method* (the official title is a little different): 4 hours in the first two quarters, 3 hours in the last quarter.

In the last Summer Quarter, I had essentially the same classes, one for General Electric Fellows, the other for Shell Merit Fellows, although in a much abridged form; the contents and the form of presentation have been developed in regular university lectures given at Stanford since 1942. I think that these classes fill an important gap in the curriculum for prospective high school teachers and, therefore, I take the liberty to say a few words about them.

(1) The aim is to give the teachers experience in genuine, nonroutine mathematical work which, at this level, cannot be "research" but just "problem solving." (To my knowledge, neither the departments of mathematics nor the schools of education offer the teachers such experience—if there are exceptions, they are certainly rare.) Problems are solved in class discussion led by the instructor. The problems are not always easy, but they are on high school level, or only slightly above it. In the first phase, the problems are grouped according to subject matter. Geometric constructions with ruler and compasses; setting up equations; binomial coefficients and arithmetic series of higher order—here are three subjects which I found particularly appropriate. In the later phases of the seminar, the problems are grouped according to method to illustrate general ideas of problem solving. At the end, the participants in the class should be given opportunity to take the place of the instructor and lead the discussion.*

(2) This is a short course in analytic geometry and calculus, including a very sketchy last chapter on differential equations. Yet the course is pretty different from the usual: connections with elementary mathematics are emphasized at the beginning, applications to science are discussed at the end, general methodical ideas are stressed all along. Pat solutions are avoided, heuristic reasoning and historical sources are often in the foreground.

* The underlying ideas have been expressed in my books *How to Solve It* and *Mathematics and Plausible Reasoning*; the difficulty is to adapt these ideas to a specific level. I am working on a book presenting the materials of the seminar; the difficulty is to make the book usable by a not specifically prepared instructor.

"Mathematics as a Language" is the title of the first chapter. The "language of algebraic formulas" is contrasted with the "language of geometric figures" which speaks to us in graphs and diagrams. Analytic geometry is introduced as a "dictionary of two languages."

"The Beginnings of the Integral Calculus" is the next chapter; problems are solved by Archimedes' mechanical method, by Cavalieri's principle, and by the "method of exhaustion" before Leibnitz's notation and the usual scheme is introduced.

There is no space here to sketch the rest of the contents, but something should be said about the reasons which led to the outlined choice of subjects. I wish to present these remarks under the guise of informal advice (such as I am inclined to give in my classes).

1. There is one infallible teaching method: you will infallibly succeed in boring your audience with your subject if you are bored with it yourself. Hence the first commandment for teachers: *Be interested in your subject.*

2. No amount of courses in teaching methods will enable you to explain understandably a point that you do not understand yourself. Hence the second commandment for teachers: *Know your subject.*

3. Our knowledge about any subject consists of "information" and of "know-how." In mathematics "know-how" is the ability to solve problems and it is much more important than mere possession of information. You have to show your students how to solve problems—can you show it if you don't know it? Hence a special commandment for mathematics teachers: *Acquire, and keep up, some aptitude for problem solving.*

4. You may be obliged to discuss many problems which have little lasting interest in themselves. Yet you should use them to develop your students (and yourself). Therefore: *Look out for such features of the problem at hand as may be useful in solving the problems to come.*

5. The number of such features is unlimited. Here is a table (see Table 1) of some which seem to me the most important in acquiring good habits of mind—these are the points that, in my opinion, deserve the most attention, and should be stressed at each reasonable opportunity, in a high school mathematics class. More space than here available and, in the first place, many examples were needed to explain the points collected in Table 1 satisfactorily.† Yet here are a few hints.

6. In any problem something must be *unknown*—otherwise there would be nothing to do, nothing to look for. Yet the unknown must be specified somehow—and it cannot be specified unless something is known or given. The essential

† See the books quoted in footnote *.

Unknown	Data	Condition
Generalization		Specialization
	Analogy	
	Strict Reasoning	
	Reasonable Guessing	
	Language	
	of	
Diagrams		Formulas

TABLE 1. SOME GENERAL AIMS OF THE HIGH SCHOOL CURRICULUM

circumstances which link the unknown to the given things, or *data*, are enumerated by the *condition*. We cannot hope to solve a worthwhile problem unless we know, and know very well, what is the unknown, what are the data, and what is the condition. The student should acquire the habit of paying proper attention to unknown, data, and condition—a habit more important for his mental development and his later studies and professional work than the knowledge of any particular mathematical fact.

7. Generalities without interesting particular cases are of little value and so are particular facts without some hope of *generalization*. What is really valuable is ready ascent from particular facts to generalizations and ready descent from generalizations to particular facts.

8. *Analogy* is the great guide of invention. The best may be missing from a mathematics curriculum in which the student never meets an impressive example of discovery by analogy.

9. It has often been said that mathematics is a good school of *strict reasoning*. In fact more is true: strict (conclusive, "logical" "deductive") reasoning is essentially confined to mathematics; it deals only with objects lifted to the logical-mathematical level. What most students need in this respect is only so much contact with strict proofs that they get into the habit of clearly distinguishing between conclusive and inconclusive reasoning, between a proof and a guess.

Yet the student should also learn to distinguish between guesses and guesses, between a good guess and a bad guess, between a good guess and a better guess. Now (this is not so well known and, therefore, it must be said with great emphasis) mathematics is also a good school of *reasonable guessing*, of plausible (inconclusive, "inductive") reasoning. There is no space here to develop the

arguments for this assertion.‡ Yet let me give a little hint for classroom use: If there is a reasonable opportunity, *start a new problem or a new subject by letting your students guess*. Having guessed, they commit themselves and have to follow developments to see whether their guess comes true.

10. What is the aim of the high school mathematics curriculum? I do not intend to compete with all the big and beautiful words which allegedly answer this question. Yet let us put the question more clearly: By what standard should we judge the success of a high school curriculum in mathematics? By essentially the same standard, I say, as we would judge the curriculum in French: by the facility that the students acquire in using the language—since mathematics is essentially a language. After graduation, your former student will go to college or into some profession. In the one place as in the other, he may face a problem capable of mathematical treatment. If he can reduce it to a neat computation, or set up an equation for it, or express it by a diagram, the result of your teaching is excellent. If, without being able to produce them, he can read graphs and formulas, and appreciate them, too, the result is still very good. If, however, he cannot read graphs, cannot read formulas, and does not care for them a bit, the result is poor.

‡ See especially the second work quoted in footnote*.

CORRECTION

Raphael M. Robinson, *The converse of Fermat's theorem*, this MONTHLY, vol. 64, 1957, pp. 703–710. The formula in Theorem 10, page 709 should be "$a^{(N-1)/2} \equiv -1 \pmod{N}$." The error was made by the editorial staff, not by the author.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California

ON THE SERIES $\sum 1/p$

LEO MOSER, University of Alberta

For the fact that the sum of the reciprocals of the primes diverges, Euler was the first to give a (somewhat incomplete) proof [1]. Variations of Euler's proofs have been given by numerous mathematicians [2] and a number of

essentially different proofs have been given recently by Erdös [3] and Bellman [4]. The purpose of this note is to add one more to this number.

We begin with a lemma concerning subseries of the harmonic series. Let $\{p\} = p_1 < p_2 < \cdots$ be a sequence of positive integers and let

$$\pi(x) = \sum_{p \leqq x} 1 \quad \text{and} \quad R(x) = \sum_{p \leqq x} 1/p.$$

LEMMA. *If* $\lim_{x\to\infty} R(x)$ *exists, then* $\lim_{x\to\infty} \pi(x)/x = 0$.

Proof. We have

$$\pi(x) = 1\{R(1) - R(0)\} + 2\{R(2) - R(1)\} + \cdots + x\{R(x) - R(x-1)\}.$$

Hence

$$\pi(x)/x = R(x) - [\{R(0) + R(1) + \cdots + R(x-1)\}/x]. \tag{1}$$

If $\lim_{x\to\infty} R(x)$ exists, the square bracket in (1) approaches this limit, so that $\pi(x)/x \to 0$ as required.

In what follows, we let the p's be the consecutive primes and assume that $\lim_{x\to\infty} R(x)$ exists. This will lead to a contradiction and thus prove our theorem.

By our assumption and the lemma, there exist integers $n > 1$ and m such that

$$\sum_{p>n} 1/p < 1/2, \tag{2}$$

$$\pi(n!m)/m < 1/2. \tag{3}$$

With such n and m form the m integers $T_i = in! - 1$, $i = 1, \cdots, m$. All prime factors of the T_i exceed n and, if $p \mid T_i$ and $p \mid T_j$, then $p \mid (T_i - T_j)$ so that $p \mid (i-j)$. Hence a prime p is a divisor of at most $(m/p)+1$ of the T_i. Since each T is divisible by at least one prime p in the range $n!m > p > n$, we deduce

$$\sum_{n!m>p>n} \left(\frac{m}{p} + 1\right) \geqq m. \tag{4}$$

From (4) we have

$$\sum_{p>n} \frac{1}{p} + \frac{\pi(n!m)}{m} \geqq 1. \tag{5}$$

But now (2), (3), and (5) yield the desired contradiction.

References

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ON THE MINIMIZATION OF MATRIX NORMS

L. MIRSKY, University of Sheffield

A well-known inequality, due to Schur [1], states that *if A is a complex $n\times n$ matrix with euclidean norm $\|A\|$ and characteristic roots $\omega_1, \cdots, \omega_n$, then*

$$\|A\|^2 \geqq \sum_{i=1}^{n} |\omega_i|^2. \tag{1}$$

Equality occurs in (1) *if and only if A is normal.* Commenting on this result, Wedderburn [2] observed: "Since replacing a matrix by a similar one corresponds to a change of coordinates when the matrix is regarded as a linear transformation, it follows from Schur's work that, when the elementary divisors of A are simple, $\|A\|$ has its minimum value when A is represented as a diagonal matrix, that is, in its normal form; and it seems probable that the normal form also gives the minimum value even if the elementary divisors are not simple." Wedderburn thus conjectured that, for a fixed matrix A, $\|S^{-1}AS\|$ is minimal when $S^{-1}AS$ is the classical canonical form of A. This would imply, in particular, that, for a nondiagonable matrix A and any nonsingular matrix S,

$$\|S^{-1}AS\|^2 \geqq \sum_{i=1}^{n} |\omega_i|^2 + 1.$$

We shall demonstrate that Wedderburn's conjecture is incorrect by establishing the following result.

THEOREM. *Let A be a complex $n\times n$ matrix with characteristic roots $\omega_1, \cdots, \omega_n$. Then*

$$\inf \|S^{-1}AS\|^2 = \sum_{i=1}^{n} |\omega_i|^2, \tag{2}$$

where the lower bound is taken with respect to all nonsingular matrices S. Furthermore, the lower bound is attained if and only if A is diagonable.

Denote by T_1 a matrix such that $T_1^{-1}AT_1$ is triangular, say

$$T_1^{-1}AT_1 = \begin{Bmatrix} \omega_1 & b_{12} & \cdots & b_{1n} \\ 0 & \omega_2 & \cdots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \omega_n \end{Bmatrix}.$$

Let δ be an arbitrary positive number $\leqq 1$, and write $T_2 = \text{diag}\,(1, \delta, \cdots, \delta^{n-1})$, $S_0 = T_1T_2$. Then

$$\|S_0^{-1}AS_0\|^2 = \sum_{i=1}^{n} |\omega_i|^2 + \sum_{1\leqq i<k\leqq n} |b_{ik}|^2\delta^{2(k-i)}$$

$$\leqq \sum_{i=1}^{n} |\omega_i|^2 + \frac{1}{2} n(n-1)b^2\delta^2,$$

where $b = \max_{i,k} |b_{ik}|$. The relation (2) now follows since, by (1),

$$\|S^{-1}AS\|^2 \geqq \sum_{i=1}^{n} |\omega_i|^2$$

for every nonsingular matrix S. Furthermore, if A is diagonable, then the lower bound of $\|S^{-1}AS\|$ is clearly attained. If, on the other hand, the bound is attained so that, for some S,

$$\|S^{-1}AS\|^2 = \sum_{i=1}^{n} |\omega_i|^2,$$

then $S^{-1}AS$ is normal and A is diagonable.

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MATRIX NIM

John C. Holladay, Los Alamos Scientific Laboratory*

The original game of Nim is the game for which a position consists of a finite set of nonnegative integers, called piles. A legal move consists of lowering the size of one and only one pile. Two players take turns moving until all of the piles are reduced to zero, in which case the player that makes the last move wins. For games such as this, where players take turns making the same type of moves (which I call termination games), it is easily shown that positions may be divided into two uniquely determined classes, called safe and unsafe ([3], Th. 9), such that a player may force a win from an opponent whose turn it is if and only if the position is safe.

In 1902, C. L. Bouton [1] showed that a position in Nim is safe if and only if expressing the pile numbers as binary numbers, each power of two is represented an even number of times. For instance, in 3-pile Nim, a safe position is (49, 36, 21). Since $49=32+16+1$, $36=32+4$ and $21=16+4+1$, we have 32 represented twice, 16 twice, 8 no times, 4 twice, 2 no times and 1 twice.

In 1910, E. H. Moore [2] defined a position for "Nim_k" (k a positive integer) as the same as for ordinary Nim (which is "Nim_1"), and a move for "Nim_k" would consist of lowering any number of piles, not to exceed k. He showed that, in this game, a position is safe if and only if, when the pile numbers are expressed as binary numbers, each power of two is represented zero times modulo $k+1$.

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My paper on termination games [3] shows that for combining termination games in a certain way, the game of Nim enters in a fundamental manner. In that paper, I also generalize "Nim_k" to positions consisting of ordinal numbers.

In this paper, still another generalization of Nim is discussed. Let n be a positive integer. Then let a position for Nim^n (superscripts, instead of subscripts, are used to avoid confusion with Moore's game) be a rectangular array of nonnegative integers, called piles, such that there are n and only n columns and a finite number of rows. Let a legal move in Nim^n consist of either (1) Lowering the sizes of any nonempty set of piles, provided they are all in the same row, or (2) Lowering the sizes of any nonempty set of piles, provided they are all in the same $n-1$ columns, or equivalently, provided that at least one column is left untouched.

A game is terminated when all of the piles are reduced to zero, and whoever terminates a game wins it. Notice that Nim^1 is the ordinary game of Nim.

THEOREM. *A position in Nim^n is safe if and only if the following two conditions are both met:*

(a) *The sum of the elements in any column is equal to the sum of the elements in any other column.*

(b) *For each row, consider the smallest pile in it. Then these minimum pile sizes, one from each row, constitute a safe position in ordinary Nim.*

Proof. That a dichotomic labeling of positions into safe and unsafe satisfies the above win-and-lose condition, it is necessary and sufficient that it satisfy the following two conditions ([3], Par. 4 and Th. 9).

I. Given a safe position, any move leads to an unsafe position.

II. Given an unsafe position, there exists a move which leads to a safe position.

Let us first show that Condition I is satisfied. Given a safe position, a move would destroy (a), the equality of column sums, unless it was of Type (1) such that each element of the row was reduced by the same amount. However, the smallest size for the row would be lowered so that Condition (b) would no longer be satisfied.

To show Condition II, consider any unsafe position. If (b) is not satisfied, choose a row R for which one may lower its minimum size to a size M to satisfy (b). If (b) is satisfied (so (a) is not), choose any row R and let M be its minimum size. Choose a column C such that the sum of the elements in C is less than or equal to any other column sum. Now the situation breaks up into the following two cases:

Case A. If any element of R which is not in C is changed to M, then the new sum for the column containing this element will be less than the sum for column C.

Case B. There exists some element of R, which is not in C, such that if its size is reduced to (or left at) M, then the new sum for this column will be greater than or equal to the sum for C.

For Case A, one may move to a safe position by making a move of Type (1). For each element of R, consider the column sum for its column were this element to be made equal to M. Choose the element which gives the maximum such column sum and make it equal to M. Now for each of the other elements of R, put its size at the value which will make its column sum equal to the column sum of the previous sentence. Then the minimum for R will be M.

For Case B, one may move to a safe position by making a move of Type (2). Leave column C unaltered. Put one of the elements in R, not in C, to M so that the resulting column sum is at least as big as the sum for C. Consider each element as being the sum of the minimum for its row and a nonnegative integer, which we will call its excess. Then the sum of the excesses for each column is greater than or equal to the sum of the excesses for C. Whenever this sum is greater than the sum for C, lower the excesses until equality is achieved. Then Condition (b) as well as Condition (a) will be satisfied and we will have obtained a safe position.

An interesting, as yet unsolved, problem is to analyze the game of Matrix Nim where instead of permitting a move of Type (2), one may lower the sizes of any nonempty set of piles, provided they are all in the same column.

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A GENERAL THEORY FOR LINEAR SYSTEMS*

R. D. SHEFFIELD, Convair, Fort Worth, Texas†

A singular linear transformation L does not have an inverse. Nevertheless, it is always possible to find a partial or pseudo-inverse M of L, that is, a linear transformation M such that $LML=L$, [1] (Proposition 2, p. 179). If L^{-1} exists, of course, then L^{-1} is a pseudo-inverse, for $LL^{-1}L=L$.

From the equation $LML=L$ one can deduce an important application of the pseudo-inverse. Suppose that the linear equation $Lx=y_0$, in which y_0 denotes a

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† Now at the University of Mississippi.

given vector, is known to be consistent; that is, there exists x_0 such that $Lx_0 = y_0$. Then since

$$LMy_0 = LMLx_0 = Lx_0 = y_0,$$

it follows that My_0 is a solution of the given linear equation.

A pseudo-inverse is also useful in solving the homogeneous equation $Lx = 0$. Since L is singular, $I - ML \neq 0$. If z is in the domain of L, then

$$L(I - ML)z = Lz - LMLz = 0;$$

in other words, $x = (I - ML)z$ is a solution of the homogeneous equation $Lx = 0$. In fact, $I - ML$ projects X onto the null space, $N(L)$, of L.

In the finite-dimensional case L is a matrix, and an explicit calculation is presented here for a pseudo-inverse M of L. This procedure is suitable for machine programming and leads to an M with the desirable feature that the nonzero columns of $I - ML$ form a basis for the null space of L.

The process makes use of the Hermite canonical form of an $n \times n$ matrix.

1.1. DEFINITION. *A matrix $H = (h_{ij})$ is said to be in Hermite canonical form provided*

$$h_{ij} = 0, \quad \textit{whenever } i > j; \tag{1}$$

$$h_{ii} = 0 \quad \textit{or } 1; \tag{2}$$

$$\textit{if } h_{ii} = 0, \quad \textit{then } h_{ik} = 0, \quad (k = 1, \cdots, n); \textit{ and} \tag{3}$$

$$\textit{if } h_{ii} = 1, \quad \textit{then } h_{ji} = 0, \quad (j = 1, \cdots, n, j \neq i). \tag{4}$$

1.2. THEOREM. *For a given $n \times n$ matrix L, there exists a nonsingular matrix M which is a product of elementary matrices such that $ML = H$ is in Hermite canonical form.*

A proof of this theorem is given in many of the standard texts (see, for example [2], Theorem 18, p. 35).

1.3. LEMMA. *If H is in Hermite canonical form, then $H^2 = H$.*

Proof. Let a_{ik} denote an arbitrary element of H^2. By 1.1, (1), $a_{ik} = \sum_{j=1}^{n} h_{ij}h_{jk}$ vanishes if $i > k$ and reduces, if $i \leq k$, to $\sum_{j=i}^{k} h_{ij}h_{jk}$. By 1.1, (2), $h_{ii} = 0$ or 1; and if $h_{ii} = 0$, then by 1.1, (3), $h_{ij} = 0$ for all j and thus $a_{ik} = 0 = h_{ik}$. If $h_{ii} = 1$, then

$$a_{ik} = h_{ik} + \sum_{j=i+1}^{k} h_{ij}h_{jk} \quad .$$

Now whenever $h_{ij} \neq 0$ for some $j > i$, it follows from 1.1, (4) that $h_{jj} = 0$, and hence from (3) that $h_{jm} = 0$ for all m and in particular for $m = k$. Hence, in any case, $a_{ik} = h_{ik}$ and $H^2 = H$.

1.4. THEOREM. *If M is the product of the elementary matrices which reduce L to H, then M is a nonsingular pseudo-inverse of L.*

Proof. By hypothesis $ML=H$, and since M is a product of elementary matrices, M^{-1} exists; therefore, $L=M^{-1}H$. Hence

$$LML = M^{-1}HMM^{-1}H = M^{-1}H^2 = M^{-1}H = L.$$

M can be computed by performing on the identity matrix the elementary row operations which reduce L to H. If, however, one is interested only in the solution of $Lx=y_0$ it is not necessary to compute M. To solve the problem one proceeds in the following manner. Form the $n\times(n+1)$ matrix $L^+=(L, y_0)$, and then perform the elementary row operations on L^+ which reduce L to H. The resulting matrix ML^+ is of the form (H, My_0), where H is of the form

$$\begin{bmatrix} 1 & a & 0 & b & d \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c & e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the only nonzero rows of H are those with a 1 in the diagonal, it follows that the rank of L, which equals the rank of H, is given by the number of 1's along the diagonal. Moreover, any column which has a 1 in the diagonal is a unit vector by 1.1, (4). Since the $(n+1)$st column of L^+ becomes My_0 as L is reduced to H and since L and M are full inverses of each other on the range of L and the range of ML, it follows that the equation $Lx=y_0$ is consistent if and only if My_0 has nonzero components only in the rows in which H has 1's. In this case, My_0 is a particular solution of $Lx=y_0$. If $H=ML$ has r 1's down the diagonal, then ML contains r unit column vectors and hence $I-ML=I-H$ contains exactly $n-r$ nonzero columns. Since $(I-ML)u=u$ for arbitrary $u\in N(L)$, it follows that these columns of $I-ML$ span $N(L)$; and since $N(L)$ is of dimension $n-r$, it even follows that they are linearly independent. Thus the complete solution of $Lx=y_0$ is given by

$$x = My_0 + (I - ML)D,$$

in which D is an arbitrary diagonal matrix containing only $n-r$ effective parameters since $I-ML$ has only $n-r$ nonzero columns.

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ON AN EXISTENCE THEOREM FOR COMPLEX-VALUED DIFFERENTIAL EQUATIONS*

PAUL R. BEESACK, Hamilton College, McMaster University

The purpose of this note is to provide an alternative proof of a theorem stated in the recent—and excellent—textbook on ordinary differential equations by Coddington and Levinson [1]. This theorem (Th. 8.4, p. 36) is stated for a system of equations involving several complex parameters. We shall state and prove the theorem for a single equation, namely

$$w' = f(t, w, \lambda), \tag{1}$$

involving only a single complex parameter λ. In addition, we strengthen slightly both the hypotheses and the conclusion in stating

THEOREM I. *Let $f(t, w, \lambda)$, f_w, f_λ be continuous on a domain*

$$D_\lambda: t \in I_t,\ w \in D,\ |\lambda - \bar{\lambda}| < c.$$

For fixed $\lambda = \bar{\lambda}$, let $w(t)$ be a solution of (1) *on a subinterval I, $a \leqq t \leqq b$, of I_t. Then there exists $\delta > 0$ such that for any point (t^0, w^0, λ^0) of U_δ, where*

$$U_\delta: a < t^0 < b,\ |w^0 - w(t^0)| + |\lambda^0 - \bar{\lambda}| < \delta,$$

there exists a unique solution $W(t; t^0, w^0, \lambda^0)$ of (1) *on I, passing through the point (t^0, w^0). Moreover, $W \in C^1$ on the domain*

$$V_\delta: a < t < b,\quad (t^0, w^0, \lambda^0) \in U_\delta,$$

and for each fixed (t, t^0), W is an analytic function of $(w^0, \lambda^0) \in U_\delta$.

In this theorem, D is a domain of the complex w-plane, the function $f = f_1 + if_2$ is complex-valued, and $\lambda = \lambda_1 + i\lambda_2$ is a complex parameter. For fixed λ, a solution of (1) is a complex-valued function $w(t) = w_1(t) + iw_2(t)$ defined on an interval I of the t-axis, of class C^1 on I (*i.e.*, $w_1(t)$, $w_2(t) \in C^1$), and satisfying the equation (1) on this interval (so that $w(t) \in D$ for $t \in I$).

In their book the authors suggest that the proof may be carried out by the method of successive approximations. This is indeed true, and as the authors remark, the proof is very similar to that of a corresponding theorem involving only real functions, the analyticity following from the uniformity of convergence. The proof we intend to give, on the other hand, makes use of the corresponding theorem for real functions—or rather, of the corresponding theorem involving a system (of two equations) involving two (real) parameters ([1], Th. 7.5, p. 30). Part of the interest in this alternative proof is due to the fact that it brings to the fore the role played by the theory of functions of two complex variables, in particular the Cauchy-Riemann equations.

For completeness, we restate the form of Theorem 7.5 [1] which we intend to use, as

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THEOREM II. *Consider the (real) system of differential equations*

$$w_1' = f_1(t, w_1, w_2; \lambda_1, \lambda_2), \qquad w_2' = f_2(t, w_1, w_2; \lambda_1, \lambda_2),$$

where the λ_i are real parameters. Let D be a domain of (t, w_1, w_2) space, and let I_λ be defined by $|\lambda_1-\bar\lambda_1|+|\lambda_2-\bar\lambda_2|<c$. By D_λ we mean the domain of $(t, w_1, w_2, \lambda_1, \lambda_2)$ space defined by

$$D_\lambda\colon (t, w_1, w_2) \in D, \qquad (\lambda_1, \lambda_2) \in I_\lambda.$$

Suppose that f_i, $\partial f_i/\partial w_j$, $\partial f_i/\partial\lambda_j$ $(i, j=1, 2)$ are all continuous on D_λ. For the fixed values $\lambda_i=\bar\lambda_i$, let $(w_1(t), w_2(t))$ be a solution of (2) on an interval $a\leqq t\leqq b$. Then there exists $\delta>0$ such that for any point $(t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)\in U_\delta$, where

$$U_\delta\colon a < \overset{0}{t} < b, \ | \overset{0}{w}_1 - w_1(\overset{0}{t}) | + | \overset{0}{w}_2 - w_2(\overset{0}{t}) | + | \overset{0}{\lambda}_1 - \bar\lambda_1 | + | \overset{0}{\lambda}_2 - \bar\lambda_2 | < \delta,$$

there exists a unique solution $W_i(t;\, t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)$ $(i=1, 2)$ of (2) on $a\leqq t\leqq b$ passing through the point (t^0, w_1^0, w_2^0) of D. Moreover, $W_i\in C^1$ on the domain

$$V_\delta\colon a < t < b, \qquad (\overset{0}{t}, \overset{0}{w}_1, \overset{0}{w}_2, \overset{0}{\lambda}_1, \overset{0}{\lambda}_2) \in U_\delta. \tag{3}$$

In order to prove Theorem I we first note ([2], Ch. IX) that the hypotheses on f imply that the eight partial derivatives $\partial f_i/\partial w_j$, $\partial f_i/\partial \lambda_j$ exist, are continuous in D_λ, and satisfy there the Cauchy-Riemann equations—which are a consequence of the equations

$$\begin{aligned} \frac{\partial f}{\partial w} &= \frac{\partial f_1}{\partial w_1} + i\frac{\partial f_2}{\partial w_1} = \frac{\partial f_2}{\partial w_2} - i\frac{\partial f_1}{\partial w_2}, \\ \frac{\partial f}{\partial \lambda} &= \frac{\partial f_1}{\partial \lambda_1} + i\frac{\partial f_2}{\partial \lambda_1} = \frac{\partial f_2}{\partial \lambda_2} - i\frac{\partial f_1}{\partial \lambda_2}. \end{aligned} \tag{4}$$

Now, writing $w=w_1+iw_2$, $\lambda=\lambda_1+i\lambda_2$, *etc.*, we replace the single complex equation (1) by the equivalent system (2). By hypothesis this system has, for $\bar\lambda=\bar\lambda_1+i\bar\lambda_2$, a solution $(w_1(t), w_2(t))$—where $w_1(t)+iw_2(t)=w(t)$—defined on I. Moreover, by our preceding remarks the remaining hypotheses of Theorem II are also satisfied. Hence there exists $\delta_1>0$ such that for any point $(t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)\in U_{\delta_1}$, there exists a unique solution $W_i(t;\, t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)$ of (2) on $a\leqq t\leqq b$ satisfying

$$W_i(\overset{0}{t};\, \overset{0}{t}, \overset{0}{w}_1, \overset{0}{w}_2, \overset{0}{\lambda}_1, \overset{0}{\lambda}_2) = \overset{0}{w}_i \qquad (i = 1, 2).$$

If we now set

$$W(t;\, \overset{0}{t}, \overset{0}{w}, \overset{0}{\lambda}) = W_1(t;\, \overset{0}{t}, \overset{0}{w}_1, \overset{0}{w}_2, \overset{0}{\lambda}_1, \overset{0}{\lambda}_2) + iW_2(t;\, \overset{0}{t}, \overset{0}{w}_1, \overset{0}{w}_2, \overset{0}{\lambda}_1, \overset{0}{\lambda}_2)$$

and $\delta=\delta_1/\sqrt{2}$, then the existence, uniqueness and continuity properties of Theorem I follow immediately from the above on noting that

$$|\overset{0}{w_1} - w_1(\overset{0}{t})| + |\overset{0}{w_2} - w_2(\overset{0}{t})| + |\overset{0}{\lambda_1} - \bar{\lambda}_1| + |\overset{0}{\lambda_2} - \bar{\lambda}_2|$$
$$\leq \sqrt{2}(|\overset{0}{w} - w(\overset{0}{t})| + |\overset{0}{\lambda} - \bar{\lambda}|).$$

It remains, however, to prove that—for each fixed $(t, t^0) - W(t; t^0, w^0, \lambda^0)$ is an analytic function of the two complex variables w^0, λ^0 for $(w^0, \lambda^0) \in U_\delta$. To prove this, we note that

$$(5) \qquad W_i(t; \overset{0}{t}, \overset{0}{w_1}, \overset{0}{w_2}, \overset{0}{\lambda_1}, \overset{0}{\lambda_2}) = \overset{0}{w_i} + \int_{t^0}^{t} f_i(t, W_1, W_2, \overset{0}{\lambda_1}, \overset{0}{\lambda_2})dt \qquad (i = 1, 2),$$

for $(t, t^0, w^0, \lambda^0) \in V_\delta$. Hence, differentiating, we have

$$(6) \qquad \begin{aligned}
\frac{\partial W_1}{\partial w_1^0} &= 1 + \int_{t^0}^{t} \left\{ \frac{\partial f_1}{\partial w_1} \frac{\partial W_1}{\partial w_1^0} + \frac{\partial f_1}{\partial w_2} \frac{\partial W_2}{\partial w_1^0} \right\} dt, \\
\frac{\partial W_1}{\partial w_2^0} &= \int_{t^0}^{t} \left\{ \frac{\partial f_1}{\partial w_1} \frac{\partial W_1}{\partial w_2} + \frac{\partial f_1}{\partial w_2} \frac{\partial W_2}{\partial w_2^0} \right\} dt, \\
\frac{\partial W_2}{\partial w_1^0} &= \int_{t^0}^{t} \left\{ \frac{\partial f_2}{\partial w_1} \frac{\partial W_1}{\partial w_1^0} + \frac{\partial f_2}{\partial w_2} \frac{\partial W_2}{\partial w_1^0} \right\} dt, \\
\frac{\partial W_2}{\partial w_2^0} &= 1 + \int_{t^0}^{t} \left\{ \frac{\partial f_2}{\partial w_1} \frac{\partial W_1}{\partial w_2^0} + \frac{\partial f_2}{\partial w_2} \frac{\partial W_2}{\partial w_2^0} \right\} dt.
\end{aligned}$$

It follows that $X(t) \equiv (\partial/\partial w_1^0) W_1(t; t^0, w_1^0, w_2^0, \lambda_1^0, \lambda_2^0)$, $Y(t) \equiv \partial W_2/\partial w_1^0$, $Z(t) \equiv \partial W_1/\partial w_2^0$, $V(t) \equiv \partial W_2/\partial w_2^0$ are solutions of the linear system

$$(7) \qquad \begin{aligned}
X' &= \frac{\partial f_1}{\partial w_1} X + \frac{\partial f_1}{\partial w_2} Y + 0 \cdot Z + 0 \cdot V, \qquad & Z' &= 0 \cdot X + 0 \cdot Y + \frac{\partial f_1}{\partial w_1} Z + \frac{\partial f_1}{\partial w_2} V, \\
Y' &= \frac{\partial f_2}{\partial w_1} X + \frac{\partial f_2}{\partial w_2} Y + 0 \cdot Z + 0 \cdot V, \qquad & V' &= 0 \cdot X + 0 \cdot Y + \frac{\partial f_2}{\partial w_1} Z + \frac{\partial f_2}{\partial w_2} V,
\end{aligned}$$

on the interval $a \leqq t \leqq b$. Here,

$$\frac{\partial f_1}{\partial w_1} = \frac{\partial}{\partial w_1} f_1(t, W_1(t; \overset{0}{t}, \overset{0}{w_1}, \overset{0}{w_2}, \overset{0}{\lambda_1}, \overset{0}{\lambda_2}), W_2(t; \overset{0}{t}, \overset{0}{w_1}, \overset{0}{w_2}, \overset{0}{\lambda_1}, \overset{0}{\lambda_2}), \overset{0}{\lambda_1}, \overset{0}{\lambda_2}),$$

etc., and we are holding t^0, w_1^0, w_2^0, λ_1^0, λ_2^0 fixed. This solution system satisfies the initial conditions $X(t^0) = 1$, $Y(t^0) = 0$, $Z(t^0) = 0$, $V(t^0) = 1$, by (6).

We now assert that the system (7) also has a solution system

$$x(t) = \frac{\partial}{\partial w_2^0} W_2(t; \overset{0}{t}, \overset{0}{w_1}, \overset{0}{w_2}, \overset{0}{\lambda_1}, \overset{0}{\lambda_2}), \quad y(t) = -\partial W_1/\partial \overset{0}{w_2},$$
$$z(t) = -\partial W_2/\partial \overset{0}{w_1}, \quad v(t) = \partial W_1/\partial \overset{0}{w_1}.$$

This follows from the fact that $\partial f_1/\partial w_1 \equiv \partial f_2/\partial w_2$, $\partial f_1/\partial w_2 \equiv -\partial f_2/\partial w_1$, according

to the first of equations (4). Thus, for example, the first of equations (7) may be rewritten as $X' = (\partial f_2/\partial w_2)X - (\partial f_2/\partial w_1)Y$. Since

$$x'(t) = \frac{d}{dt}\left(\frac{\partial W_2}{\partial w_2^0}\right) = \frac{\partial f_2}{\partial w_1}\frac{\partial W_1}{\partial w_2^0} + \frac{\partial f_2}{\partial w_2}\frac{\partial W_2}{\partial w_2^0} = -\frac{\partial f_2}{\partial w_1}y(t) + \frac{\partial f_2}{\partial w_2}x(t),$$

we see that our new system satisfies the first of equations (7). Similarly one may verify that all of equations (7) are satisfied. Moreover, we have $x(t^0)=1$, $y(t^0)=0$, $z(t^0)=0$, $v(t^0)=1$. Hence, by the uniqueness theorem for systems of linear differential equations ([1], Theorem 5.1) we conclude that $X\equiv x$, $Y\equiv y$, $Z\equiv z$, $V\equiv v$ on $a\leqq t\leqq b$. That is, for arbitrary $(t, t^0, w^0, \lambda^0)\in V_\delta$, we have

$$\frac{\partial W_1}{\partial w_1^0} = \frac{\partial W_2}{\partial w_2^0}, \qquad \frac{\partial W_2}{\partial w_1^0} = -\frac{\partial W_1}{\partial w_2^0}. \tag{8}$$

Similarly, by differentiating equations (5) with respect to λ_1^0 and λ_2^0 ,we may conclude that

$$\frac{\partial W_1}{\partial \lambda_1^0} \equiv \frac{\partial W_2}{\partial \lambda_2^0}, \qquad \frac{\partial W_2}{\partial \lambda_1^0} \equiv -\frac{\partial W_1}{\partial \lambda_2^0}. \tag{9}$$

Since all the derivatives appearing in (8) and (9) are continuous and since W is itself continuous on V_δ, it follows that for each fixed (t, t^0), W is an analytic function of (w^0, λ^0) for $(w^0, \lambda^0)\in U_\delta$.

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2. É. Picard, Traité d'Analyse, Tome II, 3rd ed., Paris, 1926.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

A SIMPLE ILLUSTRATION OF OPERATIONAL METHODS

D. W. ROBINSON, Brigham Young University

Let A and X_0 be (complex) constant matrices of orders n-by-n and 1-by-n, respectively, and consider the nth order linear homogeneous system

$$X'(t) = X(t)A, \qquad X(0) = X_0, \tag{1}$$

of differential equations with constant coefficients. It is well-known that the

AN APPLICATION OF ROTATION FORMULAS

R. LARIVIERE AND G. LOUIS, University of Illinois, Chicago

The normal form of the equation of a straight line is taught students of plane analytical geometry chiefly to enable them to write the distance from an oblique line to a point with little computation. The topic has long been a stumbling block, especially when the direction cosines of the lines are used in plane analytic geometry in preparation for their use in solid [1]. Many authors have attempted to substitute other methods of deriving the line-to-point distance [2, 3, 4]. Often these have met with indifferent success except in the hands of their originators. In at least one case the struggle has been abandoned, and the authors have returned to a presentation of the normal form as an intermediate step in the derivation of the line-to-point formula [5].

The following method of deriving this formula is an interesting application of rotation of axes. A rotation of the axes through an angle θ equal to the smallest counter-clockwise angle between the positive y-axis and the undirected line transforms the equation of the line $Ax+By+C=0$ to $x=p$. Substituting the rotation formulas in the equation and eliminating the term in y we find that $p=-C/(\pm\sqrt{A^2+B^2})$, $\tan\theta=B/A$, $\sin\theta=B/(\pm\sqrt{A^2+B^2})$, and $\cos\theta=A/(\pm\sqrt{A^2+B^2})$. The distance from the new y-axis to any point $P(x_1, y_1)$ is the new abscissa of P, $x_1\cos\theta+y_1\sin\theta$. And the distance from the line $x=p$ to the point P is $x_1\cos\theta+y_1\sin\theta-p$, namely $(Ax_1+By_1+C)/(\pm\sqrt{A^2+B^2})$.

With θ defined as above, the convention of signs implied is that the sign of the perpendicular from the origin to the line agrees with that of the y-intercept. Since $\theta<180°$ we are able to choose the sign of the radical as in the traditional procedures which use this convention.

If the convention preferred is that the perpendicular distance from the origin to the line should be positive in all four quadrants, the angle of rotation is $(180°+\theta)$ when the y-intercept is negative, or when the y-intercept is zero and the x-intercept is negative. As before $\tan\theta=B/A$, and $p=-C/(\pm\sqrt{A^2+B^2})$, but $p\geqq 0$. The distance from the line to the point P is $(Ax_1+By_1+C)/(\pm\sqrt{A^2+B^2})$.

The sign before the radical is now chosen as in the traditional procedures which use the convention that the sign of the perpendicular from the origin to the line is always positive. We may adopt these procedures since, when $C=0$ and the line passes through the origin, θ may be taken within the range $180°>\theta>0$.

References

1. F. Nowlan, Analytic Geometry, New York, 1946.
2. J. W. Cell, Analytic Geometry, New York, 1951, p. 42.
3. N. McCoy and R. Johnson, Analytic Geometry, New York, 1955, p. 52.
4. R. Middlemiss, Analytic Geometry, New York, 1955, p. 72.
5. R. S. Underwood and F. W. Sparks, Boston, 1956, p. v.

COMPUTATION OF COMMON LOGARITHMS BY REPEATED SQUARINGS

O. E. Brown, The Babcock and Wilcox Company

1. Introduction. A method for the computation of common logarithms has been derived by the writer which is so elementary that it can be understood by high school students and has the additional merit of lending itself to easy programming for modern digital computers. The method, like that described by Shanks [1] and Ostrowski [2] utilizes the arithmetic properties of the individual logarithm rather than the analytic properties of the logarithmic function.

2. Derivation. The entire method rests squarely upon the following:

THEOREM. *If* $N_1, N_2, \cdots$ *and* $a_1, a_2, \cdots$ *be two sequences of nonnegative real numbers such that* $1 \leqq N_i < 10$ *and* $a_i = 0$ *or* $a_i = 1$, *for every* i;

$$\text{(1)} \qquad \text{if } N_i^2 < 10, \qquad N_{i+1} = N_i^2, \qquad a_i = 0$$

and

$$\text{(2)} \qquad \text{if } N_i^2 \geqq 10, \qquad N_{i+1} = N_i^2/10, \qquad a_i = 1;$$

then

$$\text{(3)} \qquad \log_{10} N_1 = a_1/2 + a_2/2^2 + \cdots.$$

Proof. Let the binary representation of $\log_{10} N_1$ consist of the bits $b_1, b_2, \cdots$, *i.e.*,

$$\log_{10} N_1 = b_1/2 + b_2/2^2 + \cdots,$$

where $b_i = 0$ or 1. Then*

$$\text{(4)} \qquad N_1 = \exp_{10}(b_1/2 + b_2/2^2 + \cdots)$$

and squaring gives

$$\text{(5)} \qquad N_1^2 = \exp_{10}(b_1 + b_2/2 + \cdots).$$

Clearly, if $N_1^2 \geqq 10$ then $b_1 = 1$, while if $N_1^2 < 10$ then $b_1 = 0$ so that $b_1 = a_1$, and upon division of both sides of (5) by $\exp_{10} b_1$, we have

$$\text{(6)} \qquad \frac{N_1^2}{\exp_{10} b_1} = N_2 = \exp_{10}(b_2/2 + b_3/2^2 + \cdots).$$

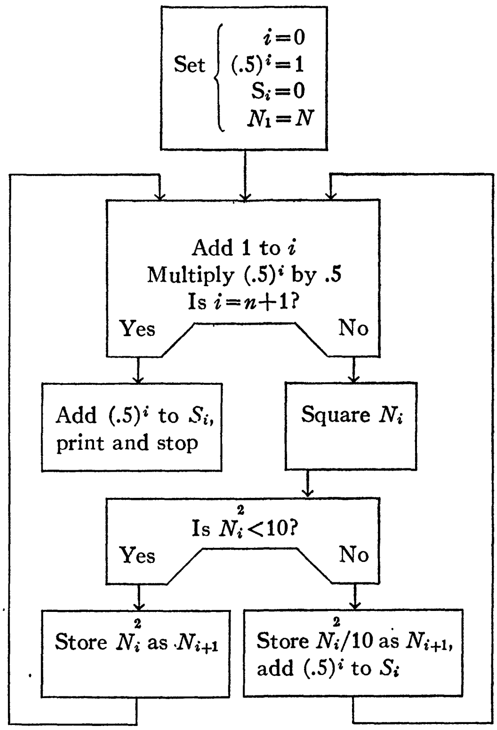
Since by (6) we have N_2 expressed in terms of $b_2, b_3, \cdots$, in the same way as (5) expresses N_1 in terms of $b_1, b_2, \cdots$, the squaring of N_2 will show that

* We write $\exp_{10} x$ for 10^x.

$b_2 = a_2$. Continuation establishes the identity of the sequence b_i with the sequence a_i and the theorem is proved.

3. Program. In order to write a computer program for the method of this paper it is only necessary to calculate each successive N_i by squaring its predecessor and dividing by 10 if need be. The a_i are selected according to (1) and (2) and $\log_{10} N_1$ accumulated by (3). A suggested block diagram is shown in Table 1.

TABLE 1



4. Accuracy. There are plainly two sources of error in the calculation. One of these is the round-off error in the squaring process. We are making what amounts to a floating decimal calculation for which estimation of errors is difficult, to say the least. However, a few trials seems to indicate that if k figures are carried in the N_i then the $\log_{10} N_1$ obtained will be correct to k figures. To be quite safe we may carry $2k$ figures of each N_i in calculating $\log_{10} N_1$ to k figures.

The other type of error which arises is that due to the necessary truncation of the series (3). Suppose that this truncation takes place after n terms, so that the deleted terms are

$$a_{n+1}/2^{n+1} + a_{n+2}/2^{n+2} + \cdots . \tag{7}$$

Since each of the a's is either 0 or 1, we may treat them as random variables whose two values have equal probabilities. The expected value of each a_i then becomes 1/2 and the truncation error (7) has the expected value $1/2^{n+1}$.

The writer has programmed the Electrodata E101 to employ this method with $n=20$. Peculiarities of that machine dictate that 10 figures are kept of each N_i. To the truncated series we have added the expected remainder term $1/2^{21}$ or .000 000 4768 as well as the round-off correction .000 005 and have printed out the resulting five-decimal logarithm. We have not, as yet, observed a case in which the five-decimal value thus obtained was wrong. We have encountered a few results which are in error by 1 in the last figure when we tried stopping after 18 squarings.

References

1. Daniel Shanks, A logarithm algorithm, Math. Tables Aids Comput., vol. 8, 1954, pp. 60–64.
2. A. M. Ostrowski, Note on a logarithm algorithm, Math. Tables Aids Comput., vol. 9, 1955, pp. 65–68.

A GROUP-THEORETIC PROOF OF WILSON'S THEOREM

WALTER FEIT, Cornell University

Wilson's theorem states that if p is a prime, then p divides $(p-1)!+1$. We give a proof based on some elementary group theory results.

Let n be the number of elements in the symmetric group S_p on p letters, whose order divides p. A well-known result of Frobenius* states that p divides n. The number of p-cycles in S_p is easily seen to be $(p-1)!$. Since every element in S_p whose order divides p is either a p-cycle or the identity, $n=(p-1)!+1$. The proof is complete.

A METHOD FOR SOLVING AN EXACT DIFFERENTIAL EQUATION

L. L. PENNISI, University of Illinois, Chicago

Suppose that the following conditions are satisfied:

$$(1)\qquad \begin{gathered} P(x, y, z)dz + Q(x, y, z)dy + R(x, y, z)dz = 0, \\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}; \end{gathered}$$

$$(2)\qquad Q = Q_1 + Q_2 \textit{ and } \frac{\partial Q_2}{\partial x} = 0, \qquad R = R_1 + R_2 \textit{ and } \frac{\partial R_2}{\partial x} = 0, \qquad \frac{\partial R_2}{\partial y} = 0;$$

$$(3)\qquad Q_1(x_0, y, z) = 0 \textit{ and } \int_{y_0}^{y} \frac{\partial Q_2}{\partial z}\, dy = R_1(x_0, y, z).$$

Then the solution of equation (1) *is given by*

* See W. Burnside, Theory of Groups of Finite Order, Cambridge University, 1911, p. 49.

$$(4) \qquad F(x, y, z) = C \qquad (C = \text{constant}),$$

where

$$(5) \qquad F = \int_{x_0}^{x} P dx + \int_{y_0}^{y} Q_2 dy + \int_{z_0}^{z} R_2 dz.$$

Proof. We shall show that $(\partial F/\partial x) = P$, $(\partial F/\partial y) = Q$ and $(\partial F/\partial z) = R$. Observe from equation (5) that $(\partial F/\partial x) = P$. Also

$$\frac{\partial F}{\partial y} = \int_{x_0}^{x} \frac{\partial P}{\partial y} dx + Q_2 = \int_{x_0}^{x} \frac{\partial Q_1}{\partial x} dx + Q_2$$
$$= Q_1(x, y, z) - Q_1(x_0, y, z) + Q_2 = Q_1 + Q_2 = Q.$$

$$\frac{\partial F}{\partial z} = \int_{x_0}^{x} \frac{\partial P}{\partial z} dz + \int_{y_0}^{y} \frac{\partial Q_2}{\partial z} dy + R_2 = \int_{x_0}^{x} \frac{\partial R_1}{\partial x} dx + R_1(x_0, y, z) + R_2$$
$$= R_1(x, y, z) - R_1(x_0, y, z) + R_1(x_0, y, z) + R_2 = R_1 + R_2 = R.$$

Remark. Almost all problems given in textbooks satisfy conditions (3) with $x_0 = y_0 = 0$. At times, however, a given exact differential equation may be separated into two or more exact differential equations in each of which the above method may be applied with different values of x_0 and y_0. The sum of these solutions gives the solution of our original equation.

Example 1. Solve the exact differential equation

$$(z^2y^2 - z^2)xdx + (x^2z^2 + z^2)ydy + (x^2y^2 - x^2 + y^2 - 1)zdz = 0.$$

Observe that $P = (z^2y^2 - z^2)x$, $Q_1 = x^2z^2y$, $Q_2 = z^2y$, $R_1 = (x^2y^2 - x^2 + y^2)z$ and $R_2 = -z$. Taking $x_0 = y_0 = 0$, we see that conditions (3) are satisfied. Hence

$$F = \int_0^x (z^2y^2 - z^2)xdx + \int_0^y z^2ydy - \int_0^z zdz.$$

Therefore, $(1/2)(x^2y^2z^2 - x^2z^2 + z^2y^2 - z^2) = C$, is the desired solution.

Example 2. Solve the exact differential equation

$$(e^x \cos y + yz)dx + (xz - e^x \sin y)dy + (xy + z)dz = 0.$$

This equation may be separated into two exact equations

$$(6) \qquad e^x \cos y dx - e^x \sin y dy = 0,$$

$$(7) \qquad yzdx + xzdy + (xy + z)dz = 0.$$

For equation (6), we have $P = e^x \cos y$, $Q_1 = e^x \sin y$, $Q_2 = R_1 = R_2 = 0$. Conditions (3) are satisfied for $x_0 = -\infty$ and $y_0 = 0$. Hence $F_1 = \int_{-\infty}^{x} e^x \sin y dx = e^x \sin y$. For equation (7), $P = yz$, $Q_1 = xz$, $Q_2 = 0$, $R_1 = xy$, $R_2 = z$. Conditions (3) are satisfied for $x_0 = y_0 = 0$. Thus $F_2 = \int_0^x yzdx + \int_0^z zdz = xyz + \frac{1}{2}z^2$. Hence the solution is $e^x \cos y + xyz + \frac{1}{2}z^2 = C$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1301. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the angles of a triangle ABC are consecutive terms of a geometric progression of common ratio 3, then

$$\cos B \cos C + \cos C \cos A + \cos A \cos B = -1/4.$$

E 1302. *Proposed by M. S. Klamkin, AVCO, Research and Development, Lawrence, Mass.*

A square is divided into two parts by an arbitrary diameter through its center. Determine the locus of the centroid of one of the equal areas.

E 1303. *Proposed by M. J. Hellman, Rutgers University*

Solve the differential equation $dy/dx = \mathcal{R}(z^n)/\mathcal{I}(z^n)$, where $z = x + iy$, n is a positive integer, $\mathcal{R}(z^n)$ denotes the real part of z^n, and $\mathcal{I}(z^n)$ denotes the imaginary part of z^n.

E 1304. *Proposed by W. B. Carver, Cornell University*

Let A_1, A_2, A_3 be the vertices of any triangle and let the arc of the circumcircle from A_i to A_j be trisected by the points T_{ij} and T_{ji}, T_{ij} adjacent to A_i and T_{ji} adjacent to A_j, $i, j = 1, 2, 3$. Further, let A_1T_{32} and A_3T_{12} intersect in P_2, A_1T_{23} and A_2T_{13} intersect in P_3. Show that $T_{31}T_{21}$ is parallel to P_2P_3.

E 1305. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Given a positive integer n and a symmetric 3×3 matrix $A = (a_{ij})$ with integral elements, when does there exist an $n\times 3$ matrix B such that $B^*B = A$ (here B^* denotes the transpose of B) and every element of B is either 0 or 1? Describe a method for finding all solutions B.

SOLUTIONS

A Line Touching a Sphere and a Cube

E 1271 [1957, 432]. *Proposed by Ward Cheney and A. A. Goldstein, Convair-Astronautics, San Diego, California*

In E_3 let L denote a line not through the origin and not parallel to any coordinate plane. Does the point of contact of L with the smallest sphere having

center at the origin necessarily lie in the same octant as the point of contact of L with the smallest cube having center at the origin and edges parallel to the coordinate axes?

Solution by W. B. Carver, Cornell University. No. For example, let the line L be $(x-1)/4=-(y-3)/4=z-8$. The sphere touching this line is the sphere $x^2+y^2+z^2=74$, the point of contact being $(1, 3, 8)$. The cube touching this line is the cube bounded by the planes $x=\pm 7$, $y=\pm 7$, $z=\pm 7$, the point of contact being $(-3, 7, 7)$.

Also solved by Leon Bankoff, Michael Goldberg, A. R. Hyde, C. S. Ogilvy, and the proposers.

The proposers pointed out that the answer to the corresponding problem in E_2 is "yes," and is also "yes" in E_3 if L denotes a plane, or more generally in E_n if L denotes a hyperplane.

An Inequality for the Triangle

E 1272 [1957, 432]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If A, B, C are the angles of a triangle, show that

$$(\sin A/2 + \sin B/2 + \sin C/2)^2 \leqq \cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2.$$

Solution by Michael Goldberg, Washington, D. C. Since $\sin\theta$ increases with θ for $0<\theta<\pi/2$, but the rate of increase decreases with θ, the maximum value of $\sin A/2+\sin B/2+\sin C/2$ for $A/2+B/2+C/2\leqq\pi/2$ occurs when $A/2=B/2=C/2$. Similarly, since $\cos^2\theta$ decreases with θ for $0<\theta<\pi/2$, but the rate of decrease increases with θ, the minimum value for $\cos^2 A/2+\cos^2 B/2+\cos^2 C/2$ for $A/2+B/2+C/2\leqq\pi/2$ is attained when $A/2=B/2=C/2$. But for $A/2=B/2=C/2=\pi/6$, the right and left members are equal. Hence the sought relation.

Also solved by Leon Bankoff, A. P. Boblétt, D. A. Breault, Calvin Foreman, A. R. Hyde, P. C. Keller, M. A. Laframboise, D. C. B. Marsh, and the proposer.

A Criterion for a Polygon to be Regular

E 1273 [1957, 432]. *Proposed by Alan Wayne, Baldwin, New York*

Prove or disprove the following proposition (which is true for $n=3$ and $n=4$): A plane n-gon with incircle of radius r and circumcircle of radius R is regular if and only if $r=R\cos(\pi/n)$.

Solution by Howard Eves, University of Maine. Let p be the perimeter of the n-gon. Now it is known that of all n-gons inscribed in a circle of radius R, the regular n-gon has the maximum perimeter. Hence $p\leqq 2nR\sin(\pi/n)$, equality holding if and only if the n-gon is regular. It is also known that of all n-gons circumscribed about a circle of radius r, the regular n-gon has the minimum perimeter. Hence $p\geqq 2nr\tan(\pi/n)$, equality holding if and only if the n-gon is regular. It now follows that $r/R\leqq\cos(\pi/n)$, equality holding if and only if the n-gon is regular.

The proposition is somewhat similarly established in Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum* (Springer, 1953) pp. 6–8.

Also solved by Michael Goldberg (paraphrasing Tóth's proof), and D. C. B. Marsh. William Moser also gave the Tóth reference. Late solution by D. A. Breault.

An Inequality from Information Theory

E 1274 [1957, 432]. *Proposed by P. G. Kirmser, Kansas State College*

Given $p_i>0$, $q_i>0$, and $\sum_{i=1}^n p_i=\sum_{i=1}^n q_i$, show that

$$\sum_{i=1}^{n} p_i \ln p_i \geqq \sum_{i=1}^{n} p_i \ln q_i.$$

Solution by Bernhard Marzetta, Basle, Switzerland. For $t>0$ we have

$$t \ln t = \int_1^t (t/s)ds \geqq \int_1^t ds = t - 1.$$

Multiplying both sides of $(p_i/q_i) \ln (p_i/q_i) \geqq p_i/q_i - 1$ by q_i, and summing over i, we get $\sum_{i=1}^n p_i \ln (p_i/q_i) \geqq 0$. Equality can occur only when $p_i=q_i$ for all i.

Also solved by J. L. Botsford, L. R. Bragg, P. L. Duren, Calvin Foreman, Michael Goldberg, D. S. Greenstein, A. R. Hyde, M. S. Klamkin, Morton Kupperman, L. M. Lewandowski, Leon Lifton, D. C. B. Marsh, S. C. Port, D. A. Robinson, H. W. E. Schwerdtfeger, D. L. Smith, Chih-yi Wang, and the proposer. Late solution by R. H. Hou.

Kupperman remarked that this result is well known in information theory, where the p_i's and q_i's are probabilities, so that $\sum_{i=1}^n p_i=\sum_{i=1}^n q_i=1$. He pointed out that a proof may be found in L. Brillouin, *Science and Information Theory* (Academic Press, 1956), pp. 13–14.

Beta Functions

E 1275 [1957, 504]. *Proposed by M. S. Klamkin, AVCO Research and Advanced Development, Lawrence, Massachusetts*

Solve for x:

$$\int_0^x s^{8/3}(1 - s)^{4/3}ds = \int_0^1 t^{8/3}(1 + t)^{-6}dt.$$

Solution by Calvin Foreman, Baker University. Set $t=s/(1-s)$ to obtain

$$\int_0^x s^{8/3}(1 - s)^{4/3}ds = \int_0^{1/2} s^{8/3}(1 - s)^{4/3}ds.$$

Since the integral on the left is a monotonically increasing function of x, the only solution is $x=1/2$.

Also solved by Julian Braun, Tien Chi Chen, Thomas Erber, S. H. Greene, Nathaniel Grossman, Emil Grosswald, J. R. Hendricks, J. R. Holdsworth, A. R. Hyde, E. S. Keeping, A. G. Konheim, D. C. B. Marsh, Bernard Marzetta, D. L. Muench, W. Renga, E. T. Sheffield, Chih-yi Wang, David Zeitlin, and the proposer. Late solution by W. S. Lawton.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4773. *Proposed by Paul Erdös, University of Toronto*

Let $n_1 \leqq n_2 \leqq \cdots$ be a sequence of positive integers. Assume $n_k^{1/2^k} \to \infty$. Show that $\sum_{k=1}^{\infty} 1/n_k$ is irrational. (It is easy to construct $\sum 1/n_k$ rational with $n_k > A^{2^k}$ for every fixed A.)

4774. *Proposed by Donald Greenspan, University of Maryland*

Give an example of a simple closed curve in Euclidean 3-space which satisfies the following two conditions: 1. No three points of the curve lie on a straight line. 2. No four points of the curve lie on a circle.

4775. *Proposed by Leonard Carlitz, Duke University*

Prove the formulas:

$$\sum_{r=0}^{m} \frac{A_r A_{m-r} A_{n-r}}{A_{m+n-r}} \frac{2m+2n-4r+1}{2m+2n-2r+1} = 1, \tag{1}$$

$$\sum_{r=0}^{m} (-2)^r \frac{A_r A_{m-r} A_{n-r} A_{k-r}}{A_{m+n-r}} \frac{4k-4r+1}{4k-2r+1} = A_{m/2} A_{n/2}, \tag{2}$$

where $m \leqq n$, $A_r = 1\cdot 3 \cdots (2r-1)/r!$, $A_0 = 1$, $A_x = 0$ for x not an integer, and, in (2), $m+n=2k$.

4776. *Proposed by D. J. Newman, AVCO Research & Development*

If $|\alpha| < 1/e$, then $\sum_{n=0}^{\infty} (z+\alpha n)^n/n!$ represents an entire function.

4777. *Proposed by R. P. Pakshirajan, Indian Statistical Institute, Calcutta*

Prove that

$$\lim_{n\to\infty} \frac{n}{2^{3n}} \left\{ \binom{n}{0}^3 + \binom{n}{1}^3 + \cdots + \binom{n}{n}^3 \right\} = \frac{2}{\pi\sqrt{3}} \cdot$$

SOLUTIONS

Divergent Integrals

4728 [1957, 201]. *Proposed by R. P. Boas, Jr., Northwestern University*

A. M. Rodov has propounded a proof that if $f(x)$ is continuous and the first of the following integrals converges, then the second diverges

$$\int_1^\infty f(x)dx, \qquad \int_1^\infty x^{-2}\{f(x)\}^{-1}dx.$$

(a) Construct a counter-example. (b) More generally, show that if $g(x)$ and $\phi(x)$ are positive and $\int_1^\infty \phi(x)dx$ diverges, then at least one of

$$\int_1^\infty \phi(x)g(x)dx \quad \text{and} \quad \int_1^\infty \{\phi(x)/g(x)\}dx$$

diverges.

I. *Solution by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.* (a) With $f(x)=(-1)^n/x$, $n=[\sqrt{x}]$, both integrals are easily seen to be convergent.

(b) By the Schwartz inequality,

$$\int_1^\infty \phi(x)g(x)dx \cdot \int_1^\infty \phi(x)/g(x)dx \geqq \left[\int_1^\infty \phi(x)dx\right]^2.$$

When $\int_1^\infty \phi(x)dx$ diverges, then at least one of the two integrals on the left must diverge.

II. *Solution by D. P. Squier and David Zeitlin, Remington Rand Univac, St. Paul, Minn.* (a) The function $f(x)=\sin x/x|\sin x|^{1/2}$ gives a counter-example.

(b) Since $g(x)+1/g(x)\geqq 2$, we have

$$\int_1^m \phi(x)g(x)dx + \int_1^m \frac{\phi(x)}{g(x)}dx = \int_1^m \phi(x)\left[g(x)+\frac{1}{g(x)}\right]dx \geqq 2\int_1^m \phi(x)dx.$$

Letting $m\to\infty$, we conclude that at least one of the given integrals diverges.

Also solved by R. P. Agnew, Robert Breusch, M. F. Friedell, Ben Fusaro, R. R. Goldberg, A. R. Harvey, G. G. Lorentz, Alexander Peyerimhoff, Chih-yi Wang, and the proposer.

Irrational Numbers

4729 [1957, 201]. *Proposed by Paul Erdös, University of Toronto*

Let b_k, $k=1, 2, \cdots$, be any sequence of nonnegative integers such that $\limsup b_k^{1/k}<2$. Assume further that $\sum_{k=1}^n b_k\to\infty$, $\liminf \sum_{k=1}^n b_k/n=0$. Prove that $\sum_{k=1}^\infty b_k/2^k$ is irrational.

Solution by Robert Breusch, Amherst College. Let b_k $(k=1, 2, \cdots)$ be a sequence of nonnegative integers such that

(1) $\limsup b_k^{1/k} < 2,$ (2) $\sum_{k=1}^{n} b_k \to \infty,$

(3) $$\sum_{k=1}^{\infty} b_k/2^k = A/B, \ A \text{ and } B \text{ positive integers.}$$

It will be shown that

(4) $$\liminf \frac{1}{n} \sum_{k=1}^{n} b_k > 0.$$

We may assume $B=1$, for the sequence (Bb_k) will also satisfy (1) and (2), and $\liminf \sum_{k=1}^{n} (Bb_k)/n > 0$ implies (4).

(1) assures the existence of two positive numbers r and k_0 such that

(5) $$r < 1, \quad \text{and} \quad b_k < (2r)^k \quad \text{for} \quad k \geqq k_0.$$

Let s be an integer >1 such that

(6) $$2r^s < 1, \quad \text{and} \quad 2r^s < \left(\frac{1-r}{2r}\right)^{1/k_0}.$$

Let n be a positive integer $\geqq sk_0$, and

(7) $$m = [n/s]; \quad \text{thus} \quad m \geqq k_0.$$

For every positive integer t, let $C_t = \sum_{k=t+1}^{\infty} b_k/2^{k-t}$; then by (3) C_t is an integer, and by (2) $C_t > 0$; thus

(8) $$C_t \geqq 1.$$

Let $t \leqq m$. Then certainly $t < n$, and

(9) $$C_t = \sum_{k=t+1}^{\infty} b_k/2^{k-t} = \sum_{k=t+1}^{n} b_k/2^{k-t} + \sum_{k=n+1}^{\infty} b_k/2^{k-t}.$$

By (5), (6) and (7),

$$\sum_{k=n+1}^{\infty} b_k/2^{k-t} < 2^t \sum_{k=n+1}^{\infty} r^k \leqq 2^m \cdot r^n \cdot \frac{r}{1-r} \leqq (2r^s)^m \cdot \frac{r}{1-r} < \frac{1}{2}.$$

Therefore, by (8) and (9),

$$\sum_{k=t+1}^{n} b_k/2^{k-t} > \tfrac{1}{2}$$

for every positive integer $t \leqq m$.
Adding these inequalities for $t=1, 2, \cdots, m$, we obtain

$$\frac{1}{2} m < \sum_{t=1}^{m} \sum_{k=t+1}^{n} b_k/2^{k-t} < \sum_{k=2}^{n} b_k \sum_{t=1}^{k-1} 1/2^{k-t} < \sum_{k=1}^{n} b_k.$$

Therefore $\sum_{k=1}^{n} b_k/n > m/2n = [n/s]/2n$, and thus

$$\liminf \frac{1}{n} \sum_{k=1}^{n} b_k \geqq \frac{1}{2s} \cdot$$

Also solved by G. G. Lorentz.

Probabilities in an Imperfect Sorting Process

4731 [1957, 201]. *Proposed by D. S. Stoller, Los Angeles, Calif.*

Consider an imperfect sorting process acting on a very large number of items, each of which belongs in one and only one of k bins. Let Q_i represent the probability that if an item belongs in bin i, it is sorted into bin i, and if it does not belong in bin i, it is sorted into some other unspecified bin. Find Q, the probability that an item is sorted into the correct bin.

Solution by A. R. Hyde, West Hartford, Conn. If P_i and R_i denote, respectively, the probabilities that an item belongs in bin i and that it is assigned to bin i, then

$$Q_i = P_iR_i + (1 - P_i)(1 - R_i).$$

The summation over k is $\sum Q_i = 2\sum P_iR_i + k - 2$. Hence

$$Q \equiv \sum P_iR_i = 1 - \tfrac{1}{2}(k - \sum Q_i).$$

Also solved by D. S. Greenstein and the proposer.

Concurrent Lines

4735 [1957, 277]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let $A_1A_2A_3A_4A_5$ be a simple 5-point plane figure, and let d be any line in the plane of the figure. Let the common point of the line d and the side a_i opposite to A_i be denoted by B_i, and the common point of the lines A_iB_{i+1}, B_iA_{i+1} by C_{i+3}. Then the five lines A_iC_i have a point D in common.

Solution by E. J. F. Primrose, The University, Leicester, England. There is a unique polarity P for which each A_i is the pole of the opposite side a_i (Coxeter, *The Real Projective Plane*, 5.65). We consider the 4-point $A_1C_1B_3B_4$. The pole of A_1B_3 for P is A_3, so A_1B_3 and C_1B_4 are conjugate lines, and similarly A_1B_4 and C_1B_3 are conjugate lines. By the dual of Hesse's theorem (Coxeter, 5.55), A_1C_1 and B_3B_4 are conjugate lines, so A_1C_1 passes through D, the pole of d for P. By a similar argument, all the lines A_iC_i pass through D.

Also solved by W. B. Carver, J. W. Clawson, R. Deaux, and the proposer.

Quadratic Residues

4737 [1957, 277]. *Proposed by E. P. Starke, Rutgers University*

For any modulus m, let $V(m)$ be the number of values of $r(0 \leqq r \leqq m-1)$ for

which the congruence $x^2 \equiv r \pmod{m}$ has at least one solution. Determine the form of $V(m)$.

Editorial Note I. In R. C. Buck, *The measure theoretic approach to density* (Amer. J. Math., vol. 68, 1946, Th. 2, pp. 563–564) it is proved that the function $V(m)$ is defined completely by the following properties:

$$\text{(i)} \qquad \text{If } (a, b) = 1, \qquad V(ab) = V(a) \cdot V(b);$$

$$\text{(ii)} \qquad V(p^n) = p^{n+1}/(2p+2) + \begin{cases} (p+2)/(2p+2) & \text{for } n \text{ even,} \\ (2p+1)/(2p+2) & \text{for } n \text{ odd;} \end{cases}$$

$$\text{(iii)} \qquad V(2^n) = 2^{n-1}/3 + \begin{cases} 4/3 & \text{for } n \text{ even,} \\ 5/3 & \text{for } n \text{ odd.} \end{cases}$$

Here p is an odd prime.

Also solved by P. T. Bateman, Robert Breusch, W. B. Carver, and A. S. Hendler.

Editorial Note II. Breusch points out that the conditions of problem 4713 [1956, 678] require that $V(n) \leqq [n^{1/2}]+1$. With the explicit expression for $V(n)$ here provided, it is easy to show that n must be 2, 3, 4, 5, 8, 9, 12, or 16. By inspection, all of these, with the exception of 9, satisfy 4713.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Introduction to the Geometry of Complex Numbers. By Roland Deaux. Translation by Howard Eves. Ungar, 1957. 208 pp. \$6.50.

This book is a revision and translation of one in French, similarly titled, published in Brussels in 1947, and reviewed more fully by S. B. Jackson in this MONTHLY, vol. 55, p. 260. Chapter 1 (40 pages) covers the geometric representation of complex numbers, of arithmetic operations on them, and in particular the geometric meaning of the anharmonic ratio. Chapter 2 (71 pages) discusses the analytic geometry of the straight line, the conic sections, epicycloids and hypocycloids, and those unicursal curves which arise from conics by inversion. Chapter 3 (80 pages) covers the direct and indirect circular transformations (homographies and antigraphies).

The author limits himself closely to the development of the relations between Euclidean plane geometry and complex numbers without allusions to other

possible geometries, to other methods of proof, or to questions of motivation, and in this limited task the author has done a very full and careful job.

The revisions incorporated in this edition concern the last topics in Chapters 2 and 3. The translation is fully satisfactory, as is the physical make-up of the volume. Only a few minor misprints were noted.

The appearance of this book in English does make it more useable in American colleges. It should prove a useful reference book for all teachers of college geometry courses and a welcome addition to the material available for collateral assignments in such courses.

E. H. CUTLER
Lehigh University

Raum und Zahl. By Kurt Reidemeister. Springer, Berlin, 1957. vii+151 pp. DM 19–80.

This book introduces several branches of mathematics, not only for their intrinsic interest but as background for philosophical discussions. The first chapter, on the origin of geometrical thought, contains a set of axioms for the affine plane, a description of some theorems of closure, and a well-chosen quotation from Plato's *Menon*, where Socrates is teaching his slave that the square on the diagonal of a given square has twice the area of the given square. This is followed by a delightful chapter on linkages, instruments for trisecting a given angle and for drawing ellipses and other special curves. The chapter on analytic geometry shows how translations, rotations and dilatations are represented by linear transformations of a complex variable, and how the scope is extended by considering linear fractional transformations. The fourth chapter gives a system of axioms for Euclidean geometry, using distance as a primitive concept. (One of the axioms is the theorem of Pythagoras.) This development leads naturally to the idea of reflection and to questions of orientation. In particular, if two figures (such as congruent but oppositely oriented screws) are related by a reflection but not by a motion, in what sense can they be said to be equivalent? From the standpoint of pure geometry they are alike in all their properties, but the recent researches of Lee and Yang seem to indicate that in physics they may be distinguished. This takes us a further step in the same direction as the remark of Weyl (*Symmetry*, Princeton, 1952, p. 129) to the effect that, whereas Euclidean geometry is invariant under the group of similitudes, its physical counterpart is invariant only under the group of congruent transformations, which "does not include the dilatations."

The fifth chapter describes some classical paradoxes and introduces the theory of transfinite ordinals and combinatorial topology. The sixth (on geometry and logic) begins with Hjelmslev's idea of representing the points and lines of the Euclidean (or non-Euclidean) plane by involutory transformations that leave them invariant, namely the half-turn p about the point and the reflection g in the line, so that the condition for incidence is $pg=gp$. This is followed by remarks about Russell's paradox and about affine geometry over an arbi-

trary field. The seventh chapter clarifies some prevalent obscurities in the foundations of differential and integral calculus. The eighth is a tribute to Gauss. The ninth (on geometry and number theory) develops the theory of algebraic numbers, leading to rigorous proofs of the impossibility, by Euclidean constructions, of duplicating the cube and trisecting an angle of 60°. The tenth and last chapter (Prolegomena to a critical philosophy) is the text of a lecture to the "Kongress des Internationalen Forums" (Zürich, 1954).

There are 31 nicely drawn figures. In view of the wealth of material contained in this remarkable book, an index would have been welcome. There is a small misprint (d for a) on page 49. The nearly consistent use of roman type (instead of italics) in formulas is unusual.

H. S. M. Coxeter
University of Toronto

Introduction to Logic. By Patrick Suppes. Van Nostrand, Princeton, 1957. 18+312 pp. $5.50.

This book emphasizes the application of logic to mathematics and the sciences. The first part develops a system of inference for the first-order predicate logic and presents some principles from the theory of definition. In the second part, successive chapters on sets, relations, and functions leads to a final chapter which indicates how a mathematical or scientific theory can be axiomatized within set theory.

Woven into the text through examples and exercises are the elementary theories of groups and of Boolean algebras, and portions of the theories of arithmetic, of ordering and of measurement. The two major examples of axiomatization studied in the last chapter deal with probability and with particle mechanics. These well-chosen examples and exercises add greatly to the value of the book.

Other valuable features are the chapters on the transitions from ordinary language to logical symbolism and from formal logic to informal mathematics, and the treatment of isomorphism, categorical theories, and representation theorems in the last chapter.

The author has tried to build his theory of inference so that formal derivations will closely parallel the informal proofs of mathematics. As a new device for achieving naturalness of deduction in arguments involving existential quantifiers, he has proposed the use of what he calls "ambiguous names." Experience will show us whether or not this device is actually helpful, but it does seem to have some drawbacks. It complicates the rules of formation of the system, and some of the usual properties of the consequence relation do not hold for formulas containing ambiguous names. It is not clear whether the claim of completeness is intended to apply to the entire system or only to formulas not containing ambiguous names.

The book makes no sharp distinction between semantics and syntax. It does show how the method of interpretation can be applied to prove arguments

invalid, premises consistent, and axioms and primitive terms independent, and it does have a brief chapter on use and mention. But it does not state fully the author's conception of the name relation, and sections that depend closely on this tend to be unclear. Specifically, the definition of "term" (p. 45), the discussion of propositions (p. 123), and the remarks on division by zero (pp. 163–169) are not up to the general level of clarity of the book.

G. N. RANEY
Pennsylvania State University

Employment Opportunities for Women Mathematicians and Statisticians. Women's Bureau Bulletin No. 262, U. S. Department of Labor. Washington, Government Printing Office, 1956. vi+37 pp. 25 cents.

Is "Math" in the Stars for You? Women's Bureau Leaflet 28, U. S. Department of Labor. Washington, Government Printing Office, 1957. 6 pp. 5 cents.

These two documents are intended to encourage girl high school students to continue with the study of mathematics. Many of the statements about employment opportunities and prospective financial rewards for well trained mathematicians apply equally well to men. Both pamphlets are useful for guidance purposes. Booklet 262 contains interesting tables and a bibliography.

H. M. GEHMAN
University of Buffalo

Digital Computer Programming. By D. D. McCracken. Wiley, New York, 1957. vii+253 pp. $6.50 (Profession edition, $7.75).

For all intents and purposes this is the first published textbook devoted exclusively to the subject of digital computer programming. Since programming is still an art rather than a science it is not surprising that the book consists essentially of a collection of techniques which have been developed over the past ten years to enable one to artfully program digital computers. These techniques are listed and described in chapters: 1. Computing Fundamentals; 2. Coding Fundamentals; 3. Binary and Octal Number Systems; 4. Decimal Point Location Methods; 5. Address Computation; 6. Loops in Computing; 7. Flow Charting; 8. Index Registers; 9. Subroutines; 10. Floating Decimal Methods; 11. Input-Output Methods; 12. Magnetic Tape Programming; 13. Program Checkout; 14. Relative Programming Methods; 15. Interpretive Programming Methods; 16. Double Precision Arithmetic; 17. Miscellaneous Programming Techniques; 18. Automatic Coding; and Miscellaneous Appendices.

There is a set of exercises at the end of almost every chapter. However, these sets do not appear to have been chosen to challenge the reader but merely to "exercise" him.

A synthetic digital computer, called TYDAC, is used to provide a vehicle for coding many of the exercises.

The general contents and tone of the book would make it an admissible text for a one-term college course at the junior level or for an industrial course for employees who plan to be associated with digital computers.

This reviewer feels that the treatment of the subject would have been greatly strengthened by commencing with a more thorough treatment of flow charts or algorithms which, after all, is where digital computing begins. Such a treatment would have permitted him to discuss some of the characteristics of problems which make their solution by digital computers possible; and, if possible, the circumstances which make such solution practical. These are matters which vex many students.

The author has a clear and simple style which is ideal for a text. Pertinent examples are painstakingly described in each of the chapters. The typography is superb.

The author and publisher should be commended for getting into print the first comprehensive methods text on the important topic of how to *code* programs for a digital computer.

A. J. PERLIS
Carnegie Institute of Technology

Norte de Problemas (Spanish). By J. Rey Pastor and J. Gallego-Diaz. Editorial Dossat, S. A., Madrid, 1956. xii+359 pp. 210 ptas.

The title indicates that this is a guide book of problems. The authors have collected some 360 problems in algebraic analysis, metrical geometry, and trigonometry, principally from entrance examinations of various Spanish colleges. Detailed and lucid solutions of the problems are given. This is an excellent book for those looking for enrichment material for their better students in elementary college mathematics.

RAYMOND L. CASKEY
Oklahoma State University

BRIEF MENTION

Niels Henrik Abel, Mathematician Extraordinary. By Oystein Ore. University of Minnesota Press, 1957. 277 pp. $5.75.

It is a real privilege to have a biography of an eminent mathematician written by the able pen of Professor Ore. This worthwhile contribution to the very limited literature on the biography of mathematicians will interest scholars everywhere.

Table de factorisation des nombres N^4+1 *dans l'intervalle* $3000<N\leqq 6000$. By Albert Gloden. Chez l'auteur: rue Jean Jaures 11, Luxembourg, 1957. 25 pp. 125 fr.

It is a pleasure to announce this extension of Gloden's 1946 table.

Physics and Metaphysics of Music and Essays on the Philosophy ofMathematics. By Lazare Saminsky. Nijhoff, The Hague, 1957. 151 pp. 10.45 gld.

A series of philosophical essays by a well-known composer.

Archive for Rational Mechanics and Analysis. Edited by C. Truesdell. Springer-Verlag, Berlin. 96 DM per volume of five numbers.

The announcement of a new international journal is always noteworthy. The distinguished editorial board of this one holds high promise for the future.

The Development and Meaning of Eddington's Fundamental Theory. By Noel B. Slater. Cambridge University Press, New York, 1957. xii+299 pp. $7.50.

This correlation of Eddington's published and unpublished manuscripts of "Fundamental Theory" provides a scholarly addition to the history of mathematical physics.

Abbreviated Proceedings, Oxford Mathematical Conference, Trinity College, Oxford, April 8–18, 1957. Technology (The Times Publishing Company Limited) London, 1957. 111 pp. $0.41 postpaid.

This low-cost booklet printed on a nonprofit basis provides an excellent and thought-provoking glimpse of the current thinking of our British colleagues. There is much in it of interest and it certainly merits the small investment.

The Shakespearean Ciphers Examined. By William F. and Elizebeth S. Friedman. Cambridge University Press, New York 1957. xvii+303 pp. $5.00.

Here, at last, is an authoritative and scientific examination of the popular pastime of "proving" that somebody else wrote Shakespeare. It is a scholarly work and a handy reference when the subject comes up.

Spheroidal Wave Functions. By Carson Flammer. Stanford University Press, Stanford, California, 1957. ix+220 pp. $8.50.

This monograph is intended for applied mathematicians, mathematical physicists and mathematical engineers, rather than the pure mathematician. The tables of spheroidal eigenvalues, expansion coefficients and the functions themselves are, of course, incomplete, but nevertheless should provide a valuable addition to the library of any applied mathematician interested in spheroidal waves. Somewhat more extensive tables are reportedly available in the book by Stratton, Morse, Chu, Little, and Corbato, which bears the same title. The reviewer would like to call particular attention to the excellent topography and legible print used in the Flammer work, which is one of the Stanford Research Institute monographs published by the Stanford University Press and, in England, by Oxford University Press.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Lloyd J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

INSTITUTE OF MATHEMATICAL SCIENCES TEMPORARY MEMBERSHIPS

The Institute of Mathematical Sciences at New York University offers temporary memberships to mathematicians and other scientists holding the Ph.D. degree who intend to study and do research in the fields in which the Institute is active. These fields include functional analysis, ordinary and partial differential equations, mathematical physics, fluid dynamics, electromagnetic theory, numerical analysis and digital computing, and various specialized branches, such as linear programming, hydromagnetics, and reactor theory.

The temporary membership program is designed primarily as a means of alleviating the present critical shortage of scientists trained in mathematical physics, applied mathematics, and related fields of mathematical analysis. The program is being supported by the National Science Foundation and also by funds contributed by industrial firms.

Temporary members may participate freely in the research projects, the advanced graduate courses and the research seminars of the Institute, and they will have the opportunity of using the computational facilities which include an IBM 704 computer and a Univac.

The temporary members will receive a stipend commensurate with their status.

Membership will be awarded for one year, but it may be renewed. Special arrangements can be made for applicants who expect to be on leave of absence from their institutions.

The temporary membership program is currently in its second year of operation. During this academic year thirteen temporary members are in residence.

Requests for information and for application blanks should be addressed to the Membership Committee, Institute of Mathematical Sciences, 25 Waverly Place, New York 3, N. Y.

PERSONAL ITEMS

Dr. Warren Weaver, Vice-President for Natural and Medical Sciences of the Rockefeller Foundation, has been awarded the Public Welfare Medal of the National Academy of Sciences for "eminence in the application of science to the public welfare."

Mr. Rohit Parikh has been awarded the Putnam Prize Scholarship by Harvard University for the 1957 Putnam Competition.

Agricultural and Mechanical College of Texas: Mr. R. L. Nolen has been appointed Assistant Professor; Mr. J. L. Fulbrigh, Mr. D. M. Gibson, Jr., Mr. R. A. Knapp, and Mr. C. B. Moehlman, have been appointed Instructors; Mr. W. B. Oldham and Mr. S. M. Wood have been appointed Junior Instructors; Assistant Professor W. S. McCulley has been promoted to Associate Professor; Mr. F. N. Huggins has been promoted to Assistant Professor; Associate Professor T. R. Nelson has retired.

Illinois Institute of Technology: Assistant Professor W. G. Franzen, Aquinas College, and Mr. C. E. Stewart have been appointed Instructors; Dr. Abe Sklar has been promoted to Assistant Professor.

Knox College: Dr. J. R. Mayor, Director, American Association for the Advancement

of Science, was awarded an Alumni Achievement Award at the June commencement in recognition of his work with the Association; Mr. Carl Ohman, National Security Agency, has been appointed Assistant Professor; Mr. Walter Gingery, Lecturer, has retired.

Mississippi State College: Mr. W. Q. Gaddis has been appointed Instructor; Mr. C. W. Gammill has been appointed Acting Instructor; Assistant Professor W. O. Spencer has been promoted to Associate Professor.

Ohio State University: Dr. Theodore Hildebrandt, Scientist, Oak Ridge Institute of Nuclear Studies, has been appointed Part-time Assistant Professor; Dr. Morton Brown, Graduate Student, University of Wisconsin, Dr. Harald Holmann, University of Maryland, and Dr. Angelo Margaris, Oberlin College, have been appointed Instructors; Associate Professor D. R. Whitney has been promoted to Professor; Visiting Associate Professor E. O. Kreyszig has been promoted to Professor; Assistant Professor Erwin Kleinfeld has been promoted to Associate Professor; Dr. D. R. Hughes, Lecturer, has been promoted to Assistant Professor.

San Antonio College: Dr. P. R. Culwell has been appointed Chairman of the Department of Mathematics; Professor W. J. Hallmark, formerly Chairman of the Department, has been appointed Dean of Men.

University of Chicago: Dr. Eldon Dyer, Member, Institute for Advanced Study, and Dr. Sigurdur Helgason, Lecturer, Princeton University, have been appointed Research Lecturers; Dr. S. Z. Sternberg, Temporary Member, Institute of Mathematical Sciences, New York University, has been appointed Instructor; Associate Professor I. E. Segal has been promoted to Professor; Professor A. A. Albert has been appointed to membership in the Steering Group of the General Sciences Panel Advisory to the Assistant Secretary of Defense for Research and Engineering.

University of Denver: Assistant Professor Margaret O. Marchand, Bemidji State Teachers College, Minnesota, has been appointed Visiting Assistant Professor; Assistant Professor O. M. Rasmussen has been promoted to Associate Professor.

University of Illinois: Dr. Rafael V. Chacon, Ohio State University, and Dr. E. A. Wijsman, University of California, Berkeley, have been appointed Assistant Professors; Dr. Robert Kelman, University of California, Berkeley, has been appointed Instructor; Assistant Professors Joseph Landin and E. J. Scott have been promoted to Associate Professors; Dr. Corinne Hattan and Dr. W. M. Zaring have been promoted to Assistant Professors; Dr. Reinhold Baer has retired; Professor C. R. Blyth is on leave of absence and will spend the year doing statistical research at Stanford University; Professor D. G. Bourgin is on leave of absence and is spending the year in travel and research in Europe; Professor J. R. Buchi is spending the academic year at Notre Dame University; Professor J. L. Doob has been elected to the National Academy of Sciences.

University of Minnesota, College of Science, Literature, and the Arts: Dr. Hans Vilhelm Radstrom, Royal Institute of Technology, Sweden, and Mr. Erik Albrecht Sparre Andersen, former President of the Life and Reinsurance Company, DANA, Denmark, have been appointed Visiting Professors; Professor R. W. Brink has retired with the title of Professor Emeritus.

University of Minnesota, Institute of Technology: Dr. Lawrence Markus, Princeton University and Dr. D. A. Pope, University of California at Los Angeles, have been appointed Assistant Professors; Dr. D. G. Aronson, University of Illinois, and Dr. N. G. Meyers, University of Indiana, have been appointed Instructors; Professor S. N. Roy, University of North Carolina, has been appointed Visiting Professor; Assistant Professors Eugenio Calabi and H. Yamabe have been promoted to Associate Professors; Dr. Warren Stenberg has been promoted to Assistant Professor; Professor P. C. Rosenbloom has returned after spending the year as Visiting Professor at Harvard University; Assistant Professor W. D. Munro is on sabbatical leave during 1957–58 at Numerical Analysis Research, University of California at Los Angeles.

University of South Dakota: Mrs. Joyce Harrell has been appointed Instructor; Dr. Paul Haeder has been promoted to Assistant Professor.

University of Toronto: Dr. R. B. Potts, University of Adelaide, South Australia, has been appointed Associate Professor; Dr. J. H. Chung, Computation Centre, University of Toronto, and Dr. A. H. Wallace, University College of North Staffordshire, England, have been appointed Assistant Professors; Dr. Paul Erdos has been appointed Visiting Professor; Mr. R. A. Ross, Dr. F. A. Sherk, and Dr. Vaughan Weston have been appointed Lecturers; Assistant Professor G. F. D. Duff has been promoted to Associate Professor; Dr. W. A. J. Luxemburg and Dr. Ralph Wormleighton, Lecturers, have been promoted to Assistant Professors; Professor H. S. M. Coxeter has received the honorary degree of Doctor of Laws from the University of Alberta.

Washington University: Mr. J. H. Hoelzer, Knolls Atomic Power Laboratory, Schenectady, New York, has been appointed Instructor; Associate Professor Harvey Cohn has been promoted to Professor; Assistant Professor Allen Devinatz has been promoted to Associate Professor; Associate Professor H. M. Schaerf is on sabbatical leave for the fall semester of 1957; Associate Professor H. Margaret Elliott is on leave as a National Science Foundation Science Faculty Fellow at Massachusetts Institute of Technology.

Vassar College: Professor Abba V. Newton has been appointed Chairman of the Department of Mathematics; Dr. Sara Ripy, University of Kentucky, has been appointed Instructor.

Assistant Professor T. J. Bartlett, University of Denver, has a position at Boeing Airplane Company, Seattle, Washington.

Assistant Professor Joanne Elliott, Barnard College, has been promoted to Associate Professor.

Mr. C. T. Fike, Teaching Fellow, University of North Carolina, has been appointed Mathematician at Oak Ridge Institute of Nuclear Studies, Tennessee.

Professor C. H. Fischer, University of Michigan, has been named by Secretary of Health, Education and Welfare, Marion B. Folsom, to a 12-member advisory council which is to review the long-range financial position of the Social Security System.

Mr. R. R. Fossum, Teaching Assistant, University of Oregon, has been appointed Mathematician at Electronic Defense Laboratory, Mountain View, California.

Dr. L. L. Gavurin has been appointed Instructor at Brooklyn College.

Mr. G. R. Gibson, Student, University of Buffalo, has been appointed Assistant, Process Development Laboratory, Dunlop Tire and Rubber Company, Buffalo, New York.

Associate Professor Harold Glander, Carroll College, has been promoted to Chairman, Department of Mathematics.

Mr. M. L. Glasser is now Research Instructor at the University of Miami.

Mr. Sidney Glusman is teaching at Seward Park High School, New York, New York.

Miss Bernice Goldberg, Numerical Analyst, General Electric Company, Evandale, Ohio, has been appointed Research Associate at the AEC Computing Center, New York University.

Assistant Professor E. L. Grindall, Ordnance Research Laboratory, Pennsylvania State University, has been promoted to Associate Professor of Engineering Research.

Assistant Professor Emil Grosswald, University of Pennsylvania, has been promoted to Associate Professor.

Mr. R. K. Gruenewald, Teacher, Eliot School, St. Louis, Missouri, is now a teacher at Woodward School, St. Louis, Lecturer at Washington University, and Lecturer in Education at St. Louis University.

Miss Rose A. Grundman, University of Illinois, has been promoted to Assistant Professor.

Dr. R. C. Gunning has been appointed Assistant Professor at Princeton University

Mr. L. W. Gunter, Western Michigan College, has a position as Laboratory Technician, Kalamazoo Vegetable Parchment Company, Michigan.

Associate Professor J. D. Haggard, Kansas State Teachers College, Pittsburgh, has been promoted to Professor.

Associate Professor A. J. Hall, San Francisco State College, has been promoted to Head of the Department of Mathematics.

Associate Professor J. R. Hanna, University of Wichita, has been promoted to Professor.

Dr. E. H. Hanson, Director, North Texas State College, has a position as Engineering Specialist at Chance Vought Aircraft, Dallas, Texas.

Mr. G. B. Hare has been appointed Instructor at Walla Walla College.

Mr. H. H. Harman, Social Scientist, Rand Corporation, Santa Monica, California, has been appointed Head, Production Department, System Development Corporation, Santa Monica.

Associate Professor V. C. Harris, San Diego State College, has been promoted to Professor.

Assistant Professor Elizabeth M. Haskins, State Teachers College, Fitchburg, Massachusetts, has been promoted to Associate Professor.

Miss Georgia M. Haswell, Dean of Women, Pfeiffer College, is now Professor of Mathematics at the College.

Assistant Professor C. A. Hayes, Jr., University of California at Davis, has been promoted to Associate Professor.

Reverend C. J. Heid, St. Vincent College, has been promoted to Assistant Professor and Chairman, Department of Mathematics.

Assistant Professor A. F. Herbst, La Verne College, has been promoted to Associate Professor.

Mr. F. W. Herlihy, Vice-President, Herlihy Mid-Continent Company, Chicago, Illinois, has been promoted to President.

Mr. A. H. Herrington, Superintendent, Quitman Public Schools, Georgia, has been appointed Applied Science Representative for the I.B.M. Corporation, Atlanta, Georgia.

Dr. P. S. Herwitz, Mathematician, I.B.M. Corporation, Washington, D. C., is now Project Engineer, Project Rancho, I.B.M. Corporation, Poughkeepsie, New York.

Mr. D. M. Hess, Student, Fordham University, is now Sales Engineering Trainee, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Mr. G. A. Heuer, Concordia College, has been promoted to Assistant Professor.

Mr. J. B. Hildebrand, Student, Albion College, is now Teacher, Southfield Public School System, Detroit, Michigan.

Assistant Professor A. P. Hillman, State College of Washington, has been appointed Assistant Professor at the University of Santa Clara.

Mr. S. B. Hobbs, Associate Engineer, Sperry Gyroscope Company, Great Neck, New York, is employed as Senior Engineer, The Martin Company, Denver, Colorado.

Mr. J. R. Hodges, Graduate Student, Peabody College, has been appointed Instructor at Little Rock University.

Mr. K. E. Hofer, Jr., Student, Illinois Institute of Technology, is now an Assistant Engineer at Armour Research Foundation, Chicago, Illinois.

Mr. S. A. Hoffman, Assistant Instructor, University of Pennsylvania, is now Associate Development Engineer, Burroughs Corporation, Research Center, Paoli, Pennsylvania.

Dr. Walter Hoffman, Wayne State University, has been promoted to Assistant Professor.

Dr. V. E. Hoggatt, Jr., San Jose State College, has been promoted to Assistant Professor.

Dr. D. B. Holdridge, Teaching Assistant, California Institute of Technology, is employed as a research engineer at the Jet Propulsion Laboratory, California Institute of Technology.

Mr. F. X. Holzhauer, Research Engineer, Ford Motor Company, Dearborn, Michigan, is Department Manager, Computation Laboratory, Continental Aviation and Engineering Corporation, Detroit, Michigan.

Mr. R. C. Huber, Mathematical Engineer, Renner, Philadelphia, Pennsylvania, is employed as Manager of Projects, Technical Service Corporation, Glen Burnie, Maryland.

Assistant Professor Burrowes Hunt, Reed College, has been promoted to Associate Professor.

Mrs. Verba W. Iturralde, Texas Western College, is now a teacher at El Paso Public Schools, Texas.

Assistant Professor H. G. Jacob, Johns Hopkins University, has been appointed Associate Professor at Louisiana State University.

Mr. B. J. Jansen, St. John's University, Minnesota, is Computer Analyst, Remington Rand Univac, St. Paul, Minnesota.

Assistant Professor C. M. Jensen, Augustana College, has been appointed Associate Professor at Mankato State College, Minnesota.

Miss June R. M. Jensen, Polytechnic Institute of Brooklyn, has been promoted to Assistant Professor.

Professor P. W. M. John, University of New Mexico, is employed as Research Statistician, California Research Corporation, Richmond, California.

Professor C. G. Jaeger, Chairman, Department of Mathematics, Pomona College, has been re-elected Mayor of Claremont, California.

Associate Professor D. A. Johnson, University of Minnesota, has been promoted to Professor of Education.

Dr. H. H. Johnson, Graduate Student, University of California, has been appointed Instructor at Stanford University, Palo Alto, California.

Mr. M. S. Johnson, Mathematician, Melpar, Inc., Falls Church, Virginia, is employed as Research Engineer, Jet Propulsion Laboratory, Pasadena, California.

Associate Professor A. W. Jones, Rensselaer Polytechnic Institute, is now Systems Engineer for Bell Telephone Laboratories, New York, New York.

Associate Professor J. W. Kaiser, Kent State University, has been promoted to Professor.

Dr. Rosella Kanarik, Teacher, Bancroft Junior High School, has been appointed Instructor at Los Angeles City College.

Mr. H. E. Kanter, Student, Carnegie Institute of Technology, has a position as Mathematician at the RAND Corporation, Santa Monica, California.

Associate Professor O. J. Karst, Stevens Institute of Technology, has been appointed Associate Professor of Education at New York University.

Professor L. O. Kattsoff, University of North Carolina, has been appointed Professor at Harpur College.

Associate Professor W. E. Koss, Agricultural and Mechanical College of Texas, has a position at Louisiana Polytechnic Institute.

Professor Emeritus C. N. Moore, University of Cincinnati, has been appointed Visiting Professor at Antioch College.

Associate Professor J. A. Nyswander, University of Michigan, has retired with the title Associate Professor Emeritus.

Professor Walter Prenowitz, Brooklyn College, lectured on geometry at a summer institute for high school teachers of mathematics at the University of North Carolina and is a visiting lecturer for the Association during February, 1958.

Professor S. E. Rauch, University of California, Goleta, is the recipient of fellowships

from the Guggenheim Foundation and the National Science Foundation for advanced study at the Massachusetts Institute of Technology.

Professor Abraham Robinson, University of Toronto, has been appointed Professor at the Hebrew University of Jerusalem, Israel.

Dr. Robert Stanley, University of South Dakota, has been appointed Assistant Professor at State College of Washington.

Assistant Professor R. G. Stanton, University of Toronto, has been appointed Professor and Head, Department of Mathematics, Waterloo College, Ontario.

Associate Professor W. G. Stokes, Austin Peay State College, has been appointed Associate Professor at Northwestern State College, Louisiana.

Mr. C. S. Stuckey, Research Mathematician, National Cash Register Company, Dayton, Ohio, has been appointed Acting Assistant Professor at the University of Cincinnati.

Professor Andre Weil, University of Chicago, is at the Institute for Advanced Study.

Professor Jekuthiel Ginsburg, Director of the Institute of Mathematics and Head of the Mathematics Department of Yeshiva University, died on October 7, 1957. He was founder and Editor of *Scripta Mathematics*. He was a member of the Association for twenty-two years.

Professor Emeritus C. A. Reagan, Friends University, died on September 14, 1957. He was a member of the Association for thirty-six years.

Assistant Professor Emeritus Eugene Stephens, Washington University, died on August 4, 1957.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ITINERARIES OF VISITING LECTURERS, 1957–58

R. D. Schafer

Hunter Coll.	New York, N. Y.	Oct. 14
Douglass Coll. and Rutgers Univ.	New Brunswick, N. J.	Oct. 15–17
Georgetown Univ., Trinity Coll., and Dunbarton Coll.	Washington, D. C.	Oct. 21–25
Skidmore Coll.	Saratoga Springs, N. Y.	Nov. 11
Syracuse Univ.	Syracuse, N. Y.	Nov. 12–13
Hobart and William Smith Colls.	Geneva, N. Y.	Nov. 14
St. John's Coll.	New York, N. Y.	Nov. 15
Univ. of Vermont	Burlington, Vt.	Nov. 18
Univ. of Montreal	Montreal, Quebec, Canada	Nov. 19–20
Univ. of New Hampshire	Durham, N. H.	Nov. 21–22
Dickinson Coll.	Carlisle, Pa.	Dec. 2–3
State Teachers Coll.	Shippensburg, Pa.	Dec. 4
Bucknell Univ.	Lewisburg, Pa.	Dec. 5–6
Temple Univ. and Ogontz Center of Pennsylvania State University	Philadelphia, Pa.	Dec. 9–10
Villanova Univ.	Villanova, Pa.	Dec. 11–12

J. G. Kemeny

Central State Coll.	Edmond, Okla.	Dec. 2–4
Univ. of Oklahoma	Norman, Okla.	Dec. 5–7
Southern Illinois Univ.	Carbondale, Ill.	Dec. 9
Knox Coll., Monmouth Coll., and Western Illinois State Coll.	Galesburg, Monmouth, and Macomb, Ill.	Dec. 10–14
Nebraska Wesleyan Univ.	Lincoln, Neb.	Dec. 16
Kearney State Teachers Coll.	Kearney, Neb.	Dec. 17–18
Fort Hays Kansas State Coll.	Hays, Kan.	Dec. 19
Bethel Coll.	Newton, Kan.	Dec. 20
Coll. of St. Teresa and St. Mary's Coll.	Winona, Minn.	March 3
St. Olaf Coll. and Carleton Coll.	Northfield, Minn.	March 4–6
Gustavus Adolphus Coll.	St. Peter, Minn.	March 7
Univ. of Minnesota	Duluth, Minn.	March 10
Univ. of Alberta	Edmonton, Alberta, Canada	March 12–14
Concordia Coll.	Moorhead, Minn.	March 17–18
North Dakota Agricultural Coll.	Fargo, N. D.	March 19
South Dakota State Coll.	Brookings, S. D.	March 20
Univ. of Kansas and environs	Lawrence, Kan.	March 21–24

Walter Prenowitz

Emory Univ.	Emory University, Ga.	Feb. 3–5
North Georgia Coll.	Dahlonega, Ga.	Feb. 6–7
Florida Agricultural and Mechanical Coll.	Tallahassee, Fla.	Feb. 10–11
Philander Smith Coll.	Little Rock, Ark.	Feb. 17–19
Memphis State Univ. and Southwestern Coll. of Memphis	Memphis, Tenn.	Feb. 20–22

H. S. M. Coxeter

Univ. of New Mexico	Albuquerque, N. M.	March 24–25
Univ. of Arizona	Tuscon, Ariz.	March 27–28
Univ. of Nevada	Reno, Nev.	March 31–April 2
Fresno State Coll.	Fresno, Calif.	April 9–11
Reed Coll.	Portland, Ore.	April 14–15
Univ. of Washington	Seattle, Wash.	April 16–18
Univ. of Alaska	College, Alaska	April 21–23
Whitman Coll.	Walla Walla, Wash.	April 28–30
Washington State Coll.	Pullman, Wash.	April 30–May 2
Northwest Nazarene Coll.	Nampa, Idaho	May 5–6
Colorado State Coll. (Rocky Mountain Section of M.A.A.)	Greeley, Colo.	May 9–10

P. R. Halmos

Beloit Coll.	Beloit, Wis.	April 14–15
North Central Coll.	Naperville, Ill.	April 16
Olivet Nazarene Coll.	Kankakee, Ill.	April 17–18
Western Michigan Univ.	Kalamazoo, Mich.	April 21
Albion Coll.	Albion, Mich.	April 22–23
Univ. of Dayton	Dayton, Ohio	April 24
Univ. of Cincinnati and Xavier Univ.	Cincinnati, Ohio	April 25–28
Ohio Univ.	Athens, Ohio	April 29–30

EMPLOYMENT LISTING FOR RETIRED MATHEMATICIANS

Mathematicians, either retired or about to retire, who have taught at colleges or universities and will be available next year for employment as teachers or as mathematicians in industry, are invited to be listed without charge by the Mathematical Sciences Employment Register Committee. Before February 28 they should send to the committee chairman, Professor J. S. Frame, Mathematics Department, Michigan State University, East Lansing, Michigan, the following information: Name, date of birth, highest degree and where obtained, most recent teaching position or employment, present address, and date available. They should indicate availability for academic or industrial employment or both, and mention geographical or other preferences.

After March 10 interested employers may obtain a copy of this list of available retired mathematicians by writing Professor Frame, enclosing a self-addressed stamped envelope.

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, Ball State Teachers College, Muncie, May 3, 1958.

IOWA, Drake University, Des Moines, April, 18, 1958.

KANSAS, Kansas State Teachers College, Emporia, April 12, 1958.

KENTUCKY, University of Kentucky, Lexington, April 26, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Randolph-Macon Woman's College, Lynchburg, Virginia, April 26, 1958.

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April 19, 1958.

MICHIGAN, University of Michigan, Ann Arbor, March 22, 1958.

MINNESOTA, St. John's University, Collegeville, May 17, 1958.

MISSOURI, University of Missouri, Columbia, April 26, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Denison University, Granville, April 26, 1958.

OKLAHOMA, Central State College, Edmond, April 18–19, 1958.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April 18–19, 1958.

UPPER NEW YORK STATE, École Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

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in
CALCULUS AND ANALYTIC GEOMETRY

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Professor H. M. Gehman
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

This is one part of a campaign to promote and disseminate mathematical exposition. The campaign is an extracurricular project of the MAA's Committee on the Undergraduate Program (C.U.P.). Note that distribution of C.U.P. publications is now handled by the office of the association.

Watch these pages of the Monthly for announcements of forthcoming C.U.P. publications.

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CONTENTS

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EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. James, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

ADVERTISING CORRESPONDENCE should be addressed to F. R. Olson, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

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(Devoted to the Interests of Collegiate Mathematics)

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TEACHING STATISTICAL INFERENCE IN ELEMENTARY MATHEMATICS COURSES*

S. S. WILKS, Princeton University

I am delighted with this opportunity to contribute to the ferment now going on concerning the underclass mathematics curriculum. At first I thought there was probably no surer way to increase the turmoil than to throw in a few thoughts and suggestions about including some statistical inference topics in redesigned freshman and sophomore mathematics courses! However, after carefully examining some of the new material which has been prepared at the freshman level, particularly *Universal Mathematics* by the Duren Committee of the Mathematical Association, *Principles of Mathematics* by Allendoerfer and Oakley, and *Finite Mathematics* by Kemeny, Snell and Thompson, I think the stage is already set for inserting some of the basic ideas of statistical inference. In fact, with the excellent introductions to the theory of sets and probability in these books the job is already under way! Statistical inference is so thoroughly intertwined with probability theory that I expect to be using the phrase "probability and statistics" about as frequently as "statistical inference."

In discussing this topic I shall divide my remarks into five parts. First, a short discussion of the substance and origin of statistical inference. Second, how the ideas are now being passed on in our educational system. Third, when and where the elementary concepts of statistical inference should be taught. Fourth, a suggested set of statistical inference ideas for the main freshman-sophomore sequence of mathematics courses. Fifth, summary and conclusions.

1. What is statistical inference and where did it come from? Statistical inference is a body of concepts and mathematical methods for making inferences about populations of objects from random samples drawn from these populations. These populations occur in all fields of inquiry. We have all sorts of populations of human beings and other living organisms, mass-produced articles, transactions, and repeated experiments, trials and tests. The inferences come in the form of statements with specified margins of uncertainty expressed in terms of probability. The statements refer to numerical characteristics of the populations derived from information contained in samples drawn from the populations. This field has been developed during the last thirty or forty years in the course of dealing with problems of experimentation and investigation in many different fields of science, industry, business and government. The body of specific results and methods which has thus grown up is large and widely scattered in the literature of the natural and social sciences and technology. The underlying ideas and concepts, however, are relatively few in number and they are essentially mathematical. The key results of statistical inference to date, in most cases, have been obtained by persons highly com-

* Invited address at the Thirty-eighth Summer Meeting of the Mathematical Association of America, August 28, 1957.

petent in mathematics working closely with scientists and investigators in these various fields. These results are published largely in the journals of statistics and mathematical statistics, of which there are six or seven important ones in the world. Applications of basic results in statistical inference and adaptations of them to specific problems in a given field are usually made by investigators in that field and are incorporated in their research or technical papers.

2. How has statistical inference been handled by our educational system? At the high school level, education in topics in this area is virtually nonexistent, although the new College Entrance Examination Board Commission on Mathematics is now taking the initiative in getting some of it started. At the college and university level knowledge in this area has been transmitted mainly outside of mathematics departments through *introductory* statistics courses at the upper class and graduate levels in economics, sociology, psychology, biology, and other departments or schools. The method of teaching even the most basic concepts at both levels in any of these departments is largely intuitive and nonmathematical. The ideas are almost completely couched in the language of illustrative examples peculiar to that particular department. The result is that the student does not learn these fundamental concepts in a precise manner and with an appreciation of their full generality.

Let me say, however, that these statistics courses are steadily improving because of the better statistical training being received by the younger men in those departments who are now teaching these courses. But I do not believe that the problem of teaching the basic ideas of statistical inference will ever be satisfactorily solved by introductory statistics courses taught in this intuitive manner at the upper class or graduate level in departments of a college or university scattered all over the campus.

In the first place the persons teaching these courses have, in general, a secondary interest in statistics. Their primary competence and interest is usually in some substantive area within their department. Their scholarship and research interests, as far as statistical inference is concerned, tend to be in the application of ready-made statistical methods to problems in their own field.

In the second place there is a fundamental need to teach the basic ideas of probability and statistical inference early in college—some of them even in high school. Teachers in the various departments and schools who are competent and interested in statistics would then be in a much better position to introduce students majoring in these departments to the important nonmathematical as well as mathematical aspects of the real statistical problems and methods peculiar to those departments. There are plenty of statistical problems and methods in economics, psychology, sociology, and biology to justify the existence of non-introductory statistics courses of special interest to students in those fields. As a matter of fact, the same can be said of departments of physics, chemistry and engineering, whose students have had even less training in probability and sta-

tistics because of their dependence on the traditional freshman and sophomore mathematics courses for the basic mathematical training of these students. But I will say more on this subject later.

I do not think it is an overstatement to say that probability and statistical concepts are involved in the problems and thinking of modern scientific and technological society as much as or more than any other body of mathematical ideas. At one end of the spectrum we find the average citizen confronted with the intelligence scores of his children, insurance problems, advertising and sales claims, public opinion polls, *etc.* He should be introduced to at least the rudiments of probability and statistics at the high school level. At the other end of the spectrum there is the scientist in almost every field who designs his experiments and analyzes and interprets the results by probability and statistical methods. Yet the situation we now have in the teaching of the fundamentals of probability and statistics could hardly be more chaotic. There is virtually none of it at the high school level. At the college level the situation, as I have tried to describe, can perhaps best be summarized by saying that it is similar to what we would have if every department in the university which needed chemistry for its curriculum taught its own introductory chemistry course. Or if every department which needed mathematics taught its own introductory mathematics course. This is justifiable for the later and more specialized courses heavily loaded with the subject matter of that department, but it would be ridiculous to handle the general introductory courses in this manner.

3. When and where should the elementary concepts of statistical inference be taught? In my opinion the basic concepts of statistical inference should be taught in the freshman and sophomore years in college. Some of the more elementary ideas of probability and statistics should eventually be taught in high school mathematics. I have some thoughts on this subject but I will not discuss them here.

By getting these ideas early in college the student in the natural and social sciences and engineering will be in a position to benefit from them in some of his upper class courses, reading or experimental work, in much the same way that students of physics and engineering now utilize their knowledge of traditional freshman and sophomore mathematics in the work of their upper class years.

Since the ideas of probability and statistics occur in so many different fields they should be taught to a wide range of students—not only to students interested in going into the social sciences but into the natural sciences and engineering, as well as those planning to go into business schools, medical schools and even law.

One of the outstanding examples of an unfulfilled need for some training in probability and statistics is in engineering education. The statistical quality control movement which started during the war and which has now pretty well permeated American industry caught the engineering profession completely untrained in even the rudiments of probability and statistics. In most places the

engineering schools rely on mathematics departments for their foundational mathematical training and the traditional courses contained no probability and statistics. The American Society for Quality Control, which was organized to promote this movement, has devoted a tremendous amount of effort, in cooperation with individuals in many colleges and universities of the country, trying to meet the minimum training needs in probability and statistics for practicing engineers. While the educational program in engineering schools is being improved to meet this need it is far from satisfactory. The main trouble is that special probability and statistics courses for engineers are upper class electives in most colleges and universities and are taken by a relatively small fraction of engineering students. Many of the engineering students who do not take such a course find it necessary to learn something about the subject somehow or other immediately after they get into industry. Whenever a course in engineering probability and statistics is given it is found necessary to spend at least a fourth of a semester on basic material on probability and statistical inference before really getting into the topics of particular engineering interest. This basic material should have been taken at the freshman-sophomore level.

If the basic concepts of probability and statistics are to be taught to a wide range of students, then where should they be taught? In my opinion they should be taught in the major freshman-sophomore sequence of mathematics courses. That is the sequence now required of students going into engineering and the physical sciences. Revision of these courses along the lines of the three books I have already mentioned together with the inclusion of some topics in statistical inference would make them better foundational mathematics courses for students who intend to go into the physical sciences and engineering. Not only that, it would also make them good basic courses for students intending to go into the social and biological sciences and it could be expected that eventually departments in these sciences would recommend or require their students to take the revised courses.

Some persons will raise the question at this point as to whether the basic concepts of probability and statistics should not be taught at the freshman level in a statistics department. The creation of statistics departments is a fairly recent development in some of the larger universities. The success with which this movement extends to the smaller colleges and universities will depend inversely on how well the mathematics departments meet the growing needs for the teaching of statistics in those institutions. Almost all the work being done by the relatively few statistics departments now in existence is at the upperclass and graduate levels and, in general, the work is excellent. However, it will be some time before they are in a position to operate effectively at the underclass level. Even when they develop underclass programs they will have to depend on mathematics departments to provide foundational training in such topics as set theory, analytic geometry and calculus or do it themselves, which would be an unjustifiable duplication of effort. This means, in effect, that it may be awkward for a statistics department to begin its teaching program before the

second term freshman or first term sophomore year. But until the statistics departments now in existence, or which will be established, do set up underclass programs this would not vitiate the argument that some of the basic concepts of statistical inference should be taught in the major freshman-sophomore sequence in the mathematics departments of the hundreds of smaller colleges of the country which do not have separate statistics departments. Such a program would provide the best potential now, and in the foreseeable future, for the country as a whole, for disseminating these ideas to the broadest possible range of college students who can benefit from them. Some may argue that this material is unfamiliar to mathematics teachers and hence cannot be effectively taught by them. It may be unfamiliar, but our experience at Princeton is that it can be quickly picked up. I am confident that good mathematics teachers can master this material during the course of a summer institute such as those which have been sponsored by the National Science Foundation during the last two summers. Needless to say, the teacher who can best teach these probability and statistics concepts at the freshman and sophomore level is like the teacher who can best teach the concepts of differential and integral calculus at this level: he not only has a firm grasp of the mathematical concepts and results but he has a genuine feeling and appreciation for good examples and illustrations. There is a great abundance of such examples suitable for illustrative purposes in probability and statistics.

4. What topics in statistical inference should be taught in elementary college mathematics courses? Any treatment of statistical inference concepts at the freshman level must be preceded by at least two or three weeks of elementary probability theory which must in turn be preceded by a similar period of treatment of elementary set theory. Excellent presentations of these two topics are given in the three books mentioned at the beginning of this paper. In such a treatment the student should come out with a good idea of a finite sample space (the set of all possible outcomes of an operation or experiment) and how to set up such spaces in a variety of problems. He should know something about events and their manipulation and representation by Venn diagrams; probability measure, conditional probability, independence, rules for determining probabilities of unions and intersections of events from the probabilities of the constituent events; evaluation of probabilities of events in problems whose sample points are assigned equal probabilities; random numbers and their use as a mechanism for experimentally realizing specified probabilities on a finite sample space.

The coverage of elementary probability should be extended to a thorough discussion of the concept of a random variable or chance variable and its assocated distribution function, with a variety of illustrative examples. Or we can think of this extension as an introduction to statistical inference. Particular attention should be paid at least to some simple cases of the hypergeometric distribution which occurs when drawing from finite populations without replace-

ment; the binomial distribution which occurs in repeated independent trials and how this distribution is a limiting case of the hypergeometric distribution; and the Poisson distribution as a certain limiting case of the binomial distribution. These distributions and their applications should be well illustrated with examples from games of chance, genetics, psychology, bacterial and radioactive counting, traffic, industrial sampling and other fields. Abbreviated tables, particularly of the binomial and Poisson distributions, could be inserted at this point to facilitate the working of exercises. These distributions are all associated with discrete random variables which naturally arise from finite sample spaces. The concept of continuous chance variables and their distributions, at least in cumulative form, should be introduced and illustrated with simple geometric examples. While a bit of calculus would be helpful at this point it is really not yet necessary if the cumulative form of the distribution is used. The normal distribution, together with brief tables, should be introduced at this point as an approximation to the binomial distribution even though the approximation cannot be rigorously developed at this level.

The next ideas we need to introduce are those of the mean and variance of the distribution of a random variable. For the case of discrete random variables these are straightforward and the definitions can be applied to find the mean and variance of random variables having the hypergeometric, binomial and Poisson distributions by a little algebraic manipulation. However, a full understanding of the mean and variance of a continuous random variable and their application to even the simpler distributions requires elementary integral calculus and should be inserted at the appropriate point in the freshman-sophomore sequence.

Having defined the mean and variance of a discrete random variable it is a short step by an elementary argument to establish the Chebyshev inequality which states that for any positive λ the probability is at least $1-1/\lambda^2$ that a chance variable differs from its mean by less than $\lambda\sigma$, where σ is the square root of the variance of the random variable. Application of the Chebyshev inequality to the binomial distribution yields the result that if p is the probability of a success on a single trial and if x is the number of successes in n independent trials then for any positive ϵ the probability is at least

$$1-\frac{p(1-p)}{\epsilon^2 n}$$

that the relative frequency x/n differs from p by less than ϵ. The student is then introduced to a simple case of the law of large numbers, namely, that by taking n sufficiently large the probability that x/n lies within the interval $(p-\epsilon, p+\epsilon)$ can be made arbitrarily near 1.

Furthermore, since p lies on the interval (0, 1), $p(1-p)$ has a maximum of 1/4. Hence, the probability is at least $1-1/(4\epsilon^2 n)$ that x/n differs from p by less than ϵ. This is equivalent to stating that for any $\epsilon>0$ the probability is at least $1-1/(4\epsilon^2 n)$ that p, even if it is unknown, is bracketed by the observable

interval $(x/n-\epsilon, x/n+\epsilon)$. Thus, we have the concept of a confidence interval for an unknown population parameter introduced in a natural setting at a very elementary level. If we wish to approximate the probability that this interval brackets p, calculations must be made from tables of the binomial distribution. For large n the student can be told how to approximate the probability by using the normal distribution with a comforting remark that the proof of this procedure depends on methods of analysis beyond the scope of the course!

Equipped with the concepts already referred to, the student is now in a position to consider the problem of sampling from finite populations. More precisely, suppose every object in a population of N objects has a value of a variable x. Consider all possible samples of n objects which could be drawn from the population. Every sample has a mean $\bar{x}$; and all possible sample means have a distribution. If the drawing of a sample is such that all possible samples are forced to be equally probable it follows by elementary algebraic methods that the mean of this distribution is exactly the same as the mean μ of the x's of all objects in the population and that the variance of the distribution is $\sigma^2(1/n-1/N)$, where σ^2 is the variance of the x's of all objects in the population, all equally weighted. For very large N relative to n the student will see that the variance of the sample means is approximately σ^2/n.

The problem of the mean and variance of the distribution of means of samples of size n for indefinitely large N can also be approached by setting up product sample spaces and probability distributions associated with independent random variables. Thus, in general it is an elementary operation to show that if x_1 and x_2 are two independent discrete random variables having means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, then $a_1x_1+a_2x_2$ is a random variable whose distribution has mean $a_1\mu_1+a_2\mu_2$ and variance $a_1^2\sigma_1^2+a_2^2\sigma_2^2$. The result can be extended to n independent discrete random variables by induction. (The analogous results for the case where x_1 and x_2 are independent continuous random variables depends, of course, on integral calculus.) In particular, if we have independent discrete random variables $(x_1, \cdots, x_n)$ all having means equal to μ and variances equal to σ^2, then $(x_1, \cdots, x_n)$ is called a sample of size n from an infinite population having mean μ and variance σ^2. By elementary methods the student can be shown that the mean value of the sample mean $\bar{x}(=(x_1+\cdots+x_n)/n)$ is μ, the population mean, and the variance of $\bar{x}$ is σ^2/n. He can be told at this stage that if the population is normal, the distribution of $\bar{x}$ is normal with mean μ and variance σ^2/n, with a promise of proof in sophomore calculus. He can also be told that if n is large $\bar{x}$ has an approximately normal distribution, but that the proof, which he could expect to encounter as an upperclassman or graduate student, requires nonelementary methods of analysis.

There are, of course, other applications of the results concerning the mean and variance of a linear function of two or more independent random variables. Some of these can be handled as examples and exercises. An important case is the problem of the mean and variance of the difference of means of samples from two different populations. Another important case is the sample sum

$x_1+x_2+\cdots+x_n$ whose distribution the student will readily find to have mean $n\mu$ and variance $n\sigma^2$. A variety of simple and interesting exercises can be devised concerning sample means and sample sums.

After a discussion of the distribution theory of sample means and sample sums it is a natural and easy step to spend a class hour or two on confidence intervals for population means. As in the case of the binomial distribution the Chebyshev inequality provides an elementary introduction to the subject. Thus, if the population variance σ^2 is known it is found that for any given $\epsilon>0$ one can immediately state that the probability has a value at least equal to

$$1-\frac{\sigma^2}{\epsilon^2}\left(\frac{1}{n}-\frac{1}{N}\right)$$

that the population mean μ is bracketed by the interval $(\bar{x}-\epsilon,\ \bar{x}+\epsilon)$. From this an elementary case of the law of large numbers will be evident. The student can be told how to approximate the probability that μ is bracketed by this interval by using the normal distribution, at the same time assuring him that the method can be rigorously established by methods of analysis beyond the course. I am sure it will be some time before we can honestly pretend to present a rigorous proof of the central limit theorem, even in the case of simple random sampling, to freshmen and sophomores! If σ is unknown he can also be told in a similar vein, without proof, about the possibility of replacing σ^2 by the sample variance s^2 in the procedure for determining approximate confidence limits and obtain results still valid for large n. At this juncture he should be told something about the "Student" t ratio which is so widely used in problems of applied statistics and how confidence limits for μ are obtained from it. The underlying assumptions can be precisely stated but the proof is definitely nonelementary. Finally, before leaving the subject of sample means some attention should be paid to the problems of obtaining confidence limits of the difference of two population means which is so important in simple experiments involving an experimental and a control group which occurs in many fields of science.

The next basic concept in statistical inference which can now be introduced is the idea of testing a statistical hypothesis. An experimental setup involving repeated independent trials, each ending only in success or failure, is an ideal starter for discussing this topic. The distribution which yields all the necessary probabilities is, of course, the binomial distribution. In this case one can easily set up the mathematical structure for interpreting the results of simple experiments for such problems as whether a medical or agricultural treatment has an effect, whether a die is biased or whether a person can discriminate between two brands of beer or cigarettes. Examples of testing whether lots of mass-produced articles have more than an allowable fraction of defectives can be devised for hypothesis-testing involving the Poisson distribution. As far as problems involving sample means are concerned, there is a very short bridge between confidence interval theory and hypothesis-testing. The classical Pearson chi-square test

could be introduced and discussed briefly at this point but without proof.

In discussing statistical tests the notion of the power of the test is basic and should be brought out. The ideas of the power of a statistical test can be historically and most effectively introduced by considering an industrial acceptance sampling plan whereby a lot of size N having Np defectives is accepted if a random sample of size n has c or fewer defectives. The exact probability of acceptance for any N, n and c can be determined from a hypergeometric distribution and illustrated for small N, n and c. For large values of N and n, and small p the student can be shown that this probability can be approximated satisfactorily by the Poisson distribution. The probability of acceptance as a function of p, the fraction of defectives in the lot, called the operating characteristic function of the sampling plan, provides an excellent introduction to the concepts of producer's risk (probability of rejecting lots of satisfactory quality) and consumer's risk (probability of accepting lots of unsatisfactory quality). Extension of the ideas of producer's and consumer's risks to the fundamental type I and type II errors in any problem of hypothesis-testing is now straightforward and can be illustrated by a variety of examples based on problems involving the binomial, Poisson, and normal distributions. The operating characteristic function and the extension of this idea to any statistical test becomes a natural vehicle for introducing the basic idea of statistical decision theory which might be discussed briefly.

Finally, it is my opinion that some of the more important rudimentary statistical inference concepts involved in problems of two or more random variables should be introduced in underclass mathematics courses. The idea of a regression function, that is the mean value of one random variable for a specific value of the other random variables, is basic. Particular attention should be paid to the case of two random variables and linear regression functions. Estimation of regression coefficients and regression functions by the method of least squares, especially for the linear case, is important. This of course involves differential calculus. Some simple problems of hypothesis testing in problems of pairs of random variables should be included. The problem of testing the independence of two random variables from the information in a sample should be mentioned. The rank correlation test is simple enough to be rigorously developed at this level. The two by two contingency table should also be discussed briefly.

Almost all of the statistical inference topics I have indicated together with the necessary probability background have been taught in a special second-term freshman course at Princeton for the last eight years. The course is taken by 150–175 freshmen every spring who have taken one term of freshman mathematics including polynomial calculus during the fall term. The course is taken as a terminal course by students who go into the social and biological sciences and humanities. It has been taught effectively and with keen interest by instructors with special interests in all fields of pure and applied mathematics who were not familiar with the material at the outset. Unfortunately students of the physical sciences and engineering do not get exposed to these ideas at the

underclass level. Very few get exposed to them at any level. However, I believe the material could be very effectively taught if it were spread throughout a modernized freshman-sophomore mathematics sequence, the scheduling of the topics depending on the introduction and development of mathematical concepts and machinery. I believe it would not only improve the sequence for students of physics, chemistry and engineering, but it would make the course more effective for students who intend to go into mathematics and the social and biological sciences.

5. Summary and Conclusions. As we all know, widespread discussion of the underclass college mathematics curriculum has been going on for several years. Out of all of this discussion will come some freshman and sophomore courses which will undoubtedly look quite different from the traditional courses. Some of the directions of change are now quite evident. These directions of change look promising from the point of view of introducing the basic concepts of probability and statistics, now occurring so widely in all branches of science and technology. Good progress, it seems to me, has been made toward introducing fundamental material on the theory of sets and elementary probability. I am especially thinking about the three books I mentioned earlier in this paper. However, I would like to put in a plea for extending the material on elementary probability to include basic concepts of statistical inference more or less along the lines I have indicated. The main reasons for including this material in the main freshman-sophomore mathematics sequence can be summarized as follows:

1. The basic concepts of statistical inference occur as a natural and important pay-off extension of probability theory and are essentially mathematical concepts and can be effectively taught in a mathematics department.

2. The present system of teaching the basic concepts of statistical inference through elementary and largely nonmathematical courses at the upperclass or graduate level in various departments is inadequate. In the first place, the persons teaching these courses have, in general, only a secondary interest in statistics; they are usually primarily interested in some substantive area in their own department. Under this system the students do not get the basic statistical ideas with enough precision and generality; they are too much couched in the language and examples of a single discipline. In the second place, the students do not study the material early enough in their college careers. In many departments, including not only the social and biological sciences but the physical sciences and engineering, students need some of the basic concepts and methods of probability and statistics in their upperclass courses and experimental work.

3. These concepts occur in almost every field of social and natural science, engineering and technology, and should be a part of the general education of any student going into any of these areas. The only department in the colleges and universities of the country now sufficiently institutionalized to teach these basic concepts as broadly as they deserve to be taught is the mathematics de-

partment. While statistics departments have recently been organized in some of the large universities there are only a few of them and they are operating at present mainly at the upperclass and graduate levels. Eventually they will develop underclass programs. However, there are hundreds of colleges and universities in the country which do not have statistics departments, and most of the smaller ones may never have them. In the meantime, the best hope for getting these basic ideas taught effectively and to a sufficiently broad range of students in all of these colleges and universities is through their main freshman-sophomore sequence of mathematics courses. Even though relatively few mathematics teachers are now familiar with the basic concepts of statistical inference, I am confident that they will find them interesting and that they can pick them up quickly by teaching the material or by attending a summer institute such as those which have been sponsored by the National Science Foundation.

4. Teaching the basic ideas of probability and statistical inference in a modernized freshman-sophomore mathematics sequence will enable the various departments of a college or university now teaching statistics courses to concentrate on the nonintroductory probability and statistical problems and methods, including the nonmathematical aspects of statistical problems, more peculiar to the subject matter of those departments. This plan would assume a knowledge of the basic ideas of probability and statistics taught in such a freshman-sophomore mathematics sequence. The departments in a position to benefit would not only be economics, sociology, biology, and psychology but would include physics, chemistry, and engineering.

The probability material now in a modernized freshman-sophomore mathematics sequence plus the extension to statistical inference along the lines I have outlined could be organized, I think, so as to occupy between one-sixth and one-fifth of the time of a two-year sequence. This does not seem excessive for a body of mathematical ideas which, during the last thirty or forty years, have come to play a key role in nearly every field of science, technology and observational inquiry. The problem of effectively integrating these concepts into a new freshman-sophomore mathematics sequence is, of course, no easy task. But it deserves the most earnest consideration of all who are interested in a modern underclass mathematics curriculum.

THE MATHEMATICS OF SENTENCE STRUCTURE*

JOACHIM LAMBEK, McGill University

The definitions [of the parts of speech] *are very far from having attained the degree of exactitude found in Euclidean geometry.*

—Otto Jespersen, 1924.

1. Introduction. The aim of this paper is to obtain an effective rule (or algorithm) for distinguishing sentences from nonsentences, which works not only for the formal languages of interest to the mathematical logician, but also for natural languages such as English, or at least for fragments of such languages. An attempt to formulate such an algorithm is implicit in the work of Ajdukiewicz.† His method, later elaborated by Bar-Hillel [2], depends on a kind of arithmetization of the so-called *parts of speech*, here called *syntactic types*.‡

The present paper begins with a new exposition of the theory of syntactic types. It is addressed to mathematicians with at most an amateur interest in linguistics. The choice of sample languages is therefore restricted to English and mathematical logic. For the same reason, technical terms have been borrowed from the field of high school grammar.

Only a fragmentary treatment of English grammar is presented here. This should not be taken too seriously, but is meant to provide familiar illustrations for our general methods. The reader should not be surprised if he discovers considerable leakage across the line dividing sentences from nonsentences. It is only fair to warn him that some authorities think that such difficulties are inherent in the present methods.§ We take consolation in the words of Sapir: "All grammars leak."

The second part of this paper is concerned with a development of the technique of Ajdukiewicz and Bar-Hillel in a mathematical direction. We introduce a calculus of types, which is related to the well-known calculus of residuals.**

* This paper was written while the author held a Summer Research Associateship from the National Research Council of Canada. The present discussion of English grammar, in its final form, owes much to the careful reading and helpful criticism of earlier versions by Bar-Hillel and Chomsky.

† An English translation of his paper [1] is available in mimeographed form at the University of Chicago.

‡ Historically, these types can be traced back to the *semantic types* attributed by Tarski [21, p. 215] to E. Husserl and S. Lesniewski. A similar technique for logical systems was developed independently by Church [8]. Closely related is also the work by Curry [11] on *functional characters*. These correspond approximately to syntactic types for languages in which functors are always written on the left of their arguments.

§ Chomsky [6; 7] believes that such methods can describe only a small proportion of the sentences of a natural language and that other sentences should be obtained from these by certain transformations.

** See [3, XIII]. The calculus presented here is formally identical with a calculus constructed by G. D. Findlay and the present author for a discussion of canonical mappings in linear and multilinear algebra.

The decision problem for this system is solved affirmatively, following a procedure first proposed by Gentzen for the intuitionistic propositional calculus.††

The methods described here may be applied in several fields. For the teaching of English they provide a rigorous version of the traditional activity known as *diagramming* and *parsing*. For introductory logic courses they offer an effective definition of *well-formed formulas*. For the mechanical translation of languages [**16**], they may help with the syntactic analysis of the input material and indicate how to arrange the output into grammatical sentences of the target language. For the construction of an auxiliary language, they tell how to achieve a completely regular syntax; this is of special importance when the auxiliary is to act as an intermediate language in mechanical translation.

2. Syntactic types. While linguists are primarily interested in speech rather than in written texts, we shall here confine attention to the latter, if only to escape the difficult task of breaking up continuous discourse into discrete words. By a *word* we shall understand a word-form: Such forms as *work*, *works*, *worked* and *working* are different words; but the word *work* occurs twice in *we work best when we like our work*, although it functions as a verb in the first place and as a noun in the second. To describe the function of a word or expression we ascribe to it a certain *syntactic type*. This concept will now be defined; it corresponds approximately to the traditional *part of speech*.

We begin by introducing two *primitive* types: s, the type of sentences, and n, the type of names. For the sake of simplicity, we here restrict *sentence* to denote complete declarative sentences, ruling out requests and questions (as well as most replies, which are usually incomplete). By a *name* we understand primarily a proper name, such as *John* or *Napoleon*. But we shall also assign type n to all expressions which can occur in any context in which all proper names can occur. Thus type n is ascribed to the so-called class-nouns *milk, rice*, $\cdots$, which can occur without article, and to compound expressions such as *poor John, fresh milk*, $\cdots$.‡‡ We do not need to assign type n to the so-called count-nouns *king, chair*, $\cdots$, which require an article, nor to the pronoun *he*, as it cannot replace *John* in *poor John works* or *milk* in *John likes milk*.

From the primitive types we form compound types, by the recursive definition: If x and y are types, then so are x/y (read *x over y*) and $y\backslash x$ (read *y under x*). The meaning of these two kinds of division will be made clear by two examples.

The adjective *poor* modifies the name *John* from the left, producing the noun-phrase *poor John*. We assign to it type n/n.

The predicate (intransitive verb) *works* transforms the name *John* from the right into the sentence *John works*. We assign to it type $n\backslash s$.

In general, an expression of type x/y when followed by an expression of type

†† See [**13**; **10**, II; **15**, XV]. Curry [**11**, appendix] has also observed the close analogy between the theory of functional characters and the propositional calculus.

‡‡ There is a difficulty here: Of course we cannot check *all* admissible name contexts (whose number is infinite) to see whether *poor John* can be fitted in. Our assignment of types is tentative and subject to future revision.

y produces an expression of type x, and so does an expression of type $y\backslash x$ when preceded by an expression of type y. We write symbolically

$$(x/y)y \to x, \qquad y(y\backslash x) \to x. \tag{I}$$

3. Type list for a fragment of English. We shall illustrate the assignment of types to English words by considering a number of sample sentences.

(1) *John works*

$$\begin{array}{cc} n & n\backslash s \end{array}$$

This remains a sentence if *John* is replaced by any other name, hence *works* has type $n\backslash s$.

(2) (*poor John*) *works*

$$\begin{array}{ccc} n/n & n & n\backslash s \end{array}$$

Here *poor John* takes the place of the name in (1); in fact *poor John* can occur in any context in which all names can occur, hence it has type n. Moreover, so has *poor Tom*, *poor Jane*, $\cdots$, thus *poor* has type n/n.

(3) (*John works*) *here*

$$\begin{array}{ccc} n & n\backslash s & s\backslash s \end{array}$$

The word *here* transforms (1), or any other sentence, into a new sentence, hence it has type $s\backslash s$. The question may be raised whether *here* can be attached to a sentence such as (3) itself. While *John works here here* is open to stylistic objections, we shall consider it grammatically well-formed.

(4) *John* (*never* *works*)

$$\begin{array}{ccc} n & (n\backslash s)/(n\backslash s) & n\backslash s \end{array}$$

Since *John* can be replaced by any name here, *never works* has type $n\backslash s$, and so has *never sleeps*, $\cdots$; hence *never* has type $(n\backslash s)/(n\backslash s)$. It may be argued that (3) could also have been grouped *John* (*works here*) suggesting the type $(n\backslash s)\backslash(n\backslash s)$ for *here*. It will be shown later that every adverbial expression of type $s\backslash s$ also has type $(n\backslash s)\backslash(n\backslash s)$.

(5) (*John works*) (*for Jane*)

$$\begin{array}{cccc} n & n\backslash s & (s\backslash s)/n & n \end{array}$$

This indicates that *for Jane* should have the same type as *here* in (3), namely $s\backslash s$, and since *Jane* can be replaced by any other name *for* has type $(s\backslash s)/n$.

(6) (*John works*) (*and* (*Jane rests*))

$$\begin{array}{ccccc} n & n\backslash s & (s\backslash s)/s & n & n\backslash s \end{array}$$

This illustrates how *and* can join two arbitrary sentences to form a new sentence; its type is therefore $(s\backslash s)/s$.

(7) $$\begin{matrix} John & (likes & Jane) \\ n & (n\backslash s)/n & n \end{matrix}$$

Here *likes Jane* has the same type as ***works*** in (1), hence *likes* has type $(n\backslash s)/n$. Similarly we may write *John* (*likes milk*) and even *milk* (*likes John*). The latter is a grammatical sentence, though open to semantic objections.

Example (7) raises an important point. Let us group this sentence

(7′) $$\begin{matrix} (John & likes) & Jane \\ n & n\backslash(s/n) & n \end{matrix}$$

Here *John likes* has type s/n, hence *likes* must be given the new type $n\backslash(s/n)$. We would regard the two types of *likes* in (7) and (7′) as in some sense equivalent. Abstracting from this particular situation, we write symbolically

(II) $$(x\backslash y)/z \rightleftarrows x\backslash(y/z).$$

In practical applications it is often tedious to distinguish between equivalent types, we then write $x\backslash y/z$ for either side of (II). Further examples of this convention are afforded by the types of *never*, *for* and *and* [see Table I]. To avoid multiplication of parentheses, we may also abbreviate $(x/y)/z$ as $x/y/z$, and, symmetrically, $z\backslash(y\backslash x)$ as $z\backslash y\backslash x$. However, parentheses must not be omitted in such compounds as $x/(y/z)$, $(z\backslash y)\backslash x$, $(x/y)\backslash z$ and $z/(y\backslash x)$.

Table I compares the syntactic types of the words discussed above with the traditional parts of speech and the recent classification of Fries [**12**].

TABLE I

	Word	Type	Part of Speech	Fries Class
(1)	*works*	$n\backslash s$	intransitive verb	2C
(2)	*poor*	n/n	adjective	3
(3)	*here*	$s\backslash s$	adverb	4
(4)	*never*	$n\backslash s/(n\backslash s)$	adverb	
(5)	*for*	$s\backslash s/n$	preposition	F
(6)	*and*	$s\backslash s/s$	conjunction	E, J
(7)	*likes*	$n\backslash s/n$	transitive verb	2B

It is fairly clear that in this manner we can build up a type list for a gradually increasing portion of English vocabulary. This should be subject to possible revision, as more information becomes available.

To distinguish between different forms such as ***works*** and ***work***, usually represented by a single dictionary entry, it is necessary to allow for more than two primitive types. Thus we might assign the type n^* to all noun-plurals, such as *men*, *chairs*, $\cdots$. In contrast to examples (1), (2), (5), (7) we then have

(1*) $$\begin{matrix} men & work \\ n^* & n^*\backslash s \end{matrix}$$

(2*) *poor men work*

$$n^*/n^* \quad n^* \quad n^*\backslash s$$

(5*) *John works for men*

$$n \quad n\backslash s \quad s\backslash s/n^* \quad n^*$$

(7*) *John likes girls, men like Jane, men like girls*

$$n \quad n\backslash s/n^* \quad n^* \qquad n^* \quad n^*\backslash s/n \quad n \qquad n^* \quad n^*\backslash s/n^* \quad n^*$$

This assignment successfully distinguishes between the forms *work* and *works*, *like* and *likes*, but it introduces an undesirable multiplicity of types for *poor*, *for*, *like*, and *likes*. While French distinguishes the forms *pauvre* and *pauvres*, English fails to make a corresponding distinction.

A more thorough analysis of the English verb phrase would compel us to introduce further primitive types for the infinitive and the two kinds of participles of intransitive verbs. This would lead to some revision of the type list embodied in Table I. While giving a more adequate treatment of English grammar, such a program would not directly serve the purpose of the present paper.

4. Formal systems. Suppose we have before us a string of words whose types are given. Then we can compute the type of the entire expression, provided its so-called *phrase structure* has been made visible by some device such as brackets. Consider for example

John (likes (fresh milk))

$$\underbrace{n \quad \underbrace{n\backslash s/n \quad \underbrace{n/n \quad n}_{n}}_{n\backslash s}}_{s}$$

The indicated computation can also be written in one line:

$$n((n\backslash s/n)((n/n)n)) \to n((n\backslash s/n)n) \to n(n\backslash s) \to s.$$

In the formal languages studied by logicians, this process offers an effective test whether a given grouped string of symbols is a well-formed formula. For in these languages each word (usually consisting of a single sign) has just one preassigned type, and the use of brackets is obligatory. Let us call expressions with built-in brackets *formulas*; then formulas may be defined recursively: Each word is a formula, and if A and B are formulas, so is (AB).

Brackets are usually omitted when this can be done without introducing ambiguity. Brackets are regularly omitted in accordance with Rule (II). Thus logicians write

$$\begin{array}{ccc} p & \rightarrow & q \\ s & s\backslash s/s & s \end{array}$$

rather than

$$\begin{array}{ccc} p & (\rightarrow & q) \\ s & (s\backslash s)/s & s \end{array}$$

Allowance being made for this convention, the sentence structure of a formalized language is completely determined by its type list. A number of examples will illustrate this.

1. The propositional calculus, according to one of its formulations, possesses an infinite sequence of propositional variables of type s, and two signs for negation and implication of types s/s and $s\backslash s/s$ respectively.

The Polish school of logicians prefer to write all functors on the left of their arguments; it is well-known [18, IV] that all brackets can then be omitted without introducing ambiguity. The implication sign in the Polish notation is therefore of type $s/s/s$.

2. Boolean algebra, rather redundantly formulated, contains an infinite sequence of individual variables, as well as the signs 0 and 1, all of type n, an accent (for complementation) of type $n\backslash n$, cap and cup of type $n\backslash n/n$, equality and inclusion signs of type $n\backslash s/n$.

3. Quine's mathematical logic [17], into which we here introduce a special sign for universal quantification, contains an infinite sequence of individual variables of type n, and signs for joint denial, universal quantification and membership of types $s\backslash s/s$, $s/s/n$ and $n\backslash s/n$ respectively.

4. The calculus of lambda conversion due to Church, with a special sign of type $n/n/n$ for application [18, p. 111], contains also an infinite sequence of individual variables x_i $(i=1, 2, \cdots)$ of type n, together with a parallel sequence λx_i of type $n/n/n$.

5. The syntactic calculus to be introduced in this paper contains a number of symbols for primitive types of type n, three connectives $\cdot$, $\backslash$, $/$ of type $n\backslash n/n$, and the sign $\rightarrow$ of type $n\backslash s/n$.

In the interpretation of formal languages [21, XVIII, Section 4] one usually assumes that expressions of type s denote truth values, expressions of type n denote members of a given domain of individuals, and expressions of type x/y or $y\backslash x$ denote functions from the class of entities denoted by expressions of type y into the class of entities denoted by expressions of type x.

The above discussion of formal systems is somewhat oversimplified. Thus in Quine's formulation of mathematical logic, no special symbol is used for universal quantification, and in Church's formulation of the calculus of lambda con-

version the sign for application is not written. The syntactic description of these languages in terms of types would be more complicated without the special symbols introduced here. In some languages it is important to distinguish between constants and variables of apparently the same type [see, *e.g.*, **1**]. A description in terms of two primitive types is then no longer adequate.

5. Type computations in English. Suppose we wish to compute the type of a string of English words, which are taken from a given type list. We cannot proceed quite as directly as in the formal systems discussed above, for two reasons, which we shall pause to discuss.

First, brackets do not usually occur in English texts, unless we regard punctuation as a half-hearted attempt to indicate grouping. Two ways of inserting brackets into an expression such as *the daughter of the woman whom he loved* may lead to essentially different syntactic resolutions, which may be accompanied by different meanings.

Secondly, English words usually possess more than one type. We have seen some examples of this in Section 3; others are easily found: The adverbial expression *today* has type s/s or $s\backslash s$, depending on whether it precedes or follows the sentence modified. The word *sound* may be a noun, an adjective, or a verb, either transitive or intransitive, depending on the context. Some "chameleon" words possess a type which is systematically ambiguous, allowing them to blend into many different contexts. Thus *only*, of type x/x, can probably modify expressions of any type x, and *and*, of type $x\backslash x/x$, will join together expressions of almost any type x to form a compound of the same type.

A mechanical procedure for analyzing English sentences would consist of four steps:

I. Insert brackets in all admissible ways.

II. To each word assign all types permitted by a given finite type list. (We ignore for the moment the difficulty arising from words which possess a potentially infinite number of types, as do the chameleons *and* and *only*).

III. For each grouping and type assignment compute the type of the total expression.

IV. Select that method of grouping and that type assignment which yields the desired type s.

A simple example, in which the problem of grouping does not arise, is

time	*flies*
n	n^*
$n^*\backslash s/n$	$n\backslash s$
$n^*\backslash s/n^*$	$n\backslash s/n$
	$n\backslash s/n^*$

Only the assignment

$$\begin{array}{cc} \textit{time} & \textit{flies} \\ n & n\backslash s \end{array}$$

produces a declarative sentence. This may be contrasted with

$$\begin{array}{ccccc} (\textit{spiders} & \textit{time} & \textit{flies}) & \textit{without} & \textit{clocks}, \\ n^* & n^*\backslash s/n^* & n^* & s\backslash s/n^* & n^* \end{array}$$

and

$$\begin{array}{cccccc} (\textit{TIME} & \textit{flies} & (10{,}000 & \textit{copies})) & \textit{to} & \textit{Montreal}. \\ n & n\backslash s/n^* & n^*/n^* & n^* & s\backslash s/n & n \end{array}$$

6. Pronouns. So far we have confined attention to the computation Rules (I) and (II). We have had one indication that other rules may play a role: the discussion of Example (4) suggests the rule

$$s\backslash s \rightarrow (n\backslash s)\backslash(n\backslash s).$$

To give a heuristic introduction for the consideration of further rules, we enter into a short discussion of English pronouns.

(8)
$$\begin{array}{cccccc} \textit{he} & \textit{works}, & \quad & \textit{he} & \textit{likes} & \textit{Jane} \\ s/(n\backslash s) & n\backslash s & & s/(n\backslash s) & n\backslash s/n & n \end{array}$$

Since *he* transforms such expressions as *works, likes Jane,* · · · , of type $n\backslash s$ into sentences, we assign to it type $s/(n\backslash s)$. We could of course enlarge the class of names to include pronouns, but then we should be hard put to explain why *poor he works* and *Jane likes he* are not sentences. At any rate, the assignment of type $s/(n\backslash s)$ to *he* is valid, irrespective of whether we regard pronouns as names. In fact, by the same argument, the name *John* also has type $s/(n\backslash s)$. This point will be discussed later.

(9)
$$\begin{array}{cccccc} \textit{that's} & \textit{him}, & \quad & \textit{Jane} & \textit{likes} & \textit{him}, \\ s/n & (s/n)\backslash s & & n & n\backslash s/n & (s/n)\backslash s \end{array}$$

$$\begin{array}{cccc} \textit{Jane} & \textit{works} & \textit{for} & \textit{him} \\ n & n\backslash s & s\backslash s/n & (s/n)\backslash s \end{array}$$

The expressions *that's*, *Jane likes* and *Jane works for* all have type s/n, hence we have ascribed type $(s/n)\backslash s$ to *him*. (This assignment is not quite correct:* The example *Jane likes poor John* indicates that the expression *Jane likes poor* also has type s/n, yet *Jane likes poor him* is not a sentence. Moreover the present assignment does not explain why *that's he* is a sentence in the speech of some people. We shall overlook these defects here.) We observe that the difference

* This was kindly pointed out to the author by N. Chomsky.

in form between *he* and *him* is reflected by a difference in type, indicating that the former operates from the left, while the latter operates from the right. Sapir [19, VII] has called these two forms the *pre-verbal* and *post-verbal* case of the pronoun respectively.

A difficulty arises when we try to show the sentencehood of

$$\begin{array}{ccc} he & likes & him \\ s/(n\backslash s) & n\backslash s/n & (s/n)\backslash s \end{array}; \tag{10}$$

for

$$(s/(n\backslash s))(n\backslash s/n)((s/n)\backslash s)$$

cannot be simplified any further by the Rules (I) and (II). We introduce two new rules

$$(x/y)(y/z) \to x/z, \qquad (x\backslash y)(y\backslash z) \to x\backslash z. \tag{III}$$

We may then assign type

$$(s/(n\backslash s))(n\backslash s/n) \to s/n$$

to *he likes* and type

$$(n\backslash s/n)((s/n)\backslash s) \to n\backslash s$$

to *likes him*, permitting two equivalent resolutions

$$\underbrace{(he\ likes)}_{s/n}\ \ \underset{(s/n)\backslash s}{him,} \qquad \underset{s/(n\backslash s)}{he}\ \ \underbrace{(likes\ him)}_{n\backslash s}.$$

Rules (III) also allow alternative, though equivalent, resolutions of expressions considered earlier; *e.g.*, the sentence

$$\begin{array}{cccc} (John & works) & for & Jane \\ n & n\backslash s & s\backslash s/n & n \end{array}$$

can now also be grouped *John* (*works* (*for Jane*)), where the predicate has type

$$(n\backslash s)((s\backslash s/n)n) \to (n\backslash s)(s\backslash s) \to n\backslash s.$$

We have seen above that the name *John* also has the type of the pronoun *he*. For the same reason, it also has the type of the pronoun *him*. We symbolize the situation by writing

$$n \to s/(n\backslash s), \qquad n \to (s/n)\backslash s$$

and more generally

$$x \to y/(x\backslash y), \qquad x \to (y/x)\backslash x. \tag{IV}$$

These new rules may actually be required for computations. Suppose that

from sample sentences such as *books by him bore* we arrived at the type $n^*\backslash s/n'$ for *by*, where n' is short for $(s/n)\backslash s$. The phrase *books by John* then requires the computation

$$n^*(n^*\backslash s/n')n \rightarrow (s/n')n \rightarrow (s/n')n' \rightarrow s$$

which utilizes rules (I), (IV) and (I) in this order.

While Ajdukiewicz [1] makes use of (III), Rules (IV) suggest that the mathematical apparatus used hitherto may have to be expanded.

7. Syntactic calculus. By an *expression* we shall mean a string of words. Let us suppose that to certain expressions there have been assigned certain primitive types. If A has type x and B has type y, we assign to the expression AB the type xy, also written $x\cdot y$. We assign type z/y to all expressions A such that AB has type z for any B of type y. We assign type $x\backslash z$ to all expressions B such that AB has type z for any A of type x. We write $x\rightarrow y$ to mean that any expression of type x also has type y. We write $x\rightleftarrows y$ to mean that $x\rightarrow y$ and $y\rightarrow x$.

The following rules are now valid:

(a) $\qquad x \rightarrow x$

(b) $\qquad (xy)z \rightarrow x(yz)$ $\qquad$ (b′) $\qquad x(yz) \rightarrow (xy)z$

(c) $\qquad$ *if* $xy \rightarrow z$ *then* $x \rightarrow z/y$ $\qquad$ (c′) $\qquad$ *if* $xy \rightarrow z$ *then* $y \rightarrow x\backslash z$

(d) $\qquad$ *if* $x \rightarrow z/y$ *then* $xy \rightarrow z$ $\qquad$ (d′) $\qquad$ *if* $y \rightarrow x\backslash z$ *then* $xy \rightarrow z$

(e) $\qquad$ *if* $x \rightarrow y$ *and* $y \rightarrow z$ *then* $x \rightarrow z$

Rules (a), (b), (b′), (e) hold trivially. Rules (c′) and (d′) are symmetric duals of (c) and (d), hence it suffices to prove the latter.

Assume $xy\rightarrow z$, and let A have type x. Then for any B of type y, AB has type z; hence A has type z/y. Thus $x\rightarrow z/y$.

Conversely, assume $x\rightarrow z/y$, and let A, B have types x, y, respectively; then AB has type z. Thus $xy\rightarrow z$.

The system presented above may be viewed abstractly as a formal language with a number of primitive type symbols of type n, three connectives $\cdot$, $/$, $\backslash$ of type $n\backslash n/n$, and a relation symbol $\rightarrow$ of type $n\backslash s/n$. If we furthermore regard (a), (b) and (b′) as axiom schemes and (c) to (e) as rules of inference, we obtain a deductive system which may be called *syntactic calculus*. A number of rules are provable in the system; for example,

(f) $\qquad x \rightarrow (xy)/y,$

(g) $\qquad (z/y)y \rightarrow z,$

(h) $\qquad y\rightarrow (z/y)\backslash z,$

(i) $(z/y)(y/x) \to z/x,$

(j) $z/y \to (z/x)/(y/x),$

(k) $(x\backslash y)/z \rightleftarrows x\backslash(y/z),$

(l) $(x/y)/z \rightleftarrows x/(zy),$

(m) *if* $x \to x'$ *and* $y \to y'$ *then* $xy \to x'y'$,

(n) *if* $x \to x'$ *and* $y \to y'$ *then* $x/y' \to x'/y$.

Here (f) follows from $xy \to xy$ by (c), (g) follows from $z/y \to z/y$ by (d), (h) fol lows from (g) by (c′), (j) follows from (i) by (c). Proofs of (i), (k) and (l) are a bit longer; we omit them in view of the decision procedure established in Section 8. Proofs of (m) and (n) are arranged in tree form.

Proof of (m).

$$\dfrac{\dfrac{\dfrac{x \to x' \quad \dfrac{x'y \to x'y}{x' \to (x'y)/y}\,(\mathrm{c})}{x \to (x'y)/y}\,(\mathrm{e})}{xy \to x'y}\,(\mathrm{d}) \qquad \dfrac{\dfrac{y \to y' \quad \dfrac{x'y' \to x'y'}{y' \to x'\backslash(x'y')}\,(\mathrm{c}')}{y \to x'\backslash(x'y')}\,(\mathrm{e})}{x'y \to x'y'}\,(\mathrm{d}')}{xy \to x'y'}\,(\mathrm{e})$$

Proof of (n).

$$\dfrac{\dfrac{\dfrac{\dfrac{y \to y' \quad \dfrac{\dfrac{x/y' \to x/y'}{(x/y')y' \to x}\,(\mathrm{d})}{y' \to (x/y')\backslash x}\,(\mathrm{c}')}{y \to (x/y')\backslash x}\,(\mathrm{e})}{(x/y')y \to x}\,(\mathrm{d}')}{x/y' \to x/y}\,(\mathrm{c}) \qquad \dfrac{\dfrac{\dfrac{x/y \to x/y}{(x/y)y \to x}\,(\mathrm{d}) \quad x \to x'}{(x/y)y \to x'}\,(\mathrm{e})}{x/y \to x'/y}\,(\mathrm{d})}{x/y' \to x'/y}\,(\mathrm{e})$$

The syntactic theorems (g), (h), (i), and (k) coincide with the Rules (I), (IV), (III), and (II), respectively. An illustration of (j), or rather its symmetric dual, appeared in Section 3, where it was pointed out that every sentence-modifying adverb is also a predicate-modifying adverb, symbolically,

$$s\backslash s \to (n\backslash s)\backslash(n\backslash s).$$

Rule (l) is due to Schönfinkel [20], who observed that a function of two variables may be regarded as an ordinary function of one variable whose value is again an ordinary function, so that

$$f(a, b) = (fa)b.$$

If a, b and $f(a, b)$ have types x, y and z respectively, then f occurs in $f(a, b)$ with type $z/(xy)$ and in $(fa)b$ with type $(z/y)/x$, these two types being equivalent by (1).

8. Decision procedure. Is there an effective method for testing whether a sentence $x \rightarrow y$ of the syntactic calculus is deducible from rules (a) to (e)? This is the so-called decision problem for the syntactic calculus. It turns out that the decision procedure discovered by Gentzen [**15**, XV] for the intuitionistic propositional calculus can be adapted for the present purpose.

Following Gentzen, we define the *sequent*

$$x_1, x_2, \cdots x_n \rightarrow y$$

to stand for

$$(\cdots ((x_1 x_2) x_3) \cdots x_n) \rightarrow y,$$

where $x_1, \cdots, x_n, y$ are types. Now let x be any of the possible products of the x_i obtained from some way of grouping the string $x_1 x_2 \cdots x_n$. Then it follows by repeated application of rules (b), (b′), (m) and (e) that

$$x \rightleftarrows (\cdots ((x_1 x_2) x_3) \cdots x_n).$$

Hence the above sequent is also equivalent to the formula $x \rightarrow y$.

Let capitals denote sequences of types, possibly empty sequences. By "U, V" we mean the sequence obtained by juxtaposing U and V; if U is empty it means V, and if V is empty it means U. The following rules are consequences of (a) to (e), provided T, P and Q are not empty.

(1) $x \rightarrow x$

(2) *if* $T, y \rightarrow x$ *then* $T \rightarrow x/y$

(2′) *if* $y, T \rightarrow x$ *then* $T \rightarrow y\backslash x$

(3) *if* $T \rightarrow y$ *and* $U, x, V \rightarrow z$ *then* $U, x/y, T, V \rightarrow z$

(3′) *if* $T \rightarrow y$ *and* $U, x, V \rightarrow z$ *then* $U, T, y\backslash x, V \rightarrow z$

(4) *if* $U, x, y, V \rightarrow z$ *then* $U, xy, V \rightarrow z$

(5) *if* $P \rightarrow x$ *and* $Q \rightarrow y$ *then* $P, Q \rightarrow xy$

Note that each of Rules (2) to (5) introduces an occurrence of one of the connectives $\cdot$, $/$, $\backslash$ into the conclusion.

To derive Rules (1) to (5) from (a) to (e), we observe that (1) is the same as (a), (2) becomes (c), (2′) becomes (c′), (4) is immediate, and (5) becomes (m), if the sequences T, U, V, P, and Q are replaced by the products of the terms in them. It remains only to prove (3), since (3′) is its symmetric dual.

First let us take the case where U and V are empty sequences. We replace T by some product t of its terms. Then (3) takes the form: *if* $t \to y$ *and* $x \to z$ *then* $(x/y)t \to z$. This may be shown thus:

$$\dfrac{\dfrac{x \to z \quad t \to y}{x/y \to z/t}\,(\mathrm{n})}{(x/y)t \to z}\,(\mathrm{d})$$

Next suppose U is empty but V is not. Replace the latter by a product v of its terms. Then (3) takes the form: *if* $t \to y$ *and* $xv \to z$ *then* $((x/y)t)v \to z$. This is established thus:

$$\dfrac{\dfrac{\dfrac{xv \to z}{x \to z/v}\,(\mathrm{c}) \quad t \to y}{(x/y)t \to z/v}\,(\text{as above})}{((x/y)t)v \to z}\,(\mathrm{d})$$

Similarly we deal with the remaining two cases in which U is not empty.

Conversely, we shall deduce rules (a) to (e) from (1) to (5), so that the two sets of rules are equivalent. For the moment we assume one additional rule, the so-called *cut*,

(6) *if* $T \to x$ *and* $U, x, V \to y$ *then* $U, T, V \to y$

It will appear later (Gentzen's theorem) that this new rule does not increase the set of theorems deducible from (1) to (5).

Now (a) coincides with (1), and (e) is a special case of (6), hence it suffices to prove (b), (c) and (d). Proofs are arranged in tree form.

Proof of (b).

$$\dfrac{\dfrac{\dfrac{x \to x \quad \dfrac{y \to y \quad z \to z}{y, z \to yz}\,(5)}{x, y, z \to x(yz)}\,(5)}{xy, z \to x(yz)}\,(4)}{(xy)z \to x(yz)}\,(4)$$

Proof of (c).

$$\dfrac{\dfrac{\dfrac{\dfrac{x \to x \quad y \to y}{x, y \to xy}\,(5)}{x \to (xy)/y}\,(2) \qquad \dfrac{y \to y \quad xy \to z}{(xy)/y, y \to z}\,(3)}{x, y \to z}\,(6)}{x \to z/y}\,(2)$$

Proof of (d).

$$\cfrac{\cfrac{x \rightarrow z/v \quad v \rightarrow v}{x,\, v \rightarrow (z/v)v}\,(5) \qquad \cfrac{\cfrac{\cfrac{v \rightarrow y \quad z \rightarrow z}{z/y,\, v \rightarrow z}\,(3)}{(z/v)y \rightarrow z}\,(4)}{}}{\cfrac{x,\, v \rightarrow z}{xy \rightarrow z}\,(4)}\,(6)$$

Let us verify that we have, in fact, a decision procedure. Given a sequent $U \rightarrow x$, we attempt to construct a proof in tree form, working from the bottom up, using Rules (1) to (5), but not (6). Every upward step eliminates an occurrence of one of the connectives $\cdot$, /, \, and there are only a finite number of ways of making this step. Therefore the total number of proofs that can be attempted is finite. The sequent $U \rightarrow x$ is deducible if and only if one of the attempted proofs is successful.

9. Proof of Gentzen's theorem. If $T \rightarrow x$ and $U,\, x,\, V \rightarrow y$ are both provable according to Rules (1) to (5), we will show that $U,\, T,\, V \rightarrow y$ is also provable, so that we may adopt as a new rule of inference the cut

$$\frac{T \rightarrow x \quad U, x, V \rightarrow v}{U,\, T,\, V \rightarrow y}\,(6).$$

We prove this by reduction on the *degree* of the cut, which is defined thus: Let $d(x)$ be the number of separate occurrences of the connectives $\cdot$, /, \ in the type formula x, and let

$$d(x_1,\, x_2,\, \cdots x_n) = d(x_1) + d(x_2) + \cdots d(x_n),$$

then the degree of the above cut is

$$d(T) + d(U) + d(V) + d(x) + d(y).$$

We will now show that in any cut, whose premises have been proved without cut, the conclusion is either identical with one of the premises, or else the cut can be replaced by one or two such cuts of smaller degree. Since no degree is negative, this will establish Gentzen's theorem. We consider seven cases, which need not be mutually exclusive.

Case 1. $T \rightarrow x$ is an instance of (1); then $T = x$ and the conclusion coincides with the other premise.

Case 2. $U,\, x,\, V \rightarrow y$ is an instance of (1); then U and V are empty and $x = y$. Hence the conclusion coincides with the premise $T \rightarrow x$.

Case 3. The last step in the proof of $T \rightarrow x$ uses one of Rules (2) to (5), but does not introduce the main connective of x. Then $T \rightarrow x$ is inferred by Rule (3), (3′) or (4) from one or two sequents, one of which has the form $T' \rightarrow x$ with

$d(T') < d(T)$. The cut

$$\frac{T' \to x \quad U,\, x,\, V \to y}{U,\, T',\, V \to y}\,(6)$$

has smaller degree than the given cut. Moreover the rule which led from $T' \to x$ to $T \to x$ will also lead from $U,\, T',\, V \to x$ to $U,\, T,\, V \to x$, as may be easily verified in the different subcases.

Case 4. The last step in the proof of $U,\, x,\, V \to y$ uses one of Rules (2) to (5), but does not introduce the main connective of x. Then $U,\, x,\, V \to y$ is inferred from one or two sequents, one of which has the form $U',\, x,\, V' \to y'$. Since the inference introduces an occurrence of one connective,

$$d(U') + d(V') + d(y') < d(U) + d(V) + d(y).$$

Therefore the cut

$$\frac{T \to x \quad U',\, x,\, V' \to y'}{U',\, T,\, V' \to y'}\,(6)$$

has smaller degree than the given cut. Moreover, the same rule which led from $U',\, x,\, V' \to y'$ to $U,\, x,\, V \to y$ will lead from $U',\, T,\, V' \to y'$ to $U,\, T,\, V \to y$, as is easily verified in the different subcases.

Case 5. The last steps in the proofs of both premises introduce the main connective of $x = x'x'' = x' \cdot x''$. We may replace

$$\frac{\dfrac{T' \to x' \quad T'' \to x''}{T',\, T'' \to x'x''}\,(5) \qquad \dfrac{U,\, x',\, x'',\, V \to y}{U,\, x'x'',\, V \to y}\,(4)}{U,\, T',\, T'',\, V \to y}\,(6)$$

by

$$\frac{T'' \to x'' \qquad \dfrac{T' \to x' \quad U,\, x',\, x'',\, V \to y}{U,\, T',\, x'',\, V \to y}\,(6)}{U,\, T',\, T'',\, V \to y}\,(6)$$

where both new cuts have smaller degree.

Case 6. The last steps in the proofs of both premises introduce the main connective of $x = x'/x''$. We may replace

$$\frac{\dfrac{T,\, x'' \to x'}{T \to x'/x''}\,(2) \qquad \dfrac{V' \to x'' \quad U,\, x',\, V'' \to y}{U,\, x'/x'',\, V',\, V'' \to y}\,(3)}{U,\, T,\, V',\, V'' \to y}\,(6)$$

by

$$\frac{V' \to x'' \qquad \dfrac{T, x'' \to x' \quad U, x', V'' \to y}{U, T, x'', V'' \to y}\,(6)}{U, T, V', V'' \to y}\,(6)$$

where both new cuts have smaller degree.

Case 7. This last case is like Case 6, except that $x = x''\backslash x'$, and is treated symmetrically.

10. Algebraic remarks. The following remarks may be of mathematical interest. If we write $=$ instead of $\rightleftarrows$, the deductive system studied here becomes a partially ordered system which resembles a residuated lattice [3, XIII]. It may be mapped homomorphically onto a free group by mapping each element x onto its congruence class modulo $\equiv$, where $x \equiv y$ means that there exists a sequence $x = x_1, \cdots, x_n = y$ $(n \geqq 1)$, such that $x_i \to x_{i+1}$ or $x_{i+1} \to x_i$ $(1 \leqq i < n)$. If this group is abelianized, we obtain something very much like the group of dimension symbols, which plays an important role in the grammar of physics.

It turns out that $x \equiv y$ if and only if

$$x \to t \textit{ and } y \to t \textit{ for some } t, \tag{1}$$

or equivalently

$$z \to x \textit{ and } z \to y \textit{ for some } z. \tag{2}$$

This result is easily proved by induction on the length of the given sequence connecting x with y, once the equivalence of (1) and (2) has been established.

Assuming (1), we put

$$z = (x/((t/t)\backslash t))((t/t)\backslash y)$$

and verify (2) by computation; assuming (2), we put

$$t = (x(z\backslash z))/(y\backslash(z(z\backslash z)))$$

and verify (1) by computation.

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VARIABLE MATRIX SUBSTITUTION IN ALGEBRAIC CRYPTOGRAPHY

JACK LEVINE, North Carolina State College

1. Introduction. The use of algebraic methods in cryptography is well-known through two important papers by Hill [1], [2]. Briefly, the basic idea can be formulated in the following way. Consider the system of simultaneous congruences

$$y_i = \sum_{j=1}^{n} a_{ij}x_j \pmod{26}, \qquad i = 1, \cdots, n, \tag{1.1}$$

where the constants a_{ij} are chosen so that the determinant $|a_{ij}|$ is *prime* to 26. By means of (1.1) the set of n variables $(x_1, \cdots, x_n)$ is transformed to the set $(y_1, \cdots, y_n)$ and, conversely, the set $(y_1, \cdots, y_n)$ will be transformed to the unique set $(x_1, \cdots, x_n)$ by means of the inverse transformation which exists by the assumption on $|a_{ij}|$.

To each of the 26 letters of the alphabet we associate an integer from the set 0, 1, $\cdots$, 25, so that no two letters correspond to the same integer. For simplicity we illustrate with the correspondence (used throughout this paper)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
(1.2)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

Now to encipher a message, or plain text, by means of (1.1), first replace each letter of the text by means of its numerical equivalent, using for illustration, (1.2). Then divide the resulting sequence of numbers into groups containing n numbers each. Call these

$$p_{11}p_{12} \cdots p_{1n} \qquad p_{21}p_{22} \cdots p_{2n} \quad \cdots \quad p_{i1}p_{i2} \cdots p_{in} \quad \cdots . \tag{1.3}$$

Each group of (1.3) is then used in (1.1) for $x_1 \cdots x_n$, and the transformed set

$y_1 \cdots y_n$ calculated. Call the sequence of these sets

$$(1.4) \qquad c_{11}c_{12}\cdots c_{1n} \quad c_{21}c_{22}\cdots c_{2n} \quad \cdots \quad c_{i1}c_{i2}\cdots c_{in} \quad \cdots.$$

Convert the numbers of (1.4) into their letter equivalents by (1.2). These letters will constitute the cipher text corresponding to the given plain text.

The decipherment is accomplished by means of the inverse to (1.1).

As a concrete example, select $n=3$, and (1.1) as

$$(1.5) \qquad \begin{aligned} y_1 &\equiv x_1 + 2x_2 + 3x_3 \\ y_2 &\equiv 2x_1 + 5x_2 + 6x_3 \\ y_3 &\equiv x_1 + 2x_2 + 4x_3 \end{aligned}$$

To encipher the text CRYPTOGRAPHIC, divide into sets of three letters, adding, say, XX to complete the last set:

C	R	Y	P	T	O	G	R	A	P	H	I	C	X	X
3	18	25	16	20	15	7	18	1	16	8	9	3	24	24

The sequence (1.3) is here 3 18 25 16 20 15 $\cdots$. Substitute the first set 3 18 25 $(=p_{11}p_{12}p_{13})$ in (1.5) for $x_1x_2x_3$ to give

$$\begin{aligned} y_1 &\equiv 3 + 36 + 75 = 114 \equiv 10 = \text{J}, \\ y_2 &\equiv 6 + 90 + 150 = 246 \equiv 12 = \text{L}, \\ y_3 &\equiv 3 + 36 + 100 = 139 \equiv 9 = \text{I}, \end{aligned}$$

Here the first cipher sequence $c_{11}c_{12}c_{13}$ of (1.4) is 10 12 9, which converted to letters by (1.2) gives JLI as shown.

The complete encipherment proceeds as above, and produces

JLI WNL TFU GVP SJQ

To decipher, obtain the inverse of (1.5),

$$(1.6) \qquad \begin{aligned} x_1 &\equiv 8y_1 + 24y_2 + 23y_3 \\ x_2 &\equiv 24y_1 + y_2 \\ x_3 &\equiv 25y_1 \qquad\quad + y_3 \end{aligned}$$

(The congruences are of course taken mod 26, in which $25=-1$, $24=-2$, *etc.*) The reciprocal of a prime p, mod 26, is q, where $pq\equiv 1$ mod 26.

Now using JLI $=10\ 12\ 9$ as $y_1y_2y_3$ in (1.6) gives

$$\begin{aligned} x_1 &= 80 + 288 + 207 = 575 \equiv 3 = \text{C} \\ x_2 &\equiv 240 + 12 \qquad\quad = 252 \equiv 18 = \text{R} \\ x_3 &\equiv 250 \qquad\quad + 9 = 259 = 25 = \text{Y} \end{aligned}$$

or CRY, the first plain-text group. The rest of the plain text is found in like manner. (In actual practice we would use -1 for 25, -2 for 24, *etc.*, in (1.6).)

In Hill's papers the transformation (1.1) is generalized by the use of matrix coefficients, but the above is sufficient for our purpose.

(The author notes in passing that simultaneous equations were used by him for cryptographic purposes to a limited extent several years prior to the appearance of Hill's papers.)

2. Fixed substitution. The cryptographic method represented by (1.1) is known as a *fixed substitution system.* This means that any given plain-text group will always be replaced by the same cipher-text group. This is true because the coefficients a_{ij} remain fixed throughout the encipherment of a message.

From a cryptographic point of view there is a distinct advantage in using a *variable substitution* method, whereby the various appearances of a given plain-text group will be replaced by different cipher groups. It is our purpose to indicate some simple ways to accomplish this based on (1.1).

3. Variable substitution, first method. It is convenient to represent (1.1) as a matrix congruence

$$C \equiv AP \pmod{26}, \tag{3.1}$$

where matrices A, C, P are defined by

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \qquad P = \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ p_n \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ c_n \end{bmatrix} \tag{3.2}$$

and P, C are one-column matrices representing corresponding plain- and cipher-text groups.

Now in classical cryptography several variable substitution methods are well-known. These can be represented by the congruences

$$c_i \equiv p_i + k_i \pmod{26}, \qquad i = 1, \cdots, N, \tag{3.3}$$

where p_i is the numerical value of the ith plain-text letter according to some correspondence as (1.2), c_i is the numerical value of the corresponding cipher-text letter, N is number of letters in the message, and the sequence of numbers $k_1k_2 \cdots$ has one of the following properties:

(a) $k_1k_2 \cdots$ is a periodic sequence, say $k_1k_2 \cdots k_mk_1k_2 \cdots k_mk_1 \cdots$, where the numbers $k_1 \cdots k_m$ of the period are selected in any preassigned manner.

(b) The number k_i is chosen by the relation

$$k_i = c_{i-1} \tag{3.4}$$

so

$$c_i \equiv p_i + c_{i-1} \qquad (c_0 \text{ chosen in advance}). \tag{3.5}$$

(c) The number k_i is chosen by the relation

$$k_i = p_{i-1}, \tag{3.6}$$

so

$$c_i \equiv p_i + p_{i-1} \qquad (p_0 \text{ chosen in advance}). \tag{3.7}$$

Note that in each of the above three methods p_i is uniquely determined in the decipherment process. This is, of course, a prime requisite in any cryptographic system.

In matrix form (3.3) can be written as

$$C \equiv P + K \tag{3.8}$$

where the indicated matrices are each of one row and one column, since (3.3) represents encipherment one letter at a time.

Now to obtain a variable substitution analogous to (3.1) we generalize (3.8) to

$$C \equiv AP + BK \pmod{26}, \tag{3.9}$$

where $A = [a_{ij}]$, $B = [b_{ij}]$ are $n \times n$ matrices with fixed elements (and $|A|$ prime to 26). Matrices C and P are as given in (3.2), and K is a one-column matrix,

$$K = \begin{bmatrix} k_1 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{bmatrix}.$$

Corresponding to the three cases (a), (b), (c) above for choosing the k_i, we have:

(a′) Define matrices

$$C_i = \begin{bmatrix} c_{i1} \\ \cdot \\ \cdot \\ \cdot \\ c_{in} \end{bmatrix}, \qquad P_i = \begin{bmatrix} p_{i1} \\ \cdot \\ \cdot \\ \cdot \\ p_{in} \end{bmatrix}, \qquad K_i = \begin{bmatrix} k_{i1} \\ \cdot \\ \cdot \\ \cdot \\ k_{in} \end{bmatrix}, \tag{3.10}$$

using (1.3), (1.4), and

$$K_i = K_{i+m}, \qquad i = 1, 2, \cdots, \tag{3.11}$$

where $K_1, \cdots, K_m$ are chosen in any preassigned manner.

Then

$$C_i \equiv AP_i + BK_i, \qquad i = 1, 2, \cdots, \tag{3.12}$$

from (3.9) gives the substitution. Also,

$$P_i \equiv A^{-1}C_i - A^{-1}BK_i. \tag{3.13}$$

(b′) In this case we choose $K_i = C_{i-1}$, so

$$C_i \equiv AP_i + BC_{i-1} \qquad (C_0 \text{ chosen in advance}).$$

(c′) Choose $K_i = P_{i-1}$, so

$$C_i \equiv AP_i + BP_{i-1} \qquad (P_0 \text{ chosen in advance}). \tag{3.14}$$

To obtain involutory transformations (in which a transformation and its inverse are identical) we have from (3.12), (3.13),

$$A = A^{-1}, \qquad B = -A^{-1}B = -AB, \tag{3.15}$$

and (3.13) becomes $P_i = AC_i + BK_i$. A solution of (3.15) is

$$A^2 = I, \qquad B = A - I \qquad (I = \text{identity matrix}).$$

To obtain A such that $A^2 = I$, a formula in [2] may be used,

$$a_{ij} \equiv \delta_{ij} - \tau\lambda_i\lambda_j, \qquad \sigma\tau \equiv 2 \pmod{26}, \qquad \sigma \equiv \sum_1^n \lambda_i^2 \pmod{26}, \tag{3.16}$$

σ must be prime to 26.

We illustrate case (c′) using

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} p_{i-1,1} \\ p_{i-1,2} \\ p_{i-1,3} \end{bmatrix}. \tag{3.17}$$

To encipher CRYPTOGRAPHIC, we have

$$\begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 18 \\ 25 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 23 \\ 14 \end{bmatrix} = \begin{bmatrix} \text{S} \\ \text{W} \\ \text{N} \end{bmatrix},$$

$$\begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 16 \\ 20 \\ 15 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 18 \\ 25 \end{bmatrix} = \begin{bmatrix} 0 \\ 17 \\ 25 \end{bmatrix} = \begin{bmatrix} \text{Z} \\ \text{Q} \\ \text{Y} \end{bmatrix},$$

$$\begin{bmatrix} c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 18 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 16 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 25 \end{bmatrix} = \begin{bmatrix} \text{O} \\ \text{E} \\ \text{Y} \end{bmatrix}, \textit{ etc.}$$

The complete encipherment is SWN ZQY OEY BMG VQW, using CXX as the last plain group.

The decipherment can be obtained from the inverse to (3.17),

$$P_i = \begin{bmatrix} 8 & 24 & 23 \\ 24 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix} C_i + \begin{bmatrix} 1 & 24 & 24 \\ 6 & 2 & 25 \\ 3 & 25 & 1 \end{bmatrix} P_{i-1}.$$

4. Variable substitution, second method. We return to the basic relation (3.1) and attempt to replace the matrix A of fixed elements by a matrix with variable elements. A general situation is obtained if the elements a_{ij} of A be considered as polynomial functions of a set of parameters $t, u, v, \cdots$ in such a way that the determinant $|A|$ of A is *independent of the parameters* and is a prime number mod 26. The inverse A^{-1} of A will then exist for all parameter values, and hence $P=A^{-1}C$ can always be found. We consider one of the simpler cases here.

Any triangular matrix

$$T = \begin{bmatrix} t_{11} & 0 & \cdots 0 \\ t_{21} & t_{22} & \cdots \cdot \\ \cdot & \cdot & \cdots \cdot \\ t_{n1} & t_{n2} & \cdots t_{nn} \end{bmatrix}$$

with $t_{ij}(t, u, v \cdots), (i\neq j)$, such that t_{ii} is a prime mod 26 will have for determinant $|T| = t_{11}t_{22} \cdots t_{nn}$, a prime mod 26. If T be transformed by elementary transformations leaving $|T|$ unchanged we can obtain a general matrix A of the desired property.

For example, using

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 3 & 0 & 0 \\ 2t+1 & 2 & 5 & 0 \\ 1 & t & t+1 & 7 \end{bmatrix}, \qquad |T| \equiv 1, \bmod 26,$$

we can transform T to

$$A(t) \equiv \begin{bmatrix} 1 & 1 & t & 2t \\ t & t+3 & t & 2t^2 \\ 2t+1 & 2t+3 & 2t^2+t+5 & 2t^2+2t \\ 1 & t+1 & 2t+1 & 2t+7 \end{bmatrix}, \qquad |A(t)| \equiv 1, \bmod 26.$$

Place $C=A(t)P$. For each $P=P_i$ give t a value k_i determined in some preassigned manner,

$$C_i = A(k_i)P_i, \qquad P_i = A^{-1}(k_i)C_i. \tag{4.1}$$

Any of the methods (a′), (b′), (c′) can be used, taking for example,

$$k_i = c_{i-1,1} \text{ or } k_i = p_{i-1,1}, \text{ or } k_i = p_{i-1,1} + p_{i-1,2}, \textit{ etc.},$$

in the latter two cases.

One disadvantage of this procedure to obtain $A(t)$ is that the elements of $A^{-1}(t)$ will in general be high degree polynomials in the parameters, thus causing computational difficulties.

One way to avoid this difficulty is to assume $A(t)$ is linear in t and impose the condition that $A^{-1}(t)$ is likewise. Thus, place

$$A(t) = G + tH, \tag{4.2}$$

with G, H constant element matrices, and $|G|$ a prime mod 26. It is easily shown $A^{-1}(t)$ will be linear in t if $H=XG$, $X^2=0$. Then

$$A^{-1}(t) = G^{-1} - tG^{-1}X. \tag{4.3}$$

To obtain a general matrix X satisfying $X^2=0$, define N by

$$N = \begin{bmatrix} 0 & n_1 & & & & & & & & \\ 0 & 0 & & & & & & & & \\ & & 0 & n_2 & & & & & & \\ & & 0 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & n_q & & & \\ & & & & & 0 & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 0 \end{bmatrix}, \qquad q \leqq [n/2]. \tag{4.4}$$

N consisting of all zeros except $n_1, n_2, \cdots, n_q$ placed immediately to the right of the main diagonal terms in alternate rows as shown. The n_i are arbitrary constants. It is evident that $N^2=0$. X is now defined by

$$X = QNQ^{-1}, \tag{4.5}$$

Q being an arbitrary constant-term matrix with an inverse. From (4.5), $X^2=0$.

From (4.2) we then define $A(t)$ by

$$A(t) = G + tXG = G + tQNQ^{-1}G \tag{4.6}$$

$A^{-1}(t)$ being given by (4.3).

Example. Take

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 17 & 6 & 12 \\ 6 & 13 & 24 \\ 12 & 24 & 23 \end{bmatrix},$$

$$X = \begin{bmatrix} 24 & 13 & 18 \\ 10 & 0 & 14 \\ 20 & 0 & 2 \end{bmatrix}, \qquad G = \begin{bmatrix} 25 & 2 & 16 \\ 2 & 25 & 10 \\ 16 & 10 & 3 \end{bmatrix} (=G^{-1}),$$

$$XG = \begin{bmatrix} 4 & 7 & 22 \\ 6 & 4 & 20 \\ 12 & 8 & 14 \end{bmatrix}, \qquad G^{-1}X = \begin{bmatrix} 4 & 13 & 16 \\ 4 & 0 & 16 \\ 24 & 0 & 18 \end{bmatrix},$$

$$A(t) = \begin{bmatrix} 25 & 2 & 16 \\ 2 & 25 & 10 \\ 16 & 10 & 3 \end{bmatrix} + t \begin{bmatrix} 4 & 7 & 22 \\ 6 & 4 & 20 \\ 12 & 8 & 14 \end{bmatrix},$$

$$A^{-1}(t) = \begin{bmatrix} 25 & 2 & 16 \\ 2 & 25 & 10 \\ 16 & 10 & 3 \end{bmatrix} - t \begin{bmatrix} 4 & 13 & 16 \\ 4 & 0 & 16 \\ 24 & 0 & 18 \end{bmatrix}.$$

Using these in (4.1) gives

$$C_i = \begin{bmatrix} 25 + 4k_i & 2 + 7k_i & 16 + 22k_i \\ 2 + 6k_i & 25 + 4k_i & 10 + 20k_i \\ 16 + 12k_i & 10 + 8k_i & 3 + 14k_i \end{bmatrix} P_i, \tag{4.7}$$

$$P_i = \begin{bmatrix} 25 + 22k_i & 2 + 13k_i & 16 + 10k_i \\ 2 + 22k_i & 25 & 10 + 10k_i \\ 16 + 2k_i & 10 & 3 + 8k_i \end{bmatrix} C_i. \tag{4.8}$$

Take $k_i = p_{i-1,1} + p_{i-1,2} + p_{i-1,3}$, $(k_1 = 1)$, and encipher CRYPTOGRAPHIC(XX),

$$C_1 = \begin{bmatrix} 3 & 9 & 12 \\ 8 & 3 & 4 \\ 2 & 18 & 17 \end{bmatrix} \begin{bmatrix} 3 \\ 18 \\ 25 \end{bmatrix} = \begin{bmatrix} 3 \\ 22 \\ 1 \end{bmatrix} = \begin{bmatrix} \text{C} \\ \text{V} \\ \text{A} \end{bmatrix}, \qquad k_2 = 3 + 18 + 25 \equiv 20,$$

$$C_2 = \begin{bmatrix} 1 & 12 & 14 \\ 18 & 1 & 20 \\ 22 & 14 & 23 \end{bmatrix} \begin{bmatrix} 16 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 24 \\ 10 \\ 15 \end{bmatrix} = \begin{bmatrix} \text{X} \\ \text{J} \\ \text{O} \end{bmatrix}, \qquad k_3 = 16 + 20 + 15 \equiv 25,$$

$$C_3 = \begin{bmatrix} 21 & 21 & 20 \\ 22 & 21 & 16 \\ 4 & 2 & 15 \end{bmatrix} \begin{bmatrix} 7 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 25 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \text{Y} \\ \text{B} \\ \text{A} \end{bmatrix}, \qquad k_4 = 7 + 18 + 1 \equiv 0,$$

$$C_4 = \begin{bmatrix} 25 & 2 & 16 \\ 2 & 25 & 10 \\ 16 & 10 & 3 \end{bmatrix} \begin{bmatrix} 16 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \\ 25 \end{bmatrix} = \begin{bmatrix} \text{N} \\ \text{J} \\ \text{Y} \end{bmatrix}, \qquad k_5 = 16 + 8 + 9 \equiv 7,$$

$$C_5 = \begin{bmatrix} 1 & 25 & 14 \\ 18 & 1 & 20 \\ 22 & 14 & 23 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 24 \\ 24 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 18 \end{bmatrix} \quad \begin{bmatrix} \text{C} \\ \text{L} \\ \text{R} \end{bmatrix}.$$

The cipher-text is thus CVA XJO YBA NJY CLR.

To decipher, use (4.8),

$$P_1 = \begin{bmatrix} 21 & 15 & 0 \\ 24 & 25 & 20 \\ 18 & 10 & 11 \end{bmatrix} \begin{bmatrix} 3 \\ 22 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 18 \\ 25 \end{bmatrix} = \begin{bmatrix} \text{C} \\ \text{R} \\ \text{Y} \end{bmatrix}, \textit{ etc.}$$

There is an advantage in using an involutory transformation (4.1), *i.e.*, one such that $A(t) = A^{-1}(t)$. To obtain this we require from (4.3), (4.6),

$$(4.9) \qquad G = G^{-1}, \qquad XG = -GX.$$

Define matrix J by

$$(4.10) \qquad \begin{bmatrix} 1 & j_1 & & & & & & & & \\ 0 & -1 & & & & & & & & \\ & & 1 & j_2 & & & & & & \\ & & 0 & -1 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 1 & j_q & & & \\ & & & & & 0 & -1 & & & \\ & & & & & & & a_1 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & a_r \end{bmatrix} = J,$$

constants $j_1, \cdots, j_q$ arbitrary, and $a_1, \cdots, a_r$ all $= \pm 1$, $(r+2q=n)$.

Then by (4.4), (4.10),

$$(4.11) \qquad J^2 = I, \qquad JN = -NJ.$$

Place

$$(4.12) \qquad G = QJQ^{-1}, \qquad X = QNQ^{-1},$$

giving

$$G^2 = I, \qquad XG = -GX, \qquad (X^2 = 0),$$

satisfying (4.9).

From (4.12), $XG = QNJQ^{-1} = -QNQ^{-1} = -X$, since direct calculation shows $NJ = -N$. Hence (4.6) gives

$$A(t) = G - tX = A^{-1}(t), \tag{4.13}$$

which can also be expressed as

$$A(t) = Q(J - tN)Q^{-1} = A^{-1}(t). \tag{4.14}$$

Example. Use N, Q, X of the previous example, and

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

By (4.12), (4.13)

$$G = \begin{bmatrix} 3 & 9 & 10 \\ 6 & 19 & 2 \\ 12 & 14 & 3 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 3 + 2t & 9 + 13t & 10 + 8t \\ 6 - 10t & 19 & 2 + 12t \\ 12 + 6t & 14 & 3 + 24t \end{bmatrix} = A^{-1}(t).$$

In case $n = 2$ it can be verified that

$$A(t) = A^{-1}(t) = \begin{bmatrix} a + bt & c + dt \\ e + ft & -(a + bt) \end{bmatrix}$$

if $a = bcd' \pm 1$, $e = -b^2c(d')^2 \mp 2bd'$, $f = -b^2d'$, and b, c, d are arbitrary (d prime mod 26), and $dd' \equiv 1$ mod 26.

The modulus 26 used throughout this paper is not essential. Other moduli can be used with suitable modifications where necessary.

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THE MOTION OF A FLEXIBLE INELASTIC TUBE CONSTRAINED TO MOVE ON A ROUGH CONVEX CURVE

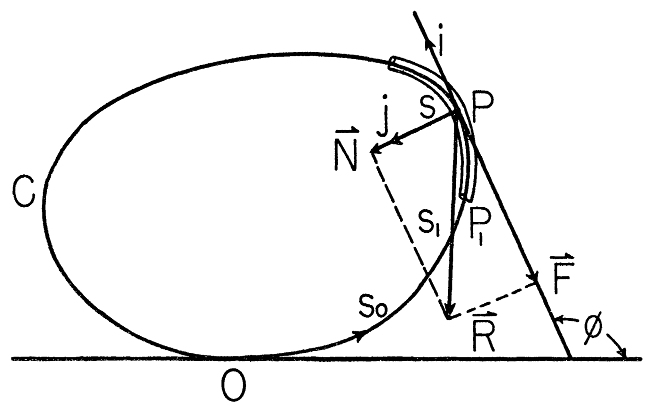
F. A. VALENTINE, University of California, Los Angeles

One might well expect that the behavior of a flexible inelastic tube constrained to move on a rough wire would be more complex than that of a bead or particle subjected to the same constraint. Indeed, such is the case. However, it is an agreeable fact that the motion of the tube has some of the same invariant properties possessed by the particle. The case of the particle has already been discussed [4 (pp. 16–20), 2, 3, 5, 6]. Since this topic is a very old one, it is to be expected that some of the results presented here as new to the author may be known to others. For the sake of the reader, a brief and concise derivation of the

known fundamental differential equations (5) is given.

DEFINITION. *The boundary of a bounded plane convex body is called a convex curve (a convex body is a convex set having interior points). A convex curve at each of whose points the curvature exists is called a differentiable convex curve C.*

We restrict ourselves to such curves in this paper. It will be helpful to refer to the following diagram.



1. Formulation of the problem. We assume that C corresponds to a fixed rough wire of length L whose coefficient of friction μ is constant. A flexible inelastic tube* A is constrained to move on C. The tube has constant linear density σ. Furthermore, we assume that *the only force acting on the tube A is the fixed constraint C*; no other external forces including gravity act on A. Let s be the arc length of that portion of C from the fixed point O to an arbitrary point P (see diagram), measuring distances positively on C counterclockwise. Moreover, let s_1 denote the value of s corresponding to the position of the terminal endpoint P_1 of the tube. The angle ϕ is the conventional angle which the tangent to C makes with the positive x-axis through O (see diagram).

Let $\vec{R}$ denote the reaction of the constraint per unit mass of the tube A. This is resolved into normal and tangential components $\vec{N}$ and $\vec{F}$ respectively. The law of friction is $|\vec{F}| = \mu|\vec{N}|$. The scalar tension in the tube is denoted by T. We define $\vec{T} \equiv T\vec{i}$, where $\vec{i}$ is the unit tangent vector to C at P (see diagram). The tension and the reaction are functions of s and s_1. We assume that $\vec{R}$ is a continuous function of s for $s_1 \leqq s \leqq s_1 + a$, $s_1 \geqq s_0$, where a is the length of the tube A, and where $a < L$.

We adopt the following initial conditions

* It will assist the reader to regard the tube as a hollow slender flexible string which is strung on the wire C. We idealize the situation so that the tube is one-dimensional, inelastic and flexible.

$$(1)\qquad t = 0, \qquad s_1 = s_0, \qquad \phi(s_0) = \phi_0, \qquad v \equiv ds_1/dt = v_0 > 0.$$

Let P be a fixed point in the tube A, and let b denote the length of the subarc of A which has the points P and P_1 as its endpoints. Designate the speed of P by $v(s_1, b)$, and denote the corresponding vector velocity of P by $\overrightarrow{V}(s_1, b)$. The speed of the terminal endpoint P_1 of the tube A is denoted by $v(s_1)$, so that $v(s_1) = v(s_1, 0)$.

2. Conclusions. Since the tube is inextensible, we have $v(s_1, b) = v(s_1) = ds_1/dt$ for $0 \leqq b \leqq a$, $s_1 \geqq s_0$.

THEOREM 1. *The speed $v(s_1)$ of the tube A subject to the initial conditions* (1) *has the form*

$$(2)\qquad v(s_1) = v_0\left(\int_{s_0}^{s_0+a} e^{\mu\phi(s)}ds\right)\bigg/\left(\int_{s_1}^{s_1+a} e^{\mu\phi(s)}ds\right),$$

where a is the length of the tube A, and where s_1 is the arc length of that portion of C from the fixed point O to the terminal endpoint P_1 of A.

Proof. Select any *fixed* point P in the tube A at the time t, and let ΔA be a segment of A of length $|\Delta s|$ and having P as one of its endpoints. The fundamental axioms of Newtonian mechanics when applied to the tube include the following statement.

Axiom. There exists a vector function $\vec{\epsilon}(s, \Delta s)$, $s_1 \leqq s \leqq s_1 + a$ such that $\vec{\epsilon} \to 0$ as $\Delta s \to 0$, and such that

$$(3)\qquad \sigma\Delta s\,\frac{d\overrightarrow{V}(s_1, b)}{ds_1}\,\frac{ds_1}{dt} = \int_s^{s+\Delta s} \sigma\overrightarrow{R}ds + \Delta\overrightarrow{T} + \vec{\epsilon}\Delta s, \qquad s = s_1 + b,$$

where $\Delta\overrightarrow{T}$ is the difference of the tensions acting at the endpoints of ΔA (see section 1 for notations). Upon dividing the first of equations (3) by Δs, we observe that $\partial\overrightarrow{T}/\partial s$ exists since $\overrightarrow{R}$ is a continuous function of s for $s_1 \leqq s \leqq s_1 + a$. Hence,

$$(4)\qquad \sigma\,\frac{d\overrightarrow{V}(s_1, b)}{dt} = \sigma\overrightarrow{R}(s_1 + b, s_1) + \left.\frac{\partial\overrightarrow{T}(s, s_1)}{\partial s}\right|_{s=s_1+b}$$

holds for $s_1 \leqq s \leqq s_1 + a$, $s_1 \geqq s_0$, $0 \leqq b \leqq a$. Since $\overrightarrow{T} \equiv T\vec{i}$, we must have

$$\frac{\partial\overrightarrow{T}}{\partial s} = \frac{\sigma T}{\partial s}\vec{i} + \frac{T}{\rho}\vec{j}, \qquad 1/\rho = d\phi/ds,$$

where $\vec{i}$ and $\vec{j}$ are unit vectors along the tangent and normal respectively to C

at P. Let $\vec{F}=-F\vec{i}$, $\vec{N}=N\vec{j}$, so that $|\vec{F}|=F$, $|\vec{N}|=N$. Since $\vec{V}(s_1, b)=v(s_1)\vec{i}(s_1+b)$, upon resolving equation (4) into tangential and normal components, we obtain

$$\tag{5} \begin{aligned} \sigma \frac{dv(s_1)}{ds_1} v(s_1) &= -\sigma F(s, s_1) + \frac{\partial T(s, s_1)}{\partial s}, \\ \frac{\sigma v^2(s_1)}{\rho(s)} &= \sigma N(s, s_1) + \frac{T(s, s_1)}{\rho(s)}, \end{aligned}$$

where $1/\rho = d\phi/ds$. Since $F=\mu N$, equations (5) imply that

$$\tag{6} \frac{\partial T(s, s_1)}{\partial s} + \mu \frac{T(s, s_1)}{\rho(s)} = \sigma\left[v(s_1)\frac{dv(s_1)}{ds_1} + \mu\frac{v^2(s_1)}{\rho(s)}\right].$$

Moreover, since the ends of the tube A are free, condition (3) implies that

$$\tag{7} T(s_1, s_1) = 0, \qquad T(s_1 + a, s_1) = 0.$$

Equation (6) and the first of (7) imply that

$$\tag{8} T(s, s_1) = e^{-\mu(\phi-\phi_1)} \int_{s_1}^{s} \left(\sigma v \frac{dv}{ds_1} + \mu \frac{\sigma v^2}{\rho}\right) e^{\mu(\phi-\phi_1)} ds,$$

where $\phi=\phi(s)$, $\phi_1=\phi(s_1)$, $v=v(s_1)$ and $1/\rho=d\phi/ds$. The integral in (8) exists since $1/\rho$ is integrable [1]. The condition $T(s_1+a, s_1)=0$ can then be satisfied if and only if the integral in (8) is zero for $s=s_1+a$. Since $v=v(s_1)$, and since $d\phi/ds=1/\rho$, the vanishing of the integral in (8) for $s=s_1+a$ implies

$$\tag{9} v\frac{dv}{ds_1} = \frac{-v^2\left[e^{\mu\phi(s_1+a)} - e^{\mu\phi(s_1)}\right]}{\int_{s_1}^{s_1+a} e^{\mu\phi} ds}.$$

Since $v_0>0$, if $v\neq 0$ for $t\geqq 0$, the unique solution of (9) subject to the initial conditions (1) satisfies the equation

$$\tag{10} \frac{dv}{ds_1} = -v\frac{J'(s_1)}{J(s_1)}, \quad \text{where } J(s_1) \equiv \int_{s_1}^{s_1+a} e^{\mu\phi} ds, \qquad v \neq 0.$$

The solution of (10) subject to (1) is precisely $v=v_0J(s_0)/J(s_1)$, which is the same as that given in equation (2) of the theorem. Equation (2) implies that $v\to 0$ and $s\to\infty$ as $t\to\infty$. Hence, our assumption that $v\neq 0$ for $t\geqq 0$ is a perfectly valid one, since (9) has a unique solution satisfying (1).

THEOREM 2. *The speed with which the tube returns to its initial position is independent of the convex curve C and of the length of the tube A (no gravity acts on A; see formulation of the problem).*

Proof. The desired speed is $v(s_0+L)$, where L is the length of C. Since

$$J(s_0 + L) \equiv \int_{s_0+L}^{s_0+L+a} e^{\mu\phi(s)}ds = \int_{s_0}^{s_0+a} e^{\mu\phi(L+x)}dx,$$

and since $\phi(L+x) = \phi(x) + 2\pi$, we get $J(s_0+L) = e^{2\pi\mu}J(s_0)$, so that

$$v(s_0 + L) = v_0 \frac{J(s_0)}{J(s_0 + L)} = \frac{v_0 J(s_0)}{e^{2\pi\mu}J(s_0)} = v_0 e^{-2\pi\mu}.$$

In the following theorem we inquire as to what happens to the speed v of the tube as a function of s_1 and the length a. Hence, the left member of (2) is written in the form $v(s_1, b, a)$ so that we have $v(s_1, b, a) = v(s_1, 0, a)$.

THEOREM 3. *The following limit*

$$v \equiv \lim_{a\to 0} v(s_1, 0, a) = v_0 e^{\mu(\phi_0-\phi_1)} \tag{11}$$

holds, where $\phi_1 = \phi(s_1)$, $\phi_0 = \phi(s_0)$, *and where a is the length of the tube. This value is the same as that of a particle constrained to C and satisfying* (1).

Proof. The limit in (11) follows immediately by an application of an appropriate form of L'Hospital's rule to the indeterminate form (2) for $a = 0$. The second sentence in the theorem follows immediately from the theory for a particle ([1], p. 17).

THEOREM 4. *If C is a circle of radius r, then the speed $v(s_1)$ of the tube is the same as that of a particle, so that $v(s_1)$ has the form given in* (11).

Proof. Equation (2) can be written in the form

$$v = v_0 e^{\mu(\phi_0-\phi_1)}\left(\int_{s_0}^{s_0+a} e^{\mu(\phi-\phi_0)}ds\right) \Big/ \left(\int_{s_1}^{s_1+a} e^{\mu(\phi-\phi_1)}ds\right). \tag{12}$$

Since for a circle we have $\phi - \phi_1 = (s - s_1)/r$, $\phi - \phi_0 = (s - s_0)/r$, the two integrals in (12) are equal. This implies that $v(s_1)$ has the form given in (11).

THEOREM 5. *The tension $T(s, s_1)$ in a tube of positive length a is identically zero for $s_1 \leqq s \leqq s_1 + a$, $s_0 \leqq s_1 \leqq s_0 + L$ if and only if C is a circle.*

Proof. Suppose $T(s, s_1) \equiv 0$ on the intervals indicated. Since $v(s_1, b) = v(s_1)$ for $0 \leqq b \leqq a$, equation (6) and the assumption $T(s, s_1) \equiv 0$ imply that

$$\frac{dv(s_1, b)}{ds_1} + \mu \frac{v(s_1, b)}{\rho(s_1 + b)} = 0, \qquad v(s_0, b) = v_0. \tag{13}$$

The solution of (13) is

$$v(s_1, b) = v_0 e^{\mu[\phi(s_0-b)+\phi(s_1-b)]}. \tag{14}$$

Since $v(s_1, b) = v(s_1, 0)$ for $0 \leqq b \leqq a$, equation (14) implies

$$\phi(s_0 + b) - \phi(s_1 + b) = \phi(s_0) - \phi(s_1).$$

This implies that

$$\frac{\phi(s_1 + b) - \phi(s_1)}{b} = \frac{\phi(s_0 + b) - \phi(s_0)}{b} \qquad b \neq 0,$$

so that

$$\left.\frac{d\phi}{ds}\right|_{s=s_1} = \left.\frac{d\phi}{ds}\right|_{s=s_0}, \qquad s_1 \geqq s_0.$$

Hence, the curvature $d\phi/ds$ is a constant, and since C is a plane convex curve, it must be a circle.

Conversely, if C is a circle, it is a simple matter to verify that the integrand in (8) must vanish, so that $T(s, s_1) \equiv 0$ on the intervals indicated.

3. Some thought provoking questions. The following questions have puzzled the writer for some time; they do not appear to possess simple answers.

(a) Suppose the tube has a finite breaking strength. What is the range of values of the initial speed v_0 so that the tube will remain unbroken during its subsequent motion?

(b) What can one say about the number of places where the tube can break apart simultaneously? What happens to the different parts of a tube which breaks apart during its motion?

(c) How does the period of time required for the tube to complete one journey vary with the length of the tube?

(d) What can be said about the speeds of tubes of different lengths other than that which was said in Theorem 2?

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THE PASCAL LINE AND ITS GENERALIZATIONS

CHARLES FOX, McGill University

1. Introduction. It is now about 300 years since Pascal (1623–1662) first enunciated his famous theorem concerning hexagons inscribed in conic sections and an extensive literature has since grown up around it. Many references to this literature will be found in [2], [5], and [7]. It would seem hardly possible to add much that is new to the subject at this stage, but the converse of Pascal's theorem seems to have attracted very little attention and a study of the theorem from this point of view leads to many interesting results. Our main aim in this article is to see how far Pascal's theorem can be extended to higher plane curves or to three dimensional surfaces and the converse of the theorem proves to be very useful in this connection.

Clifford (1845–1879) studied the intersections of coplanar lines in one of his earliest papers [3] and obtained a result related to the one we give here in Section 3. He does not seem to have gone beyond this case although at the end of the first section of this paper he writes "The present communication is confined mainly to the case $n=4$; in a second communication we hope to notice some peculiarities of the higher plane case and to state the *true* analogues of Pascal's theorem in the Geometry of Higher Dimensions." No doubt Clifford's untimely death and the long illness which preceded it prevented him from returning to this topic.

Pascal's theorem has been extended to three dimensions in at least two ways. One extension is stated by Chasles in his monumental work on geometry [2] and is proved by Salmon ([8] Sec. 144, p. 141.) This was rediscovered by Court [4] and recently extended to n dimensions by Bottema [1]. Another extension, whose converse we shall discuss in Section 6, appears to have been first stated and proved by Salmon ([8], Sec. 144b, p. 143.)

I am indebted to the referee for many of the references above.

2. The converse of Pascal's theorem. We first prove the following result:

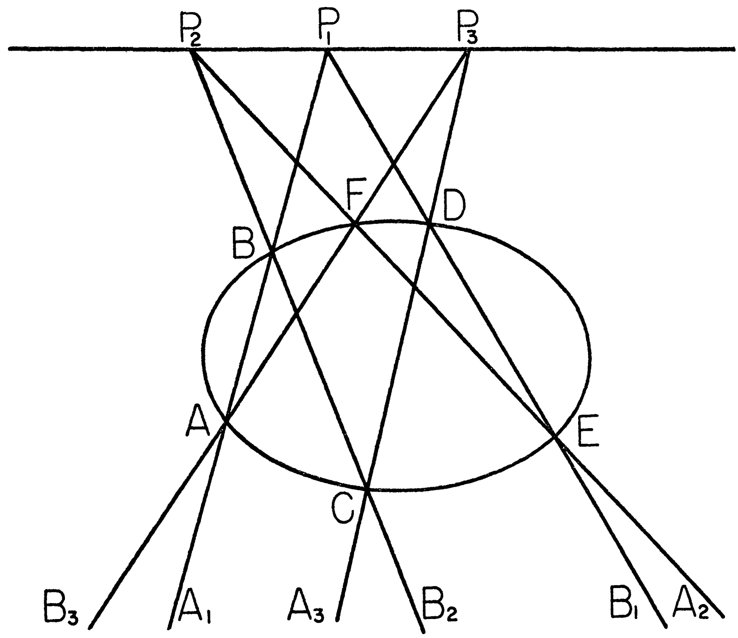
Given a conic, equation $C_2=0$, and a coplanar line, $L=0$, it is always possible to inscribe a hexagon in the conic whose Pascal line is $L=0$.

Let $A_m=a_mx+b_my+1$, $B_m=c_mx+d_my+1$, and let the point of intersection of the lines $A_m=0$, $B_n=0$ be denoted by (A_m, B_n). In order to prove the theorem we must establish the following identity:

$$C_2L = aA_1A_2A_3 + bB_1B_2B_3, \tag{1}$$

where a and b are constants. For each line $A_m=0$ and $B_n=0$ we have two constants at our disposal and so with a and b we have altogether fourteen constants at our disposal. Against this (1) is a cubic equation and a cubic in x and y possesses ten terms, so that ten equations must be satisfied among the coefficients of (1) if it is to be an identity. Hence we can establish (1) with four constants to spare.

Again, the nine points of intersection of the three A lines with the three B lines in (1) must also lie either on the conic or on $L=0$. Consider the three intersections (A_1, B_1), (A_1, B_2), and (A_1, B_3). Since the line $A_1=0$ can cut the conic in only two points, two of these, (A_1, B_2) and (A_1, B_3) will lie on the conic and the other on the line $L=0$. Similarly (A_2, B_1) and (A_2, B_3) will lie on the conic and (A_2, B_2) will lie on the line and finally (A_3, B_1) and (A_3, B_2) will lie on the conic and (A_3, B_3) on the line.



This completes the proof that a hexagon can be inscribed in a given conic so as to have a given line as its Pascal line, and we note that this can be done with four arbitrarily chosen constants.

To illustrate the theorem let us fix the line $A_1=0$; using up two of our four disposable constants; and also the directions of the lines $B_2=0$ and $B_3=0$; using up one constant for each direction. In Figure 1 the line $A_1=0$ meets the prescribed conic at A and B and the prescribed Pascal line at P_1. Suppose that the $B_2=0$ line, which must pass either through A or through B, passes through B. Since its direction is known we can find its other intersection with the conic at C, say, and also its intersection with the prescribed Pascal line, P_2, say. The line $B_3=0$ must now pass through A and, its direction being known, will cut the conic again at a point, F, say, and the Pascal line at P_3. Evidently F must be (A_2, B_3) and P_2 must be (A_2, B_2), so that P_2F is the line $A_2=0$. Let the join of P_2 and F meet the conic again at E. Similarly the join of P_3 and C must be the line $A_3=0$ and if this cuts the conic again at D then we have constructed a

hexagon $ABCDEF$, inscribed in the given conic and having $L=0$ as its Pascal line, as required. DEP_1 is the line $B_1=0$.

3. Extension to cubic curves. We now prove the following extension of this converse of Pascal's theorem:

Given a cubic curve, equation $C_3=0$, and any coplanar line, $L=0$, we can inscribe an octagon in the curve having the following properties. Using the terminology of Section 2, if the octagon consists of the lines $A_m=0$, $m=1, 2, 3, 4$, and $B_n=0$, $n=1, 2, 3, 4$, then the twelve points of intersection (A_m, B_n), $m\neq n$, lie on the cubic and the four points of intersection (A_m, B_m), $m=1, 2, 3, 4$, lie on the line $L=0$.

To prove this we must establish the identity

$$C_3L = aA_1A_2A_3A_4 + bB_1B_2B_3B_4, \tag{2}$$

where $C_3=0$ is the equation of the cubic curve, $L=0$ that of the line, and a and b are constants. A cubic in x and y has fifteen terms and against this we have eighteen constants (two for each line, a and b) at our choice. Hence (2) can be established with three constants still at our disposal.

All the sixteen intersections of the lines $A_m=0$, $m=1, 2, 3, 4$, with the lines $B_n=0$, $n=1, 2, 3, 4$, must evidently lie on either the cubic curve or on $L=0$. Since $A_1=0$ cuts the cubic in three points only we must have (A_1, B_1) on the line $L=0$ and (A_1, B_2), (A_1, B_3), (A_1, B_4) on the cubic. On dealing similarly with the other three A lines we see that of the sixteen points of intersection twelve are on the cubic and four are on the line $L=0$, as outlined in the enunciation of the theorem.

The figure is difficult to draw since we do not have enough degrees of freedom. We have three available constants and so we may choose the line $A_1=0$, using up two constants, and the direction of $B_2=0$ (which must pass through one of the three intersections of $A_1=0$ with the cubic) using up the third constant. From this stage we cannot proceed any further geometrically; we must actually use the ten coefficient equations required to establish (2) in order to compute the unknown constants of the A and B lines which still remain to be drawn.

4. Extension to quartic curves. For the plane quartic curve we can prove the following theorem:

Given a plane quartic curve whose equation is $C_4=0$ and any coplanar line, equation $L=0$, we can inscribe in the quartic a ten-sided polygon whose sides can be grouped into two sets of five lines, $A_m=0$, $m=1, 2, 3, 4, 5$, and $B_n=0$, $n=1, 2, 3, 4, 5$, with the following properties. The twenty intersections (A_m, B_n), $m\neq n$, lie on the quartic curve and the five intersections (A_m, B_m), $m=1, 2, 3, 4, 5$, lie on the line $L=0$.

The proof, as before, consists in the establishment of the identity

$$C_4L = aA_1A_2A_3A_4A_5 + bB_1B_2B_3B_4B_5, \tag{3}$$

where a and b are constants. A quintic in x and y has twenty-one terms of varying degrees and on the right hand side of (3) we have twenty-two constants at our disposal (two for each line and a, b). Hence (3) can be established with one free constant still at our disposal. Evidently the five A_m lines intersect the five B_n lines in twenty-five points all of which lie either on the quartic or on $L=0$. Among the intersections of $A_1=0$ with the five B_n lines, four will lie on the quartic and one, (A_1, B_1), say, will lie on $L=0$. On dealing similarly with the other A_m lines we see that the twenty-five intersections of the A_m lines with the B_n lines lie twenty on the quartic and five on the line $L=0$ in the manner described above.

Once again the figure is difficult to draw since we have only one free constant. Any line in the plane of the quartic can act as a "generalized Pascal line" for some ten-sided polygon inscribed in the quartic. But, once one of the vertices of the polygon is chosen, then the remaining vertices are determined. There may be several such polygons once one vertex is chosen but the figure is too difficult to draw.

5. The impossibility of extension beyond the quartic. A generalization of the previous results would be as follows:

Given a quintic, $C_5=0$, and a coplanar line, $L=0$, then a twelve-sided polygon, $A_m=0$, $m=1, 2, 3, 4, 5, 6$, $B_n=0$, $n=1, 2, 3, 4, 5, 6$, can be inscribed in the quintic so that thirty of the intersections of the A_m with the B_n lines lie on the quintic and six intersections, (A_m, B_m) $m=1, 2, 3, 4, 5, 6$, lie on the line $L=0$.

In order to prove this we must attempt to establish the identity

$$C_5L = aA_1A_2A_3A_4A_5A_6 + bB_1B_2B_3B_4B_5B_6, \tag{4}$$

where a and b are constants. Since a sextic in x and y has twenty-eight terms (4) can hold only if twenty-eight equations among the coefficients are satisfied. But we have only twenty-six disposable constants, two for each of the A_m and B_n lines and also a and b, so that (4) cannot be established in general. In other words, no line in the plane of a quintic can act as a "generalized Pascal line" for a polygon inscribed in it. Thus the Pascal theorem cannot be extended to quintics and a similar argument shows that it cannot be extended to any curve of degree greater than four.

This means that the type of generalization considered here cannot be carried beyond the quartic. Of course other types of generalization are possible, of which one example is given in [6].

6. Pascal's theorem in three dimensions. We now consider the extension of Pascal's theorem to ruled quadric surfaces, by which we mean hyperboloids of one sheet and hyperbolic paraboloids. Degenerate cases, such as cones, cylinders and pairs of planes are excluded.

Let $P_m=a_mx+b_my+c_mz+1$, $Q_n=d_nx+e_ny+f_nz+1$ and let us denote the line of intersection of the planes $P_m=0$ and $Q_n=0$ by (P_m, Q_n). We shall consider

hexagons inscribed in ruled quadric surfaces, but the hexagons will be constructed of six planes, instead of six lines, and they will intersect the surfaces in generators. To show that such a construction is possible in practice let H denote the ruled quadric and let X_1, X_2, X_3 be any three points on H, chosen so that no two lie on the same generator. The tangent planes at these points, equations $P_1=0$, $P_2=0$, and $P_3=0$, respectively, say, can form three of the planes of the hexagon inscribed in H. Another three planes can be found as follows. The plane $P_1=0$ touches H at X_1 and intersects H in two generators, g_1 and g_1', say. Similarly the planes $P_2=0$ and $P_3=0$ intersect H in the generators g_2 and g_2', g_3, and g_3' respectively, where g_1, g_2, g_3 are all generators of one kind and g_1', g_2', g_3' are all generators of the other kind. Let g_1 and g_2' intersect at Y_3, g_2 and g_3' at Y_1 and g_3 and g_1' at Y_2. Then the tangent planes at Y_1, Y_2 and Y_3 can form the other three planes of the hexagon inscribed in the ruled surface H. If the tangent planes at Y_1, Y_2, Y_3 have equations $Q_1=0$, $Q_2=0$, $Q_3=0$, respectively, then evidently g_1' is the line of intersection (P_1, Q_2), g_1 is the line (P_1, Q_3), and there are corresponding results for the other four generators. Thus among the nine lines of intersection of the three P planes with the three Q planes six are generators of H.

The following generalization of Pascal's theorem to three dimensions is proved by Salmon ([8], Sec. 144b, p. 143).

THEOREM. *If among the nine lines of intersection of the three planes* $P_m=0$, $m=1, 2, 3$, *with the three planes* $Q_n=0$, $n=1, 2, 3$, *six lines*, (P_m, Q_n), $m\neq n$, *lie on a ruled quadric surface then the remaining three lines*, (P_m, Q_m), $m=1, 2, 3$, *must be coplanar.*

We now consider the converse, which is as follows:

Given a ruled quadric, equation $H=0$, *and a plane, equation* $P=0$, *we can always construct a hexagon of six planes*, $P_m=0$, $m=1, 2, 3$ *and* $Q_n=0$, $n=1, 2, 3$, *so that six of the nine lines of intersection of the P planes with the Q planes*, (P_m, Q_n), $m\neq n$, *are generators of* $H=0$ *and the remaining three*, (P_m, Q_m), $m=1, 2, 3$, *lie on the plane* $P=0$.

To prove this we must establish the identity

$$HP = pP_1P_2P_3 + qQ_1Q_2Q_3, \tag{5}$$

where p and q are constants and H and P are given. Now a cubic in x, y, z possesses twenty terms of varying degrees so that (5) can hold only if twenty coefficient equations are satisfied. Against this we have twenty disposable constants on the right hand side, three for each plane and p, q, so that (5) can be established, but without constants to spare.

Evidently each of the nine lines of intersection of the three P planes with the three Q planes must lie either on $H=0$ or on $P=0$. Consider the three lines (P_1, Q_1), (P_1, Q_2) and (P_1, Q_3). Since $P_1=0$ cuts $H=0$ in a conic section two of these lines will lie on $H=0$ and the third, (P_1, Q_1), say, will lie on $P=0$. By dis-

cussing the other six lines of intersection similarly we see that six of the nine intersections lie on $H=0$ and three on $P=0$, in the manner described in the enunciation of the theorem.

Since there are no free disposable constants it follows that given the plane $P=0$ we can inscribe in $H=0$ only a finite number of hexagons whose plane sides have three lines of intersection on $P=0$. This is in complete contrast with the two-dimensional theorem of Section 2 in which for any given Pascal line an infinite number of hexagons can be inscribed in a given conic.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

A REMARK ON FINITELY ADDITIVE MEASURES

WILLIAM G. BADE, University of California, Berkeley

An *algebra* of sets is a family of subsets of a set S which contains the void set ϕ and is closed under complementation and finite unions. By a *finitely additive measure* we shall mean a real-valued function μ defined on an algebra Σ of sets such that

$$\mu(E_1 \cup \cdots \cup E_n) = \sum_{i=1}^{n} \mu(E_i)$$

for every finite family $E_1, \cdots, E_n$ of mutually disjoint sets in Σ. We say μ is *bounded* if there exists a real number M such that $|\mu(E)| \leq M$, $E \in \Sigma$.

If μ is nonnegative or merely bounded above or below, then it is bounded,

[1]. If Σ is a σ-algebra and μ is countably additive on Σ, then it is well known that μ must be bounded, [2, p. 17]. In a recent paper [1], E. Hewitt has remarked that it is not known whether a finitely additive measure whose domain is a σ-algebra is necessarily bounded. We shall answer this question negatively by the following theorem.

THEOREM. *If Σ is any infinite algebra of sets, there exists a finitely additive measure defined on Σ which is unbounded.*

Proof. We observe first that there exists in Σ an infinite sequence $\{E_n\}$ of mutually-disjoint nonvoid sets. For let $\{F_n\}$ be any sequence of distinct nonvoid sets in Σ. Let $E\in\Sigma$, $E\neq\phi$, $E\neq S$. We assert that one of the two sequences $\{E\cap F_n\}$ or $\{E'\cap F_n\}$ contains infinitely many distinct sets. For suppose there exists an integer N such that the sets $E\cap F_n$ all coincide for $n\geqq N$. Then since $F_n=(E\cap F_n)\cup(E'\cap F_n)$, all of the sets $E'\cap F_n$ must be distinct for $n\geqq N$. Thus there exists a nonvoid set E_1 in Σ such that infinitely many of the sets $E_1'\cap F_n$ are distinct. Applying the same argument with S replaced by E_1', we construct a nonvoid set $E_2\subseteq E_1'$, such that infinitely many of the sets $(E_1'-E_2)\cap F_n$ are distinct. The construction proceeds by induction.

Now select points $t_n\in E_n$, and for each n define the measure ρ_n by the formulas $\rho_n(E)=1$ if $t_n\in E$, $\rho_n(E)=0$ if $t_n\notin E$. Let

$$\nu = \sum_{n=1}^{\infty} \frac{\pi^n}{4^n}\rho_n.$$

Then ν is finitely additive (in fact the natural extension of ν to the σ-algebra generated by Σ is countably additive). The sequence $\mathfrak{A}=\{4^{-n}\pi^n\}$ is linearly independent over the rational field, and we may imbed $\mathfrak{A}$ in a Hamel base $\mathfrak{B}$ for the real numbers, [2, p. 100]. Now define the real valued function α on $\mathfrak{B}$ in the following way: on $\mathfrak{A}$ define $\alpha(4^{-n}\pi^n)=n$, and define $\alpha(b)=0$ for $b\in\mathfrak{B}-\mathfrak{A}$. We extend α to all real numbers by linearity. For if x is any real number, there exist unique elements $b_1,\cdots,b_m$ in $\mathfrak{B}$ and rationals $r_1,\cdots,r_m$ so that $x=\sum_{i=1}^m r_ib_i$. The formula $\alpha(x)=\sum_{i=1}^m r_i\alpha(b_i)$ gives the required extension. The set function μ defined by

$$\mu(E) = \alpha(\nu(E)), \qquad E\in\Sigma,$$

is finitely additive. Since

$$\mu(E_n) = \alpha(\nu(E_n)) = \alpha(4^{-n}\pi^n) = n, \qquad n = 1, 2, \cdots,$$

we see μ is unbounded.

The author thanks the referee for his helpful criticism.

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ON THE PRODUCT OF TWO CONTINUOUS FUNCTIONS OF BOUNDED VARIATION

NORMAN LEVINE AND JOHN E. YARNELLE, University of Pittsburgh

Introduction and preliminaries.

DEFINITION 1. *A function T on an interval I will be termed a CBV function if and only if T is both continuous and of bounded variation on I.*

We attempt in this paper to give a necessary and sufficient condition for a CBV of a CBV to be a CBV. $T_1(x)=y$ will denote a continuous function on $[\alpha, \beta]=I_x$ with images satisfying $a\leqq y\leqq b$. I_y denotes $[a, b]$. $T_2(x)=z$ will denote a second continuous transformation on $I_y=[a, b]$ with images satisfying $c\leqq z\leqq d. [c, d]=I_z$. $T_{21}(x)=T_2\{T_1(x)\}$ for $x\in I_x$. Hence $T_{21}(x)$ will always be continuous from I_x to I_z. $m(S)$ will denote the Lebesgue measure of a measurable set S.

DEFINITION 2. *The multiplicity function $N(y, T_1, I_x)$ is defined as follows:*

$$N(y, T_1, I_x) \equiv \begin{cases} n \text{ if } T_1^{-1}(y) \text{ consists of } n \text{ points for } n = 0, 1, \cdots, \\ \infty \text{ if } T_1^{-1}(y) \text{ is infinite.} \end{cases}$$

DEFINITION 3. *If $S_x\subset I_x$, then*

$$N(y, T_1, S_x) = \begin{cases} n \text{ if } T_1^{-1}(y) \cap S_x \text{ has } n \text{ points for } n = 0, 1, \cdots, \\ \infty \text{ if } T_1^{-1}(y) \cap S_x \text{ is infinite.} \end{cases}$$

$N(z, T_2, I_y)$ and $N(z, T_{21}, I_x)$ are similarly defined.

THEOREM 1. *If $y=T(x)$ is continuous on I_x, then $N(y, T, I_x)$ is a Borel measurable function of y (R, 36).**

THEOREM 2. *If $y=T(x)$ is continuous on I_x, then $T(x)$ is BV if and only if $N(y, T, I_x)$ is summable on the y-axis (R, 196).*

Main Results.

LEMMA 1. *Let T_1 be CBV on I_x, $T_2(y)$ be CBV on I_y and $T_1(I_x)\subset I_y$. Let $T_{21}(x)$ be CBV on I_x. Let $\overline{Y}$ be $\{y \mid N(y, T_1, I_x)=\infty\}$. Then $m\{T_2(\overline{Y})\}=0$.*

Proof. Let $y_1\in I_y$ and $z_1=T_2(y_1)$. $T_1^{-1}(y_1)$ is the set of all points x such that $T_1(x)=y_1$. $T_{21}^{-1}(z_1)$ is the set of all x such that $T_{21}(x)=z_1$. Then $T_1^{-1}(y_1)\subset T_{21}^{-1}(z_1)$. Hence $N(y_1, T_1, I_x)\leqq N(z_1, T_{21}, I_x)$. But for y in $\overline{Y}$, $N(y, T_1, I_x)=\infty$. Thus if $z\in T_2(\overline{Y})$, then $N(z, T_{21}, I_x)=\infty$. Then if $m\{T_2(\overline{Y})\}>0$, it follows that $\int_{I_z} N(z, T_{21}, I_x)=\infty$. Then T_{21} is not BV on I_x by Theorem 2, a contradiction.

Again, let $T_1(x)$ be CBV on I_x and $T_2(y)$ be CBV on I_y and $T_1(I_x)\subset I_y$. Let T_{21} be the product function on I_x. Let $Y_i\equiv\{y \mid N(y, T_1, I_x)=i\}$ $i=0, 1, \cdots$. Let $Z_{ji}\equiv\{z \mid T_2^{-1}(z)\cap Y_i$ has j points$\}$ $j=0, 1, \cdots$. The sets Z_{ji} are not neces-

* (R, 36) refers to Rado [3].

sarily disjoint. It is clear that $Z_{ji} \equiv \{z \mid N(z, T_2, Y_i) = j\}$. Now Y_i is a Borel set since $N(y, T_1, I_x)$ is a Borel measurable function by Theorem 1. It follows that $T_2(Y_i)$ is Lebesgue measurable (R, 31). $N(z, T_2, Y_i)$ is a measurable function (R, 201) and hence Z_{ji} is a measurable set.

THEOREM 3. *Let T_1 be CBV on I_x, T_2 be CBV on I_y, $T_1(I_x) \subset I_y$. Then T_{21} is CBV on I_x if and only if both of the following conditions hold*:

$$\text{(a)} \qquad m\{T_2(\overline{Y})\} = 0, \qquad\qquad \text{(b)} \qquad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} ijm(Z_{ij}) < \infty.$$

Proof. Sufficiency. Assume that (a) and (b) both hold. Now $z \in Z_{ji}$ implies $N(z, T_2, Y_i) = j$. Since T_2 is CBV, $N(z, T_2, I_y)$ is finite a.e. and hence $N(z, T_2, Y_i)$ is finite a.e. Therefore for each i,

$$\sum_{j=0}^{\infty} jm(Z_{ji}) = \int_{I_z} N(z, T_2, Y_i) dz$$

and

$$\sum_{i=0}^{\infty} i \left\{ \sum_{j=0}^{\infty} jm(Z_{ji}) \right\} = \sum_{i=0}^{\infty} i \int_{I_z} N(z, T_2, Y_i) dz.$$

By (b) we have $\sum_{i=0}^{\infty} i \int_{I_z} N(z, T_2, Y_i) dz < \infty$. It is easy to see that $N(z, T_{21}, I_x) = \sum_{i=1}^{\infty} iN(z, T_2, Y_i)$ for all z not in $\overline{Z} \cup T_2(\overline{Y})$ where $\overline{Z} \equiv \{z \mid N(z, T_2, I_y) = \infty\}$. But $m(\overline{Z}) = 0$ and $m\{T_2(\overline{Y})\} = 0$. Hence

$$\int_{I_z} N(z, T_{21}, I_x) dz = \int_{I_z} \sum_{i=0}^{\infty} iN(z, T_2, Y_i) dz = \sum_{i=0}^{\infty} i \int_{I_z} N(z, T_2, Y_i) dz < \infty.$$

This yields $T_{21}(x)$ BV on I by Theorem 2.

Necessity. Assume that $T_{21}(x)$ is CBV on I_x. (a) follows from Lemma 1. From Theorem 2

$$\infty > \int_{I_z} N(z, T_{21}, I_x) dz = \int_{I_z} \sum_{i=0}^{\infty} iN(z, T_2, Y_i) dz = \sum_{i=0}^{\infty} i \int_{I_z} N(z, T_2, Y_i) dz$$

$$= \sum_{i=0}^{\infty} i \left\{ \sum_{j=0}^{\infty} jm(Z_{ji}) \right\}.$$

Hence (b) follows.

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THE NELSON SLIDE RULE

BLAKE D. MILLS, JR., University of Washington

The device to be described here, known locally as the Nelson slide rule, is the by-product of an engineering student's study of mechanical computing methods. This device is presented not as a slide rule competitive with the conventional type but as an interesting study in numerical relationships. It was developed by Alfred M. Nelson while a senior in mechanical engineering at the University of Washington a few years ago. Mr. Nelson is now professionally occupied in the development of computing equipment and it has devolved upon one of his former professors to bring his slide rule to the printed page.

The conventional slide rule multiplies two numbers by adding their logarithms. Toward this end the sliding and stationary scales are each graduated logarithmically, *i.e.*, so that the distance from the left end of the scale to any given number is proportional to the logarithm of that number.

The Nelson slide rule multiplies quite differently and its scales are not graduated logarithmically. Rather, simple nonlinear scales are provided for the numbers to be multiplied and the product is read on a uniformly graduated scale. The computation is based upon relationships among "triangular" integers, where a triangular integer is the sum of all the integers from one through any selected integer. In the following discussion, *triangular* a means the sum of the integers from one through a.

This slide rule multiples two numbers a and b by subtracting triangular $a-b$ from triangular a, and then adding triangular $b-1$. Justification of this procedure will follow discussion of the slide rule and its operation.

The Nelson slide rule is shown in Figure 1, set for multiplying 8 times 3. It is seen to include two stationary scales and two additional scales on a sliding member.

The upper stationary scale, from which the desired product ab is read, is uniformly graduated.

The lower stationary scale, for the multiplier a, is graduated so that the distance from the left end of the scale to any integer a is proportional to triangular a. Accordingly, starting from zero at the left end of the scale, the lengths of successive increments are in the proportion of 1, 2, 3, 4, *etc.* The increment from zero to 1 is the same length as the increment between any two successive integers on the upper stationary scale.

The lower sliding scale, for the term $a-b$, is graduated identically to the lower stationary scale.

The upper sliding scale, for the multiplicand b, is graduated so that the distance from the left end of the scale to any integer b is proportional to triangular $b-1$. The graduation of this scale is identical to the two lower scales, but the numbers on the corresponding graduations are greater by one than on the lower scales.

To multiply with the Nelson slide rule, the multiplier a (the larger of the two

numbers being multiplied) is selected on the lower stationary scale. The difference $a-b$ is selected on the lower sliding scale and is placed opposite the selected a. The multiplicand b is selected on the upper sliding scale, and the product ab is read directly opposite on the upper stationary scale. Thus, in Figure 1, the multiplier is selected to be 8 on the lower stationary scale, the multiplicand is selected to be 3 on the upper sliding scale, the difference of 5 is set on the lower sliding scale opposite 8 on the lower stationary scale, and the product 24 is read on the upper stationary scale opposite the multiplicand 3.

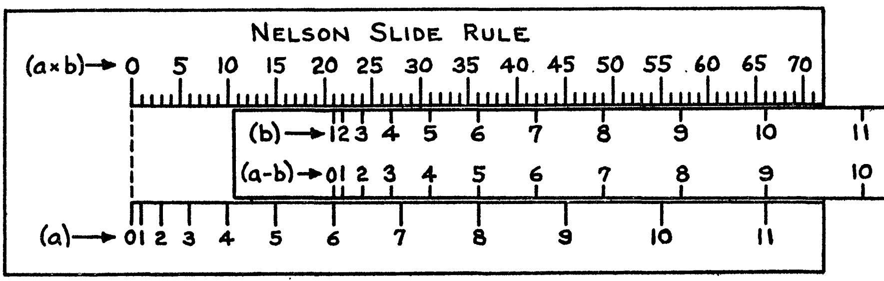

FIG. 1

The operation of the Nelson slide rule can be justified as follows:

Triangular $a=1+\cdots+a$ is an arithmetical progression whose sum is readily shown to be $(a/2)(1+a)=(\frac{1}{2})(a+a^2)$.

Triangular $a-b=1+\cdots+(a-b)=(\frac{1}{2})[(a-b)+(a-b)^2]$.
$=(\frac{1}{2})(a-b+a^2-2ab+b^2)$.

Triangular $b-1=1+\cdots+(b-1)=(\frac{1}{2})[(b-1)+(b-1)^2]$.
$=(\frac{1}{2})(b-1+b^2-2b+1)$.

Then: Triangular $a-$triangular $a-b+$triangular $b-1=$
$=(\frac{1}{2})(a+a^2-a+b-a^2+2ab-b^2+b-1+b^2-2b+1)=ab$.

NONLINEAR RECURRENCE RELATIONS FOR CLASSICAL ORTHOGONAL POLYNOMIALS

T. S. CHIHARA, Seattle University

1. In recent notes, various authors (*e.g.*, [**1**], [**3**]) have considered certain nonlinear recurrence formulas which characterize ultraspherical, Laguerre, and Hermite polynomials. It is the purpose of this note to indicate briefly a method of deriving such formulas for the classical orthogonal polynomials in general. In particular, one such formula is given for the Jacobi polynomials, which have not previously been considered.

The derivations are based on the differential equation [**2**, 10.7(1)]

$$X\frac{d^2y}{dx^2}+K_1p_1(x)\frac{dy}{dx}+\lambda_n y=0, \qquad y=p_n(x) \tag{1}$$

and Tricomi's differentiation formula [2, 10.7(4)]

$$X\frac{dp_n(x)}{dx} = (\alpha_n + nX''x/2)p_n(x) + \beta_n p_{n-1}(x), \tag{2}$$

which are satisfied by the classical orthogonal polynomials. Here, $X=1$, x, or $1-x^2$ as $\{p_n(x)\}$ denotes the Hermite, Laguerre, or Jacobi polynomials, respectively, and K_1, λ_n, α_n, β_n are constants depending on $\{p_n(x)\}$. For the precise definition of these constants, we refer to [2, Sec. 10.7] whose notation will be followed in this paper.

2. We will derive a relation involving the combination, $(p_n')^2-p_np_n''$, $p_n = p_n(x)$. A similar procedure can be used for the combination, $(p_n')^2 - p_{n+1}'(x)p_{n-1}'(x)$.

Squaring (2) and multiplying (1) by Xp_n and subtracting, we obtain

$$X^2[(p_n')^2 - p_np_n''] = [(\alpha_n + nX''x/2)^2 + \lambda_nX]p_n^2 + 2\beta_n(\alpha_n + nX''x/2)p_np_{n-1} + \beta_n^2p_{n-1}^2 + K_1p_1(Xp_n').$$

Using (2) to eliminate Xp_n' from the right side of this relation yields

$$X^2[(p_n')^2 - p_np_n''] = U_np_n^2 + \beta_n[2(\alpha_n + nX''x/2) + K_1p_1]p_np_{n-1} + \beta_n^2p_{n-1}^2, \tag{3}$$

where

$$U_n = U_n(x) = (\alpha_n + nX''x/2)^2 + K_1p_1(\alpha_n + nX''x/2) + \lambda_nX. \tag{4}$$

From the standard recurrence relation [2, 10.3(7)], we have

$$\beta_n^2p_{n-1}^2 = \beta_n^2[(A_nx + B_n)p_n - p_{n+1}]p_{n-1}/C_n$$

so that (3) becomes finally

$$X^2[(p_n')^2 - p_np_n''] = U_np_n^2 - (\beta_n^2/C_n)p_{n-1}p_{n+1} + \beta_nV_np_np_{n-1}, \tag{5}$$

where

$$V_n = V_n(x) = 2(\alpha_n + nX''x/2) + \beta_n(A_nx + B_n)/C_n + K_1p_1(x). \tag{6}$$

If $p_n(x) = H_n(x)$, $L_n^\alpha(x)$, or $C_n^\lambda(x)$, (5) reduces to formulas considered by Webster [3, (5), (6) and Theorem 3].* For the Jacobi case, $p_n(x) = P_n^{(\alpha,\beta)}(x)$, the coefficients in (5) become:

$$U_n = n\left[x^2 - \frac{2(\alpha-\beta)}{2n+\alpha+\beta}x + n + \alpha + \beta + 1 - \frac{(\alpha^2-\beta^2)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2}\right],$$

* The derivation of (2) uses the orthogonality of $\{p_n(x)\}$. Hence (5) as derived requires $\alpha > -1$, $2\lambda > -1$, while the formulas of Webster only require α, $2(\lambda-1) \neq -m$ $(m=1, 2, 3, \cdots)$.

$$\beta_n^2/C_n = \frac{4(n+1)(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

$$\beta_n V_n = -\frac{2(n+\alpha)(n+\beta)}{2n+\alpha+\beta}\left[x+\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right].$$

Since $\beta_n \neq 0$, (5) determines p_{n+1} uniquely in terms of p_{n-1} and p_n, hence can be used to characterize the Jacobi polynomials.

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NOTE ON ALTERNATING TANNERY SERIES

R. B. LEIPNIK, University of Washington

We call a limit of the form $\lim_{n\to\infty}\sum_{k=1}^{q_n} a_{k,n}$ a Tannery series. Such series often arise in applications. If $\lim_{n\to\infty} q_n=\infty$ and $\lim_{n\to\infty} a_{k,n}=a_k$, it is natural to expect that

$$\lim_{n\to\infty}\sum_{k=1}^{q_n} a_{k,n} = \sum_{k=1}^{\infty} a_k$$

whenever the latter converges.

From an example given by Bromwich* (p. 137), we see that this need not happen. Let $a_{k,n}=\log\{1+(k/n^\alpha)\}$, $q_n=n$, $\alpha>3/2$. Clearly, $a_k=\lim_{n\to\infty} a_{k,n}=\log 1=0$ so that $\sum_{k=1}^{\infty} a_k=0$. However, we have

$$\frac{k}{n^\alpha}-\frac{1}{2}\,\frac{k^2}{n^{2\alpha}} \leqq \log\left(1+\frac{k}{n^\alpha}\right) \leqq \frac{k}{n^\alpha}$$

so that

$$\frac{1}{n^\alpha}\,\frac{n(n+1)}{2}-\frac{1}{2n^{2\alpha}}\,\frac{n(n+1)(2n+1)}{6} \leqq \sum_{k=1}^{q_n}\log\left(1+\frac{k}{n^\alpha}\right) \leqq \frac{1}{n^\alpha}\,\frac{n(n+1)}{2}\,.$$

Since $\alpha>3/2$, we find $\lim_{n\to\infty} n(n+1)(2n+1)/n^{2\alpha}=0$. Hence

$$\lim_{n\to\infty}\sum_{k=1}^{q_n} a_{k,n} = \lim_{n\to\infty}\frac{n^{1-\alpha}(n+1)}{2} = \begin{cases} 0, & \alpha>2; \\ 1/2, & \alpha=2; \\ \infty, & 3/2<\alpha<2. \end{cases}$$

* T. J. I'A. Bromwich, An Introduction to the Theory of Infinite Series, 2nd ed., rev., London, 1955.

Thus the expected result materializes only for $\alpha > 2$, the limit is different but finite for $\alpha = 2$, and infinite if $3/2 < \alpha < 2$. (Bromwich considers $\alpha = 2$ only.)

Tannery proved the following useful theorem (Bromwich, p. 136) somewhat analogous to the Lebesgue domination theorem for integrals.

THEOREM 1. *If*

$$\sum_{k=1}^{\infty} \sup_n |a_{k,n}| < \infty, \qquad \lim_{n\to\infty} q_n = \infty, \qquad \lim_{n\to\infty} a_{k,n} = a_k,$$

then

$$\lim_{n\to\infty} \sum_{k=1}^{q_n} a_{k,n} = \sum_{k=1}^{\infty} a_k.$$

Unfortunately, this is far from being a necessary condition for the interchange of these limiting operations. In a probability problem, the author encountered a Tannery series, not covered by Theorem 1, whose terms alternate in a way reminiscent of Leibniz's theorem on alternating series. A result that takes care of this situation is

THEOREM 2. *If* $\{q_n\}$ *is an integer sequence with* $\lim_{n\to\infty} q_n = \infty$; $\{f_{k,n}\}$ *is a nonnegative double sequence such that for some* n_0, $f_{k,n} \geqq f_{k+1,n}$ *whenever* $1 \leqq k \leqq q_n$ *and* $n > n_0$; $\lim_{n\to\infty} f_{k,n} = f_k$ *is finite for all* k; *and* $\lim_{k\to\infty} f_k = 0$; *then*

$$\lim_{n\to\infty} \sum_{k=1}^{q_n} (-1)^{k-1} f_{n,k} = \sum_{k=1}^{\infty} (-1)^{k-1} f_k.$$

Proof. The same reasoning used in proving the ordinary Leibniz theorem shows that, for $n > n_0$, $q_n \geqq 2s+1$, $s \geqq 1$, the inequalities

$$\sum_{k=1}^{2s} (-1)^{k-1} f_{k,n} \leqq \sum_{k=1}^{q_n} (-1)^{k-1} f_{k,n} \leqq \sum_{k=1}^{2s+1} (-1)^{k-1} f_{k,n} \tag{1}$$

follow from the monotone character of $\{f_{k,n}\}$. Taking limits superior and inferior in (1) as $n \to \infty$ and using the existence of $\lim_{n\to\infty} f_{k,n} = f_k$, we find

$$\sum_{k=1}^{2s} (-1)^{k-1} f_k \leqq \underline{\overline{\lim_{n\to\infty}}} \sum_{k=1}^{q_n} (-1)^{k-1} f_{k,n} \leqq \sum_{k=1}^{2s+1} (-1)^{k-1} f_k \tag{2}$$

for $s \geqq 1$. Clearly, $f_k = \lim_n f_{k,n} \geqq f_{k+1} = \lim_n f_{k+1,n}$, so that from the assumption $\lim_k f_k = 0$ we find that $\sum_{k=1}^{\infty} (-1)^{k-1} f_k$ converges by the ordinary Leibniz theorem. Taking limits as $s \to \infty$ in (2), we have

$$\sum_{k=1}^{\infty} (-1)^{k-1} f_k \leqq \underline{\overline{\lim_{n\to\infty}}} \sum_{k=1}^{q_n} (-1)^{k-1} f_{k,n} \leqq \sum_{k=1}^{\infty} (-1)^{k-1} f_k. \tag{3}$$

Hence the conclusion of the theorem follows.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

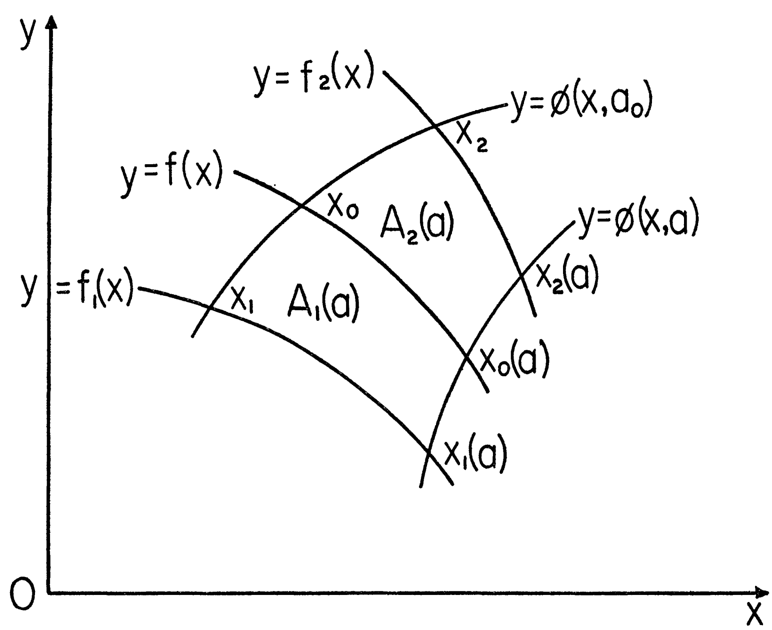
All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

ON THE BANKS OF THE NILE

H. J. HAMILTON, Pomona College

A number of years ago someone—I forget who—posed the problem of finding that common boundary between two adjacent pieces of property along the Nile for which, however high the River might flood, the ratio between the inundated portions of these properties would remain the same. Under certain simplifying assumptions, the solution is easily obtained, as we shall now show.

We suppose first that it is possible to orient axes so that the representation in the figure is achieved. Here the curve $y=\phi(x, a_0)$ is the low-water edge of the River, $y=\phi(x, a)$ is the edge at flood time (the variability of which is indicated by the parameter a), $y=f(x)$ is the desired common boundary, and $y=f_1(x)$ and $y=f_2(x)$ are the remaining relevant parts of the boundaries of the pieces of property (the inundated parts of which we call $A_1(a)$ and $A_2(a)$, respectively). The labels attached to the points of intersection among these curves which are shown are their abscissas.



Next, we suppose that, on sufficient intervals of the x- and a-axes and in sufficient regions of the $x-a$ plane—as case may be—$f_1(x)$ and $f_2(x)$ are con-

tinuous; $x_0(a)$, $x_1(a)$, and $x_2(a)$ are differentiable; and $\phi(x, a)$ and $\partial\phi(x, a)/\partial a$ are continuous.

Then, *if, for a given fixed ratio r, the equation $A_1(a)=rA_2(a)$ has a solution $f(x)$ which is continuous on a sufficient interval, this solution is given by $y=\phi(x, a)$, where a satisfies the equation*

$$\int_{x_1(a)}^{x} \frac{\partial\phi(t, a)}{\partial a}\,dt = r\int_{x}^{x_2(a)} \frac{\partial\phi(t, a)}{\partial a}\,dt.$$

To prove this, we now write the equation $A_1(a)=rA_2(a)$ in terms of integrals,

$$\int_{x_1}^{x_0}\phi(x, a_0)dx + \int_{x_0}^{x_0(a)} f(x)dx - \int_{x_1}^{x_1(a)} f_1(x)dx - \int_{x_1(a)}^{x_0(a)}\phi(x, a)dx$$
$$= r\left\{\int_{x_0}^{x_2}\phi(x, a_0)dx + \int_{x_2}^{x_2(a)} f_2(x)dx - \int_{x_0}^{x_0(a)} f(x)dx - \int_{x_0(a)}^{x_2(a)}\phi(x, a)dx\right\},$$

and differentiate with respect to a:

$$f[x_0(a)]x_0'(a) - f_1[x_1(a)]x_1'(a) - \int_{x_1(a)}^{x_0(a)} \frac{\partial\phi(x, a)}{\partial a}\,dx - \phi[x_0(a), a]x_0'(a)$$
$$+ \phi[x_1(a), a]x_1'(a)$$
$$= r\left\{f_2[x_2(a)]x_2'(a) - f[x_0(a)]x_0'(a) - \int_{x_0(a)}^{x_2(a)} \frac{\partial\phi(x, a)}{\partial a}\,dx - \phi[x_2(a), a]x_2'(a)\right.$$
$$\left. + \phi[x_0(a), a]x_0'(a)\right\}.$$

Since $f[x_0(a)]=\phi[x_0(a), a]$, the first and fourth terms in this equation cancel; and since $f_1[x_1(a)]=\phi[x_1(a), a]$, so do the second and fifth terms. Corresponding cancellations occur on the right hand side of the equation, and we are left with

$$\int_{x_1(a)}^{x_0(a)} \frac{\partial\phi(x, a)}{\partial a}\,dx = r\int_{x_0(a)}^{x_2(a)} \frac{\partial\phi(x, a)}{\partial a}\,dx.$$

Since, finally, $f[x_0(a)]=\phi[x_0(a), a]$, the conclusion follows.

Example. Taking $f_1(x)=0$, $f_2(x)=x$, and $\phi(x, a)=2x-a$, we are led to the equation

$$\int_{a/2}^{x}(-1)dt = r\int_{x}^{a}(-1)dt,$$

whose solution is $a=2(r+1)x/(2r+1)$. Substitution of this into the equation $y=2x-a$ gives $y=2rx/(2r+1)$, the validity of which is easily established geometrically.

(We would welcome any information you may have on this problem. Ed.)

NOTE ON THE EULER-MACLAURIN FORMULA

W. D. MUNRO, University of Minnesota*

When the subject of numerical integration is introduced into a first course in calculus it usually begins with the trapezoidal rule and ends with Simpson's rule. The doubly useful Euler-Maclaurin formula is generally ignored, possibly because of the difficulty at this level of presenting any of the standard derivations. A formal and heuristically appealing derivation—which seems never to appear in the texts—may be based on nothing more than the Taylor formula or series. Any number of terms can be obtained.

We seek an approximate value for $\int_a^{a+h} f(x)dx$, where h is "small." Let $F(x)$ be such that $F'(x)=f(x)$. Then

$$\int_a^{a+h} f(x)dx = F(a+h) - F(a).$$

By Taylor's expansion about a,

$$F(a+h) = F(a) + f(a)h + \frac{f'(a)h^2}{2!} + \cdots,$$

and

$$F(a+h) - F(a) = f(a)h + \frac{f'(a)h^2}{2!} + \frac{f''(a)h^2}{3!} + \cdots.$$

By expansion about $a+h$,

$$F(a) = F(a+h) - f(a+h)h + \frac{f'(a+h)h^2}{2!} + \cdots$$

and

$$F(a+h) - F(a) = f(a+h)h - \frac{f'(a+h)h^2}{2!} + \cdots.$$

Addition yields

$$\int_a^{a+h} f(x)dx = \frac{1}{2}\left\{h[f(a+h) + f(a)] - \frac{h^2}{2!}[f'(a+h) - f'(a)] + \frac{h^3}{3!}[f''(a+h) + f''(a)] + \cdots\right\}. \tag{1}$$

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Omission of all terms involving h^2 and beyond gives the trapezoidal rule. We attempt to improve this by including terms in h^2 and h^3. A second Taylor expansion about each point gives

$$\begin{aligned} f(a+h)-f(a) &= f'(a)h+\frac{f''(a)h^2}{2!}+\frac{f'''(a)h^3}{3!}+\cdots \\ &= f'(a+h)h-\frac{f''(a+h)h^2}{2!}+\frac{f'''(a+h)h^3}{3!}+\cdots \end{aligned}$$

and subtraction yields

$$\begin{aligned} -[f'(a+h)-f'(a)]+\frac{h}{2!}[f''(a+h)+f''(a)] \\ -\frac{h^2}{3!}[f'''(a+h)-f'''(a)]+\cdots=0. \end{aligned} \tag{2}$$

Using only two terms we have approximately,

$$f''(a+h)+f''(a)=\frac{2}{h}[f'(a+h)-f'(a)].$$

Substitution in (1) gives

$$\begin{aligned} \int_a^{a+h} f(x)dx=\frac{h}{2}[f(a+h)+f(a)]-\frac{h^2}{12}[f'(a+h)-f'(a)] \\ +\text{ (terms in } h^4 \text{ and beyond)}, \end{aligned}$$

which agrees with the first two terms of the Euler-Maclaurin formula.

If (2) is used with obvious extension

$$f''''(a+h)+f''''(a)=\frac{2}{h}[f'''(a+h)-f'''(a)],$$

approximately,

$$\begin{aligned} \int_a^{a+h} f(x)dx=\frac{h}{2}[f(a+h)+f(a)]-\frac{h^2}{12}[f'(a+h)-f'(a)] \\ +\frac{h^4}{720}[f'''(a+h)-f'''(a)]+\text{ (terms in } h^6 \text{ and beyond)}. \end{aligned}$$

Application to an interval (a, b) with $x_k=a+kh$, $h=(b-a)/n$ and $k=0, \cdots, n$ gives the standard form

$$\int_a^b f(x)dx = h\sum_{k=0}^{n} f(x_k) - \frac{h}{2}[f(b)+f(a)] - \frac{h^2}{12}[f'(b)-f'(a)] + \frac{h^4}{720}[f'''(b)-f'''(a)] + \cdots .$$

This is especially useful as a summation formula in the form

$$\sum_{k=0}^{n} f(x_k) = \frac{1}{h}\int_a^b f(x)dx + \frac{1}{2}[f(b)+f(a)] + \frac{h}{12}[f'(b)-f'(a)] + \cdots .$$

For example if terms through the first derivative are used to calculate $\log n! = \sum_{k=1}^{n} \log k$, we have approximately (with $h=1$, $a=1$),

$$\log n! = \sum_{k=1}^{n} \log k = \int_1^n \log x dx + \frac{1}{2}\log n + \frac{1}{12}\left[\frac{1}{n} - 1\right] = \left(n+\frac{1}{2}\right)\log n - n + \frac{1}{12n} + \frac{11}{12} \cdot$$

This gives as an approximation to $n!$ the result $e^{11/12}\sqrt{n}(n/e)^n e^{1/12n}$, which compares favorably with the well known Stirling result since $e^{11/12} = .998\sqrt{2\pi}$.

The question naturally arises as to whether a continuation of the procedure will always yield terms in agreement with those of the Euler-Maclaurin formula. Are we really generating the correct coefficients in terms of the Bernoulli numbers or is it mere coincidence for the first few? It is possible (albeit tedious) to establish that the coefficients here obey the correct generating law for the Bernouilli numbers as given, say, by $x/(e^x-1)$. A simpler device is, however, available. Let the coefficients in the two series be given by

$$\int_a^{a+h} f(x)dx = \frac{h}{2}[f(a+h)+f(a)] - \sum_{k=1}^{\infty} A_{2k}h^{2k}[f^{(2k-1)}(a+h) - f^{(2k-1)}(a)],$$

$$\int_a^{a+h} f(x)dx = \frac{h}{2}[f(a+h)+f(a)] - \sum_{k=1}^{\infty} B_{2k}h^{2k}[f^{(2k-1)}(a+h) - f^{(2k-1)}(a)].$$

Either formula is exact for a polynomial, since the series terminate. Let $f(x) = x^{2n}$, $h=1$, $a=0$, then

$$\int_0^1 x^{2n}dx = \frac{1}{2n+1} = \frac{1}{2} - [A_2(2n) + A_4(2n)(2n-1)(2n-2) + \cdots + A_{2n}(2n)!]$$
$$= \frac{1}{2} - [B_2(2n) + B_4(2n)(2n-1)(2n-2) + \cdots + B_{2n}(2n)!]$$

and a simple induction establishes $A_{2n} = B_{2n}$, all n. This also gives, simply, the relation connecting the A_{2k}, $k=1, \cdots, n$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within the three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1306. *Proposed by J. Gallego-Diaz, Vanderbilt University*

A girl entered a store and bought x flowers for y dollars (x and y integers). When she was about to leave, the clerk said, "If you buy ten more flowers I shall give you all for two dollars, and you will save 80 cents a dozen." Find x and y.

E 1307. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Let P denote the contact of AB with the circle tangent internally to the circumcircle of a triangle ABC and touching the sides AB and AC. If Q is the foot of the perpendicular from P upon AC, show that PQ is equal to the diameter of the incircle of triangle ABC.

E 1308. *Proposed by I. S. Gál, Cornell University*

Let A_1, A_2, A_3, A_4 be the vertices of a square and let P be an arbitrary point in the plane. Show that

$$\sum_{k=1}^{4} A_kP \geqq (1+\sqrt{2}) \max_{(k)} A_kP + \min_{(k)} A_kP$$

and determine the points P for which equality takes place.

E 1309. *Proposed by Martin Gardner, New York, N. Y.*

Dissect a regular pentagram (five pointed star) into no more than nine pieces which can be reassembled to form a square. Pieces may be turned over.

E 1310. *Proposed by J. L. Brown, Jr., Pennsylvania State University*

Evaluate $\int_0^\pi e^{\cos x} \sin nx \sin(\sin x)dx$, where n is a positive integer.

SOLUTIONS

A Well-concealed Endpoint Maximum

E 1276 [1957, 504]. *Proposed by C. S. Ogilvy, Hamilton College*

Find the greatest right circular cylinder coaxal with and inscribed in the solid formed by rotating around the y-axis the area bounded by the two axes, the parabola $y=9x^2-28x+24$, and the parabola's minimum ordinate.

Solution by D. C. B. Marsh, Colorado School of Mines. The volume of any inscribed coaxal right circular cylinder for the given solid is

$$V(x) = \pi x^2(9x^2 - 28x + 24) \quad \text{for} \quad 0 \leqq x \leqq 14/9.$$

Since $V(x)$ is continuous and differentiable over the closed interval, it may have extrema only where $V'(x)=0$ or at the endpoints. $V'(x)=12\pi x(x-1)(3x-4)$. There are relative minima at $x=0$ and $x=4/3$, and relative maxima at $x=1$ and $x=14/9$. By inspection of the latter, the absolute maximum is seen to occur at the endpoint for the cylinder with altitude 20/9, radius 14/9, and volume $3920\ \pi/729$.

Also solved by H. L. Baldwin, Leon Bankoff, A. P. Boblétt, A. L. Epstein, Michael Goldberg' A. G. Grace, Jr., Cornelius Groenewoud, A. R. Hyde, J. D. E. Konhauser, C. F. Pinzka, David Zeitlin, and the proposer.

Extreme Parameters in an Inequality

E 1277 [1957, 504]. *Proposed by Alexander Oppenheim, University of Malaya, Singapore, Malaya*

For each $p>0$ there is a greatest q and a least r such that

$$\frac{q \sin x}{1 + p \cos x} \leqq x \leqq \frac{r \sin x}{1 + p \cos x}$$

for $0 \leqq x \leqq \pi/2$. Determine q and r as functions of p.

Solution by W. B. Carver, Cornell University. For $x=0$ the required relation holds for all values of q and r. For $0<x\leqq\pi/2$ and for positive p we may multiply by the positive quantity $(1+p \cos x)/x$, and the required relation takes the form

$$(q \sin x)/x \leqq 1 + p \cos x \leqq (r \sin x)/x.$$

With p fixed we consider two curves, the fixed curve C,

$$y = 1 + p \cos x,$$

and the movable curve C_s,

$$y = (s \sin x)/x \text{ for } x \neq 0, \qquad y = s \text{ for } x = 0,$$

where s is a variable parameter. We are interested only in the arcs of these curves in the interval $0 \leqq x \leqq \pi/2$.

The curve C slopes downward from $y=p+1$ at $x=0$ to $y=1$ at $x=\pi/2$; and the curve C_s also slopes downward from $y=s$ at $x=0$ to $y=2s/\pi$ at $x=\pi/2$. The whole arc of the curve C_s moves up or down according as the parameter s is increased or decreased, and the problem is to find the limits q and r of the range of values of s for which the arc of C_s has any points in common with the arc of C.

The curve C and the curve C_s when $s=p+1$ (we may say the curve C_{p+1}) are tangent at $x=0$ with the common tangent having a slope 0. An examination of several terms in the expansions of $1+p \cos x$ and $[(p+1) \sin x]/x$ in powers of x shows that if $p \leqq 1/2$ the curve C_{p+1} lies below the curve C when x is near zero, and if $p>1/2$ the curve C_{p+1} is above C for x near zero.

Similarly, the curve $C_{\pi/2}$ cuts the curve C at $x=\pi/2$; and by expanding

$$1 + p \cos(\pi/2 - h) \quad \text{and} \quad [\pi \sin(\pi/2 - h)]/2(\pi/2 - h)$$

in powers of h one sees that the curve $C_{\pi/2}$ is above or below the curve C for x near $\pi/2$ according as $p<2/\pi$ or $p \geqq 2/\pi$. For $p=\pi/2-1$ the curves C_{p+1} and $C_{\pi/2}$ are, of course, the same.

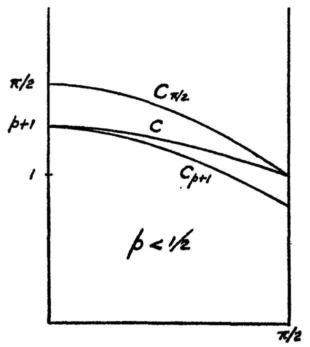

FIG. 1

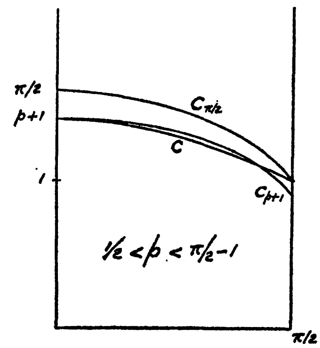

FIG. 2

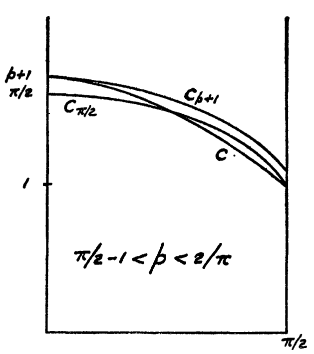

FIG. 3

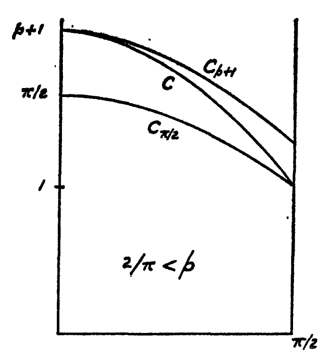

FIG. 4

The relative positions of the curves C, C_{p+1}, and $C_{\pi/2}$ for various values of p are shown in Figures 1–4. One sees that the least value of r required by the problem is

$$r = \pi/2 \quad \text{when} \quad p \leqq \pi/2 - 1,$$
$$r = p + 1 \quad \text{when} \quad p \geqq \pi/2 - 1.$$

Also the required greatest value of q is

$$q = p + 1 \quad \text{when} \quad p \leqq 1/2,$$
$$q = \pi/2 \quad \text{when} \quad p \geqq 2/\pi.$$

The difficult part of the problem is to find the greatest value q for values of p

in the interval

$$1/2 < p \leqq \pi/2 - 1 \quad \text{and} \quad \pi/2 - 1 \leqq p < 2/\pi.$$

Figures 2 and 3 indicate that when p has a value in the interval $1/2 < p < 2/\pi$ the required greatest value of q is that value s (other than $s = p+1$) for which the curve C_s has a point of tangency with the curve C. For such tangency we must have

$$(1) \qquad \begin{cases} 1 + p \cos x = (s \sin x)/x, \\ \ \ - p \sin x = s(x \cos x - \sin x)/x^2. \end{cases}$$

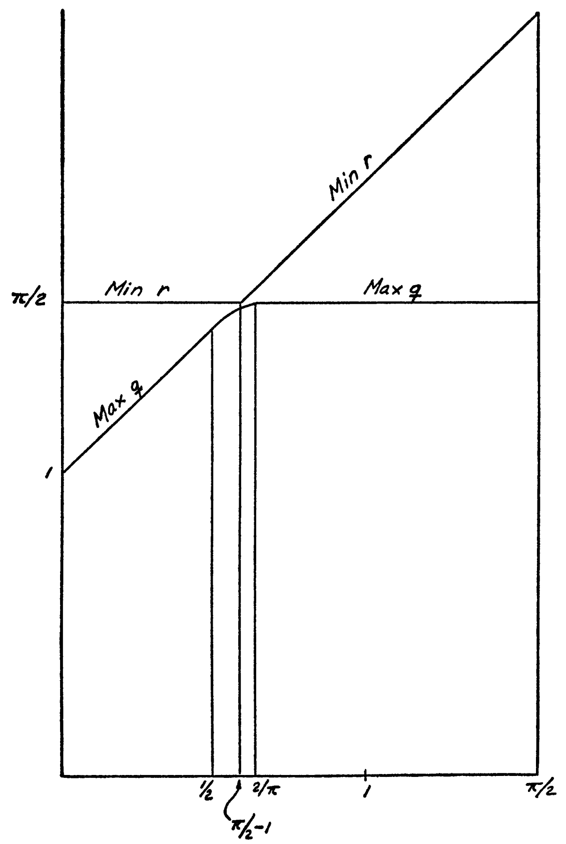

FIG. 5

Theoretically, the elimination of x between these two equations would give a relation between s and p from which we might express s (the required greatest value of q) as a function of p; but obviously this function of p will not have any simple closed form. If we solve equations (1) for s and p in terms of x we have

$$(2) \qquad \begin{cases} s = (x^2 \sin x)/(x - \sin x \cos x), \\ p = (\sin x - x \cos x)/(x - \sin x \cos x). \end{cases}$$

By then assigning numerical values of x we can calculate pairs of corresponding values of s and p, with p in the range $1/2 < p < 2/\pi$. For $x = \pi/3$, for example, we have $p = 0.5575$, $s = 1.546$. The required least r and greatest q are shown graphically as functions of p in Figure 5. Equations (2) may be regarded as parametric equations for the part of the curve Max q in the interval $1/2 < p < 2/\pi$.

Also solved by W. H. Laubach and D. C. B. Marsh (jointly), and the proposer. Late solutions by Julian Braun and Ya'akov Shima.

A Property of the Mystic Hexagram

E 1278 [1957, 504]. *Proposed by V. F. Ivanoff, San Carlos, Calif.*

If a hexagon $ABCDEF$ is inscribed in a conic, and if P is any point either on the conic or on the Pascal line of the hexagon, then

$$(\sin APB \sin CPD \sin EPF)/(\sin BPC \sin DPE \sin FBA) = \text{constant}.$$

Solution by the proposer. If P is on the conic, then

$$(\sin APB \sin CPD)/(\sin BPC \sin APD) = \text{constant}$$

and

$$(\sin DPE \sin FPA)/(\sin EPF \sin DPA) = \text{constant},$$

and the desired result follows by division. We have to use a different approach, however, to establish the result for the more general position of P.

Let $a_1 = 0$, $a_2 = 0$, $a_3 = 0$ represent equations of the lines AB, CD, EF, and let $b_1 = 0$, $b_2 = 0$, $b_3 = 0$ represent equations of the lines DE, FA, BC. Then

$$(1) \qquad a_1 a_2 a_3 = \lambda b_1 b_2 b_3,$$

where λ is an arbitrary constant, is the equation of a cubic passing through the vertices of the hexagon and through the three Pascal points of the Pascal line of the hexagon. (The three Pascal points are the points of intersection of the pairs of lines AB, DE; CD, FA; EF, BC.) By a proper choice of λ we obtain the degenerate cubic composed of the conic and the Pascal line. (To get this cubic we set the condition that the cubic (1) pass through a fourth point of the Pascal line, and from this condition find the value of λ.) Now the equation $a_1 = 0$ may be taken in the form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

where (x_1, y_1), (x_2, y_2) are the coordinates of points A and B. If (x, y) are the coordinates of point P, then the determinant in the above equation is equal to

$PA \cdot PB \sin APB$, both expressions being equal to twice the signed area of triangle PAB. In this way $a_1, a_2, a_3, b_1, b_2, b_3$ can be made to represent twice the signed areas of triangles PAB, PCD, PEF, PDE, PFA, PBC, respectively, whence, from the equation of the degenerate cubic, we have

$$\frac{PA \cdot PB \sin APB \; PC \cdot PD \sin CPD \; PE \cdot PF \sin EPF}{PB \cdot PC \sin BPC \; PD \cdot PE \sin DPE \; PF \cdot PA \sin FPA} = \text{constant}.$$

Cancellation now yields the desired result.

Cutting an $n \times n \times n$ Cube into n^3 Unit Cubes

E 1279 [1957, 504]. *Proposed by L. R. Ford, Jr., and D. R. Fulkerson, The RAND Corporation, Santa Monica, Calif.*

In the February 1957 issue of *Scientific American* the following problem appeared: "A carpenter, working with a buzz saw, wishes to cut a wooden cube, three inches on a side, into 27 one-inch cubes. He can do this easily by making six cuts through the cube, keeping the pieces together in the cube shape. Can he reduce the number of necessary cuts by rearranging the pieces after each cut?" Generalize this to find the number of cuts necessary to dissect an $n \times n \times n$ cube into n^3 one-inch cubes.

Solution by D. C. B. Marsh, Colorado School of Mines. We first note that only one dimension of any one piece may be divided at each cut, and that cuts which come as near as possible to bisecting each piece yield the optimum. Thus, if pieces are piled after each cutting, an n-unit length will be divided into unit segments by k cuts, where k is defined by $2^k \geqq n > 2^{k-1}$, and unit cubes will be attained with $3k$ cuts.

Also solved by Merrill Barnebey, P. L. Chessin, W. J. Cody, Michael Goldberg, Frank Hawthorne, Werner Held, James Jordan, H. G. Loomis, Leo Moser, C. S. Ogilvy, and the proposers. Late solution by B. D. Roberts.

It was pointed out that this problem is a special case of Leo Moser's Problem 102, *Mathematics Magazine*, vol. 25, no. 4, March–April 1952, p. 219. There it is more generally shown that the least number of planar cuts required to cut an $a \times b \times c$ block into abc unit cubes, with piling permitted, is

$$3 + [\log_2 (a-1)] + [\log_2 (b-1)] + [\log_2 (c-1)].$$

The problem can also be further generalized to an r-dimensional box of dimensions $a_1, a_2, \cdots, a_r$; here the minimal number of cuts is given by

$$r + \sum_{i=1}^{r} [\log_2 (a_i - 1)].$$

Three Squares in Arithmetic Progression

E 1280 [1957, 505]. *Proposed by V. K. Narayanan, Puthenchanthai, Trivandrum, South India*

If $x=24$, then $x+1$ and $2x+1$ are perfect squares; if $x=40$, then $2x+1$ and $3x+1$ are perfect squares; if $x=8$, then $x+1$ and $3x+1$ are perfect squares. Is there a positive integer x such that all three $x+1$, $2x+1$, $3x+1$ are perfect squares?

Solution by Leonard Carlitz, Duke University. If a positive integer x exists such that $x+1$, $2x+1$, $3x+1$ are all squares, then the numbers 1, $x+1$, $2x+1$, $3x+1$ constitute four squares in arithmetic progression. This has been shown to be impossible. See Dickson's *History of the Theory of Numbers*, vol. 2, p. 440 and p. 635 (ref. 109).

Also solved by Leon Bankoff, D. A. Breault, P. L. Chessin, Edgar Karst, Sam Kravitz D. C. B. Marsh, and Leo Moser. Late solutions by W. J. Blundon and M. S. Klamkin.

Some of the solutions submitted were not complete. Chessin and Moser employed well-known Pellian theory. The proposer pointed out that it is possible to find positive integers n such that $x+n$, $2x+n$, $3x+n$ will all be perfect squares for some positive x. Thus if $n=193$, then we may take $x=2016$, and if $n=241$ we may take $x=720$. The value $n=193$ is the smallest such n.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4778. *Proposed by R. C. Lyness, Preston, England*

Given $f(r)=\alpha^r(\beta-\gamma)+\beta^r(\gamma-\alpha)+\gamma^r(\alpha-\beta)$ in which α, β, γ, are nonzero and distinct. If n is a positive integer and $f(n+1)=0$, prove that

$$f(n+2)f(n) = \alpha^n\beta^n\gamma^n(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)f(-n).$$

4779. *Proposed by Solomon Leader, Rutgers University*

Let $f(x)=|x-k|$ for $k-\frac{1}{2}\leqq x\leqq k+\frac{1}{2}$, where k runs through the integers, and

$$g_n(x) = \sum_{\nu=0}^{n-1} 10^{-\nu}f(10^\nu x).$$

($\operatorname{Lim}_{n\to\infty} g_n(x)$ is Van der Waarden's example of a continuous, nowhere differentiable function.) Let $\|g_n\|$ be the total variation of g_n on the interval (0, 1). Find $\lim_{n\to\infty} n^{-1/2}\|g_n\|$.

4780. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

An equiproduct point of a curve is defined to be a point such that the product of the two segments of any chord through the point is constant. (1) Show that if every point inside a curve is equiproduct, the curve must be a circle. (2) What is the maximum number of equiproduct points a noncircular oval can have?

4781. *Proposed by J. L. Massera, Mathematics Institute, Montevideo, Uruguay*

Let $f(x, y, z, \cdots) = a_0(y, z, \cdots)x^n + a_1(y, z, \cdots)x^{n-1} + \cdots + a_n(y, z, \cdots)$, where the a_i are any real functions defined in any region G of the $(y, z, \cdots)$-space. Let $g(y, z, \cdots)$ be any real function defined in G and construct $f^*(x, y, z, \cdots) = f + gf_x + g^2 f_{xx} + \cdots$. Then, if $f \geqq 0$ in the cylinder $K = G \times \{x: -\infty < x < +\infty\}$, we have $f^* \geqq 0$ in K. More precisely, if the a_i do not vanish simultaneously in G, $f^* > 0$ in K except at the points where $f = g = 0$.

4782. *Proposed by V. F. Ivanoff, San Carlos, Calif.*

Given a composite function $F(x) \equiv f[g(x)]$. Denoting the nth derivative of $f(g)$ by $D^n f$, and the derivatives of $g(x)$ by $g', g'', \cdots, g^{(n)}$, show that

$$F^{(n)}(x) = \begin{vmatrix} g' & g'' & g''' & g^{(iv)} & \cdots & g^{(n)} \\ -1 & g'D & 2g''D & 3g'''D & \cdots & \binom{n-1}{1} g^{(n-1)}D \\ 0 & -1 & g'D & 3g''D & \cdots & \binom{n-1}{2} g^{(n-2)}D \\ 0 & 0 & -1 & g'D & \cdots & \binom{n-1}{3} g^{(n-3)}D \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots -1 & g'D \end{vmatrix} Df.$$

SOLUTIONS

Separable Compact Hausdorff Space

4733 [1957, 276]. *Proposed by Albert Wilansky, Lehigh University*

Give an example of a compact Hausdorff space which is separable (has a dense sequence) but not completely separable (does not satisfy the second axiom of countability).

Solution by the proposer. The space will be nonmetrizable, obviously. Let X be the half-open interval of real numbers $(0, 1]$; $\overline{X}$ its Cech compactification. Since $\overline{X} \neq X$, $\overline{X}$ is not metrizable (Hewitt). This follows also from the fact that $C(X)$ is not separable, while $C(\overline{X})$ would be if $\overline{X}$ were metric.

Also solved by G. E. Collins, M. K. Fort, Jr., Leonard Gillman, R. C. James, M. J. Mansfield,

Montford Plebstnoch, R. E. Priest, Helmut Salzmann, H. F. Thornton, and L. E. Ward, Jr.,

Editorial Note. A variety of examples were cited by the solvers. See also J. L. Kelley, *General Topology*, Ex. N, p. 104 and Ex. M, p. 164.

A Vanishing Function

4738 [1957, 369]. *Proposed by R. R. Goldberg, Pittsburgh, Pa.*

If, for all positive x, $\sum_{k=1}^{\infty} |F(kx)| < \infty$ and $\sum_{k=1}^{\infty} F(kx)=0$, then $F(x)$ vanishes identically.

Solution by D. S. Greenstein, University of Michigan. Let $S_n=\sum_{k=1}^{\infty} F(knx)$ and let $p_1, p_2, \cdots$, be the sequence of primes in ascending order. Clearly $S_n=0$ for any $n\geqq 1$. Consider

$$
\begin{aligned}
A_n &= S_1 - \sum_{i=1}^{n} S_{p_i} + \sum_{i<j\leq n} S_{p_ip_j} - \sum_{i<j<k\leq n} S_{p_ip_jp_k} + \cdots + (-1)^n S_{p_1\cdots p_n} \\
&= \sum_{(k,p_1\cdots p_n)=1} F(kx) = 0.
\end{aligned} \tag{1}
$$

Now this last sum contains the term $F(x)$ and no other term $F(kx)$ with $k\leqq p_n$. Therefore we have

$$
|F(x)| = |A_n - F(x)| < \sum_{k=p_n+1}^{\infty} |F(kx)|, \tag{2}
$$

but by the absolute convergence of S_n the last member of (2) is arbitrarily small for sufficiently large n. Hence $F(x)=0$.

Also solved by P. T. Bateman, Ward Cheney, D. D. Dix, N. J. Fine, Charles Fox, Joseph Lehner and G. M. Wing, Gideon Peyser, M. F. Smiley, A. D. Ziebur, and the proposer.

Editorial Note. Bateman provides the following example to show that the hypotheses regarding absolute convergence cannot be discarded. Let

$$
F(x) = \begin{cases} 0 & \text{if } x \text{ is not an integer,} \\ \lambda(x)x^{-1} & \text{if } x \text{ is an integer,} \end{cases}
$$

where $\lambda(x)=(-1)^u$, u being the total number of prime factors of x, when multiple factors are counted multiply. If x is irrational, clearly $\sum_{k=1}^{\infty} F(kx)=0$. If $x=a/b$, where a and b are coprime positive integers, then

$$
\sum_{k=1}^{\infty} F(kx) = \sum_{k=1,b|k}^{\infty} F(ka/b) = \sum_{n=1}^{\infty} F(na) = \sum_{n=1}^{\infty} \lambda(na)n^{-1}a^{-1} = \lambda(a)a^{-1}\sum_{n=1}^{\infty} \lambda(n)n^{-1}.
$$

Now, since $\lambda(n)=\sum_{d^2|n} \mu(n/d^2)$, clearly

$$
\sum_{n\leq N} \lambda(n)n^{-1} = \sum_{md^2\leq N} \mu(m)(md^2)^{-1} = \sum_{d^2\leq N} d^{-2} \sum_{m\leq Nd^{-2}} \mu(m)m^{-1}.
$$

Hence

$$
\sum_{n=1}^{\infty} \lambda(n)n^{-1} = 0
$$

by Landau's theorem that $\sum_{m=1}^{\infty} \mu(m)m^{-1}=0$ (*cf.*, Hardy, *Divergent Series*, Oxford, 1949, p. 380). Thus $\sum_{k=1}^{\infty} F(kx)=0$ for any positive x, even though $F(x)$ is not identically zero.

Intersecting Octahedra

4739 [1957, 369]. *Proposed by V. L. Klee, Jr., University of Washington*

Suppose C is a closed convex subset of the Euclidean space E^3 whose boundary is a regular octahedron, and that C_1, C_2, and C_3 are translates of C (*i.e.*, $C_i = C + x_i$ for some $x_i \in E^3$). Then, if each of the intersections $C_1 \cap C_2$, $C_2 \cap C_3$, and $C_3 \cap C_1$ is nonempty, must $C_1 \cap C_2 \cap C_3$ be nonempty?

Solution by Harley Flanders, University of California. The answer is yes. For suppose $C_1 \cap C_2$ and C_3 are disjoint. We note that C_3 is the intersection of the eight closed half-spaces determined by its faces, consequently there must be a plane P which contains a face of C_3 such that $C_1 \cap C_2$ lies entirely in the open half-space determined by P which does not contain C_3. We now choose any plane M perpendicular to P and form the orthogonal projections R_1, R_2, R_3 of C_1, C_2, C_3 on M. The R_i are rhombuses obtained from each other by translation and evidently lie in the impossible configuration in which each two intersect, but $R_1 \cap R_2$ and R_3 are disjoint.

The corresponding result for rhombuses in the plane reduces to a statement about disjoint segments on a line via oblique projection.

The same proof applies when the regular octahedra are merely homothetic.

Editorial Note. Although the proposer had originally an independent argument, he finds the result in a recent paper by Hanner, *Math. Scand.*, vol. 4, 1956, pp. 65–87. If the three-dimensional cartesian space E is metrized by means of the norm $N(x) = |x_1| + |x_2| + |x_3|$ then its unit cell is an octahedron. From the result stated above and another of Hanner's results, it follows that E is 4-hyperconvex but not 5-hyperconvex, in the sense of Aronszajn and Panitchpakdi (*Pacific J. Math.*, vol. 6, 1956, pp. 405–440). This solves part of their Problem 1, p. 437. From another result in Hanner's paper it follows that if a finite-dimensional Banach space is 5-hypercomplex, it is hypercomplex.

Locally Schlicht Mapping

4740 [1957, 370]. *Proposed by R. J. Dickson, Lockheed Aircraft Corporation, Burbank, Calif.*

Is every locally schlicht analytic mapping of the complex plane onto itself a schlicht mapping?

Solution by D. S. Greenstein, University of Michigan. Let $h(z)$ be any nonconstant, even, entire function. Consider the odd, transcendental, entire function

$$f(z) = \int_0^z e^{h(t)} dt.$$

Clearly $f(z)$ is locally schlicht since its derivative has no zeros. Since it is odd and transcendental, it must be an onto mapping by virtue of the Picard theorem. Also, by the Picard theorem, it is not schlicht.

Also solved by Harley Flanders and the proposer. Late solution by I. N. Baker.

Complete Metric Space

4741 [1957, 370]. *Proposed by L. A. Rubel, Institute for Advanced Study*

Prove or disprove the statement: If a metric space S is homeomorphic to its completion, then S is complete.

I. *Solution by Richard Arens, University of California, Los Angeles.* Let S^* be the following set in the plane:

$$S^* = \{(x, y): x = 0, 1, 2, \cdots; y = 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots\}.$$

Let $S=S^*-\{(0, 0)\}$. Then S^* is the completion of S. But the following mapping g, defined on S, rather obviously maps S homeomorphically on S^*:

$$g\left(m, \frac{1}{n}\right) = \left(m - 1, \frac{1}{n+1}\right) \qquad (m > 0),$$

$$g(m, 0) = (m - 1, 0), \qquad g\left(0, \frac{1}{n}\right) = (n - 1, 1).$$

Since S is not complete, the proposed statement is false.

II. *Solution by G. E. Bredon, Harvard University.* Let S be the subset of the plane which consists of the interior of the first quadrant together with the (open) positive x-axis. The completion of this is the closed quadrant and both of these are clearly homeomorphic to the closed half-plane.

Also solved by F. Cunningham, Jr., M. K. Fort, Jr., G. A. Harris, Jr., Meyer Jerison, V. L. Klee, Jr., D. J. Newman, J. R. Schoenfield, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Introduction a L'étude de L'analyse Symbolique. By Maurice Parodi. Gauthier-Villars, Paris, 1957. 246 pp. $10.20.

This is an introductory treatise on the operational calculus using the Laplace transform (the term "symbolic analysis" or "symbolic calculus" for this subject seems not to be used in English). The author has in mind applications of the transform calculus to physics and engineering, and the treatment is pitched for the mathematical technologist rather than for the mathematician. Fortunately this is not one of the books of the kind in which one finds a "proof" that the Heaviside unit step-function has a derivative equal to one at the jump discontinuity. However, one is surprised to find that what is announced as a brief outline of L. Schwartz's theory of distributions turns out to be a standard treat-

ment of the unit impulse (Dirac delta function) by means of the Stieltjes integral (pp. 43–47).

One third of the book consists of a first chapter devoted to the elementary theory of the Laplace transform as a function of a complex variable—in addition to deriving the transforms of many useful functions the text establishes basic properties: analyticity of the transform, behavior at the origin and at infinity, differentiation and integration with respect to a parameter, convolution theorem, Lerch's theorem, the Bromwich-Wagner inversion formula, and the Heaviside expansion theorem. The remainder of the work is devoted to the solution of ordinary and partial differential equations (formally), and brief treatments of the evaluation of definite integrals and the solution of some integral equations. One finds the usual large number of applications to wave propagation, transmission lines, heat flow, elasticity and electric networks. The author's choice of examples well illustrates the range and power of the transform calculus. There are no exercises for the student-reader.

THEODORE HAILPERIN
Lehigh University

Elements of Partial Differential Equations. By Ian N. Sneddon. McGraw-Hill New York, 1957. x+327 pp. $7.50.

Quoting from the author's preface: "The aim of this book is to present the elements of the theory of partial differential equations in a form suitable for the use of students and research workers whose main interest in the subject lies in finding solutions of particular equations, rather than in the general theory."

The above motivation is excellently carried out by introducing the reader to a fairly complete list of techniques to be used in attempting to solve a boundary value problem for a partial differential equation, the techniques being illustrated by examples taken mostly from Laplace's equation, the wave equation, and the diffusion equation.

Each chapter has a fine list of problems which, together with the worked examples, includes many important applications from mathematical physics and engineering. A number of selected results from the general theory is included, with references to more complete treatments. In the same way most proofs are omitted, although those which are supplied are mainly geometrical and fit in well with the intuitive approach used.

The overall impression of the book is that it is somewhat disconnected, due, no doubt, to its goal of providing methods for solving specific problems. It is implicit in the text that the reader has a good grounding in ordinary differential equations including eigenfunction expansions, complex variables and a first course in partial differential equations.

A partial list of topics: Chapter I. Pfaffian differential equations, and integrability conditions. Chapters II, III. Partial differential equations of first and second orders, some theoretical results and special methods of solution. Chapters IV, V, VI. Applications of diverse methods of solving Laplace's equation,

the wave equation and the diffusion equation. Appendix. Systems of surfaces. Answers to odd numbered problems.

GORDON LATTA
Stanford University

The Hypercircle in Mathematical Physics. By J. L. Synge. Cambridge University Press, New York, 1957. xii+424 pp. $13.50.

The aim of this book is to furnish for engineers and mathematical physicists a practical method for obtaining approximate solutions to partial differential equations which is based upon theory available only in pure mathematics. Instead of following the usual procedure of replacing the partial differential equations by a system of equations in finite differences, the author introduces a method by which the solution may be considered as a point in function space, a point of intersection of two linear subspaces orthogonal to each other. Although function space is infinite in dimensionality, use is made of two subspaces each of which has a finite number of points. The points in one subspace are chosen as extremities of vectors satisfying some of the conditions of the problem and other conditions are satisfied by the points (vectors) of the other subspace. Thus, by working in two subspaces, the solution appears as a point lying on the hypercircle of intersection of the subspaces. It is possible to calculate the precision of the result at any stage by finding the radius of the hypercircle. If the radius is zero, an exact solution is obtained.

The concepts of function space employed are developed from the beginning in an extraordinarily lucid manner. Such concepts as orthonormalization, distance in function space as exemplified by the Hilbert, Dirichlet, and Minkowski metrics, minimum property of a normal to a linear n space, and the orthogonality of two linear spaces are carefully explained for later use. A knowledge of elementary calculus and vector analysis should be enough to allow the reader to follow the development. There is a beautiful interplay between geometric and algebraic methods of attack. The author makes use of many figures which serve to convince the reader that a method of proof is often suggested by geometrical intuition. As an aid to understanding many illustrative examples are worked out, and exercises are strategically placed to enable the reader to test his comprehension.

In the one chapter of Part I the geometry of function space without a metric is skillfully portrayed. By the adoption of a positive definite metric throughout the four chapters of Part II enrichment is gained for the geometry of function space by multiplying vectors. In the applications the scalar product or metric is chosen suitable to the boundary value problem dealt with. In Chapter 2 the key to the hypercircle method is revealed after consideration of hyperplanes, hyperspheres, and hypercircles. Chapter 3 considers the Dirichlet problem for a finite domain in the Euclidean plane, the domain of the physical problem being either singly or multiply connected. For a point in function space, a vector field defined over the domain and its boundary is employed. In Chapter 4 a detailed

account is given of the hypercircle method as applied to the torsion of a cylinder, with particular attention to multiply connected cross sections, and the use of pyramid vectors. In Chapter 5, boundary value problems are connected with variational principles, and some examples of mixed boundary-value problems are discussed. In the two final chapters which make up Part III, the geometry of function space with indefinite metric and some applications to vibration problems are considered. Changes in the theory appear because the geometry is now analogous to the Minkowski geometry in the space-time of special relativity. Arithmetical bounds on a solution are no longer available and minimum principles give way to stationary principles.

This book is stimulating for both geometers and analysts, and it must surely be looked into by all who are interested in finding approximate solutions of boundary value problems.

C. E. SPRINGER
University of Oklahoma

Statistical Analysis of Stationary Time Series. By Ulf Grenander and Murray Rosenblatt. Wiley, 1957. 300 pp. $11.00.

This book is a welcome and most useful addition to the Wiley publications in Statistics. It is concerned primarily with random mechanisms producing a process that does not change with t. The probability model considered fits data arising over moderate periods of time in studies of random noise, problems in turbulence and oceanography as well as in some investigations appertaining to meteorology and econometrics.

The monograph gives an excellent and scholarly treatment of this extremely important subject. It fulfills the need for a new approach to time series analysis that differs basically from the many techniques previously used. The authors demonstrate a keen insight into the theory and the various fields of application. They have completed certain research and new results have been obtained. It is also most gratifying to find rigor and clarity in the presentation. The approach here is much more general in scope than that found in most of the earlier work in time series analysis.

There are eight Chapters in the monograph. Chapter I gives the derivation of pertinent results relative to stationary stochastic processes with illustrations taken largely from physical fields. Chapter II discusses several types of linear problems when the covariance function (spectrum) of the stationary process is known. Here one also finds reference to the construction of the optimal linear predictor. In Chapter III we find a practical slant. There is discussed a uniform, very general approach to statistical problems dealing with infinite dimensional models. In Chapter IV the authors discuss various aspects of the problem of estimating the spectral density. Chapter V is devoted to applications of the methods in the study of random noise, turbulence and storm-generated ocean waves. In Chapter VI the authors set up for their results certain statistical tests

of hypotheses or confidence regions. Large sample expressions for bias and the variance of estimates are obtained. Chapter VII gives us a better-than-usual discussion of regression analysis with a discussion of linear unbiased estimates of the coefficients. Chapter VIII contains a very interesting and inspiring assortment of problems on the maxima and zeros of time series and also prediction when the covariance function is estimated from time series. Following Chapter VIII one finds delightful and well chosen problems for each chapter in the monograph. In the reviewer's opinion, these problems with their solutions could well be considered Volume II of the monograph.

This monograph is of use and interest to the physicist, the technologist, the mathematical and applied statistician, the mathematician, and the econometrician. The authors have given the reader a wealth of information in a very clear and concise fashion. In the reviewer's opinion, it will prove to be an excellent text for much-needed courses of instruction in time series analysis as well as to stimulate much needed research in time series analysis. The reader is given ample inspiration for extensions of the analysis. It is important to recognize the nonparametric character of the techniques presented by the authors.

To study and/or read the monograph with understanding, one should have maturity in mathematical statistics and stochastic processes which implies maturity in mathematics.

FRANK M. WEIDA
The George Washington University

BRIEF MENTION

A Guide to Graduate Study. Programs leading to the Ph.D. degree. By Frederic W. Ness, Editor. Association of American Colleges, Washington, D. C., 1957. xi+335 pp. $5.00.

This valuable book should certainly be available to every person who advises senior students planning to continue for graduate study. It contains descriptions of the instructional programs leading to the Ph.D. degree at 135 colleges and universities in the United States, including admission requirements, fees, first year aid programs, fields of study, residence requirements, the number of members of the graduate faculty by department, and enrollments. Physical scientists will be considerably disappointed to note that where 50 or more institutions offer the Ph.D. program in a given field, no specific listing of the school is provided in the "Guide to Fields of Study." Accordingly, chemistry, economics, English, education, chemical engineering, history, mathematics, philosophy, physics and psychology are omitted from this list. They are, of course, listed among the offerings of the individual schools. In examining the book, the reviewer has noted a number of discrepancies between the data given and current actual practice; however, this is no doubt the fault of the persons supplying the data, not of the compiler. The book is a welcome addition.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending News items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SOUTHERN REGIONAL GRADUATE SUMMER SESSION IN STATISTICS

The fifth Southern Regional Graduate Summer Session in Statistics will be held June 16–July 26, 1958, at Oklahoma State University, Stillwater, Oklahoma. The summer sessions are rotated annually among the following institutions: Virginia Polytechnic Institute, Oklahoma State University, University of Florida and North Carolina State College.

The program may be entered at any session, and consecutive courses will be offered in successive summers. The summer work in statistics may be applied towards residence requirements at any one of the cooperating institutions, as well as certain other institutions, in partial fulfillment of residence requirements for graduate degrees. Each summer session lasts six weeks and the several courses offered carry three semester hours of graduate credit. The summer sessions are designed to carry out a recommendation of the Southern Regional Education Board's Committee on Statistics, on which the four institutions initiating the program are represented.

The sessions will be of particular interest to (1) research and professional workers who want intensive instruction in basic statistical concepts and who wish to learn modern statistical methodology, (2) teachers of elementary statistics courses who want some formal training in modern statistics, (3) prospective candidates for graduate degrees in statistics, (4) graduate students in other fields who desire supporting work in statistics, and (5) professional statisticians who wish to keep informed of advanced specialized theory and methods.

The faculty for the 1958 Summer Session will include the following Visiting Professors: H. O. Hartley, Statistical Laboratory, Iowa State College; T. Federer, Biometrics Unit, Cornell University; J. E. Freund, Department of Mathematics, Arizona State College; A. W. Wortham, Operations Research Department, Texas Instruments, Dallas, Texas.

The local staff includes: Professors C. E. Marshall, F. A. Graybill, R. D. Morrison, J. W. Hamblen, Roy Deal, and J. E. Hoffman.

Of particular interest at this summer session will be the six weekly symposia covering six important areas in statistics. They are: Sampling Survey Designs, Experimental Designs, Nonparametric Statistics, Response Curves and Surfaces, Multiple Comparisons, and High Speed Computing. Discussants will be selected from major contributors to these areas. These invited speakers together with the outstanding summer school staff will cover the respective subjects from three points of view: applications, their mathematical bases, and the problems that lie on the frontier.

Inquiries should be addressed to Dr. C. E. Marshall, Director, Statistical Laboratory, Oklahoma State University, Stillwater, Oklahoma.

NSF INSTITUTES FOR SCIENCE AND MATHEMATICS TEACHERS

The National Science Foundation has announced institutes for high school teachers of science and mathematics for the academic year 1958–59. Inquiries concerning admis-

sion to a particular Institute should be sent directly to the Director of that Institute at the address listed. These Institutes and Directors are as follows:

Chicago, University of: Prof. E. P. Northrop, Eckhart Hall, Box 23, University of Chicago, Chicago 37, Illinois. (Mathematics only).

Colorado, University of: Prof. W. E. Briggs, 318 Hellems Annex, University of Colorado, Boulder, Colorado.

Illinois, University of: Prof. Joseph Landin, Department of Mathematics, University of Illinois, Urbana, Illinois. (Mathematics only).

Iowa State Teachers College: Prof. R. A. Rogers, Department of Science, Iowa State Teachers College, Cedar Falls, Iowa.

Michigan, University of: Prof. F. D. Miller, Department of Astronomy, University of Michigan, Ann Arbor, Michigan.

North Carolina, University of: Prof. E. C. Markham, Department of Chemistry, University of North Carolina, Chapel Hill, North Carolina.

Ohio State University: Prof. J. S. Richardson, 208 Communications Laboratory, Ohio State University, Columbus 10, Ohio.

Oklahoma State University: Prof. J. H. Zant, Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma.

Oregon State College: Prof. S. E. Williamson, Department of Science Education, Oregon State College, Corvallis, Oregon.

Pennsylvania, University of: Dean W. E. Arnold, School of Education, University of Pennsylvania, Philadelphia 4, Pennsylvania.

South Dakota, State University of: Prof. C. M. Vaughn, Department of Zoology, Medical and Science Building, State University of South Dakota, Vermillion, South Dakota.

Syracuse University: Dr. A. T. Collette, 400 Lyman Hall, Syracuse University, Syracuse 10, New York.

Texas, University of: Prof. R. C. Anderson, Chemistry Department, University of Texas, Austin 12, Texas.

Utah, University of: Prof. T. J. Parmley, Room 215, Physical Science Building, University of Utah, Salt Lake City 12, Utah.

Virginia, University of: Prof. J. W. Cole, Jr., Department of Chemistry, University of Virginia, Charlottesville, Virginia.

Washington University: Prof. E. U. Condon, Department of Physics, Washington University, St. Louis 5, Missouri.

Wisconsin, University of: Prof. C. H. Sorum, Chemistry Department, University of Wisconsin, Madison, Wisconsin.

SYMPOSIUM ON NUMERICAL APPROXIMATION

A Symposium on Numerical Approximation sponsored by the Mathematics Research Center, U.S. Army, will be held April 20–23 at the University of Wisconsin. The topics include linear approximation, interpolation, Tchebycheff and other extremal approximations, expansions and algorithms.

One-hour surveys (including a survey of recent Russian literature) and thirty-minute research papers will be presented. There will be opportunity for formal and informal discussion. Approximately twenty speakers will participate. Among these are the following guests from abroad: L. Collatz, L. Fox, Z. Kopal, C. P. Miller, A. Ostrowski, and E. L. Stiefel.

Workers in the field interested in attending the Symposium are urged to write to Professor R. E. Langer, Director, Mathematics Research Center, U.S. Army, University of Wisconsin, 1118 W. Johnson Street, Madison 6, Wisconsin.

PERSONAL ITEMS

Associate Professor M. W. Milligan of Adams State College is on leave and has been awarded a Danforth Foundation Teacher Study Grant at Oklahoma State University.

Associated Colleges of Claremont, California: Harvey Mudd College, a new School of Science and Engineering, is a member of the Associated Colleges.

University of Kentucky: Dr. C. J. Scriba, University of Giessen, Germany, has been appointed Research Instructor; Professor H. H. Downing, formerly Head of the Department of Mathematics and Astronomy, has retired with the title of Professor Emeritus.

University of Pennsylvania: Mr. M. V. U. Krishna, University of Southern California, Mr. Kamini Patwary, Graduate Assistant at American University, and Mr. U. V. Ward, Graduate Student at Wayne State University, have been appointed Instructors; Assistant Professor George Butcher has been granted a National Science Foundation Fellowship and is on leave of absence during 1957–1958.

Assistant Professor Smbat Abian of the University of Tennessee has been appointed Assistant Professor at Queens College.

Associate Professor M. A. Al-Bassam, College of Engineering, Baghdad, Iraq, has been appointed Associate Professor at Lamar State College of Technology.

Dr. H. A. Antosiewicz of the National Bureau of Standards is on leave and has accepted a position as Editorial Consultant for Mathematical Reviews, American Mathematical Society, Providence, Rhode Island.

Mr. H. F. Bechtell, Jr., University of Wisconsin, has been appointed Assistant Professor at Grove City College.

Dr. Evelyn Boyd of the National Bureau of Standards has accepted a position as Research Mathematician for the Data Processing Center, Service Bureau Corporation, New York, New York.

Professor Leonard Bristow, University of Santa Clara, has been appointed Assistant Professor at San Jose State College.

Assistant Professor E. E. Capel, University of Miami, has accepted a position at Westinghouse Laboratories, Pittsburgh, Pennsylvania.

Associate Professor D. E. Carscallen, Wabash College, has retired.

Mr. W. H. Colbert, Jr., University of Nevada, has accepted a position as Engineer for Northrop Aircraft Inc., Hawthorne, California.

Mr. R. D. Depew has been transferred by IBM Corporation from Lincoln Laboratory, Lexington, Massachusetts, to Kingston, New York, as Associate Mathematician in Systems Development.

Dr. Gus DiAntonio, Bell Aircraft Corporation, has accepted a position as Senior Engineer for the Martin Company, Baltimore, Maryland.

Mr. H. L. Farris of Black, Sivalls & Bryson, Inc. is now Editorial Assistant for the Society of Exploration Geophysicists.

Mr. N. V. Fellers, Jr., American Institute for Foreign Trade, Phoenix, Arizona, is now associated with Ebasco International Corporation, New York, New York.

Mr. C. T. Fike, University of North Carolina, has accepted a position as Director of the ORACLE Applications Program, Oak Ridge Institute of Nuclear Studies, Oak Ridge, Tennessee.

Dr. L. R. Ford of RAND Corporation has accepted a position as Director of Operations Research, General Analysis Corporation, Sierra Vista, Arizona.

Mr. J. H. Griesmer, Princeton University, has accepted a position as Associate Mathematician for International Business Machines Research Center, Ossining, New York.

Dr. R. L. Helmbold, Carnegie Institute of Technology, has accepted a position as Operations Analyst with Technical Operations, Inc., Fort Monroe, Virginia.

Assistant Professor S. P. Hoffman, Jr., Polytechnic Institute of Brooklyn, has been appointed Assistant Professor at Trinity College, Connecticut.

Dr. M. L. Keedy, University of Nebraska, is now Associate Director of the Junior High School Mathematics Research Study, University of Maryland.

Professor B. C. Keeler, Webb Institute of Naval Architecture, has been appointed Dean of the Faculty.

Mr. C. D. Keim, previously a student at Gannon College, is now Technical Assistant at Lincoln Laboratory, Western Electric Corporation, Lexington, Massachusetts.

Dr. A. B. Lehman, Massachusetts Institute of Technology, has been appointed Research Assistant at Case Institute of Technology.

Dr. Nathaniel Macon of the General Electric Company has been appointed Associate Professor at Alabama Polytechnic Institute.

Associate Professor D. M. Merriell of Robert College has been appointed to an assistant professorship at Santa Barbara College, University of California.

Dr. D. M. Mesner, Purdue University, has accepted a position as Research Associate in the Statistical Engineering Laboratory, National Bureau of Standards, Washington, D. C.

Mr. Duncan Morrill, U. S. Army, has a position as Assistant Engineer for Western Electric Company, Winston-Salem, North Carolina.

Professor C. E. Moulton, Shurtleff College, has been appointed Associate Professor and Chairman of the Department of Mathematics, MacMurray College.

Mr. A. A. Mullin, Massachusetts Institute of Technology, has been appointed a member of the Mathematics Department of the Digital Computer Laboratory, University of Illinois.

Dr. G. M. Petersen, University College of Swansea, has been appointed to an associate professorship at the University of New Mexico.

Dr. R. P. Peterson, Jr., Operations Research, Bank of America, has accepted a position as Mathematician for the Matson Navigation Company, San Francisco, California.

Professor B. J. Pettis, Tulane University, has been appointed Professor at the University of North Carolina.

Professor Donald Everett Richmond, Williams College, is on leave and has been appointed Visiting Professor at Dartmouth College.

Mr. J. N. Rogers, University of Wisconsin, has accepted a position as Analyst with General Electric Company, Vallecitos Atomic Laboratory, Pleasanton, California.

Assistant Professor H. L. Rolf, Baylor University, has been appointed Assistant Professor at Georgetown College.

Dr. Berthold Schweizer, Illinois Institute of Technology, has been appointed Assistant Professor at San Diego State College.

Mr. Malcolm Smith, Motorola, Inc., has accepted a position as Senior Research Engineer at CONVAIR, Pomona, California.

Associate Professor M. R. Spiegel, Hartford Graduate Center, Rensselaer Polytechnic Institute, has been promoted to Professor.

Assistant Professor W. L. Strother, University of Miami, has been promoted to Associate Professor.

Associate Professor D. R. Sudborough, Central Michigan College, has been appointed Assistant Professor at San Jose State College.

Mr. John Todd, National Bureau of Standards, has been appointed Professor at the California Institute of Technology.

Dr. Bryant Tuckerman, Institute for Advanced Study, has accepted a position as Mathematician with the International Business Machines Corporation, Ossining, New York.

Dr. J. B. Tysver, University of Michigan, has accepted a position as Research Specialist at the Boeing Airplane Company, Seattle, Washington.

Dr. J. A. Ward has been made Chief of the Digital Computer Branch, Air Force Missile Development Center, Holloman Air Force Base, New Mexico.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 117 persons have been elected to membership by the Board of Governors on applications duly certified.

PAUL AIZLEY, A.B. (Harvard) Grad. Asst., University of Arizona.

SAMUEL I. ALTWERGER, B.S. (C.C.N.Y.) Head, Dept. of Math., Newton H.S., Elmhurst, N. Y.; New School of Social Research.

WILLARD E. ANDERSON, B.S. (Huron) Grad. Asst., South Dakota State College.

EDWARD J. BARBEAU, JR., Student, University of Toronto.

RALPH E. BEALS, Student, University of Kentucky.

JOHN A. BEEKMAN, M.S. (Iowa) Actuarial Dept., Minnesota Mutual Life Ins. Co., St. Paul, Minnesota.

WILLIAM F. BETHANY, B.S. (Memphis) Instr., Memphis State University.

WILLIAM C. BROWN, B.A. (Queens) Vice President & Actuary, The Colonial Life Ins. Co. of America, East Orange, New Jersey.

ORLANDO L. BYERS, JR., M.S. (Michigan) Chm., Dept. of Math., Edsel Ford H.S., Dearborn, Michigan.

MORTON CHALFIN, Computer, Sylvania Electric Products, Waltham, Massachusetts.

RUSSELL M. COLLINS, JR., B.S. (M.I.T.) Actuarial Dept., Minnesota Mutual Life Ins. Co., St. Paul, Minnesota.

HARVEY J. COOPERSMITH, B.S. (Brooklyn) Instr., Fairleigh Dickinson University.

CARMELO S. CRUZ, B.A. (Puerto Rico) Teacher, Juncos H.S., Juncos, Puerto Rico.

CHARLES H. CUNKLE, Ph.D. (Missouri) Asst. Professor, Colorado State University.

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THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section was held at Georgetown University, Washington, D. C. on December 7, 1957 with 90 members in attendance. Professor R. P. Bailey, Chairman, presided at the meetings.

At the business session the Section considered the matter of inviting travelling lecturers to strengthen its programs. The Section voted to continue its Sectional Mathematics Contest in 1958.

The following papers were presented at the meeting:

1. *Report of the Commission on Mathematics*, by Executive Director A. E. Meder, Commission on Mathematics, New York, New York. (By invitation.)

See this MONTHLY, Report of the November (1957) Meeting of the New Jersey Section.

2. *On a "folk theorem" in probability theory*, by Professor Eugene Lukacs, The Catholic University of America, introduced by the Secretary.

E. J. McShane coined the term "folk theorem" to describe theorems which are known to many people through verbal communication but which have never been published with a formal proof. In this note we discuss such a theorem in probability theory. It is known that the characteristic function of every purely discrete distribution is almost periodic; the "folk theorem" to which we refer asserts that also the converse is true. We prove however a somewhat more restrictive statement: "A distribution function is purely discrete if, and only if, its characteristic function is almost periodic and has an absolutely convergent Fourier series." The validity of the folk theorem depends on the solution to the problem whether the characteristic function $f(t)$ of a singular distribution can be an almost periodic function.

3. *The Maryland-District of Columbia-Virginia Mathematics Contest*, by Professor D. B. Lloyd, District of Columbia Teachers College.

On account of an elaborate award system which it has built up, this section will repeat its own contest program this year. This is its fourth year of successful operation. However the Section will use the test questions of the National Contest Committee and administer the test on the same date. Awards consisting of Savings Bonds, Clock Radios, Slide Rules, and other small items, are contributed by local concerns. Both local, state, and over-all awards are made, as well as certificates to the winner at each school. Committee members are: W. H. Norris, Chairman; Ferrell Atkins, P. L. Chessin, Bro. Dunston Hayden, S. B. Jackson, Fr. Koehler, D. B. Lloyd, Carol McCamman, K. Meals, W. G. Youden.

4. *Vertices of space curves*, by Professor C. S. Wolfe, U. S. Naval Academy.

The purpose of this paper is to show the relationship between various definitions of a vertex and give a necessary and sufficient condition for the existence of a vertex. On a space curve C satisfying certain conditions a point P is defined to be a vertex if and only if the osculating sphere at P has contact of order at least four with C at P. An interesting fact is that this definition is not equivalent to the condition that the radius of the osculating sphere be an extreme value at P. An example is given and the corresponding results in two dimensions are derived.

5. *An equivalent linearization solution of a fourth-order nonlinear system*, by Dr. C. H. Murphy, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

The usual approach to the solution of a fourth-order nonlinear system is handicapped by tedious algebraic manipulations and the fact that the associated phase plane is four dimensional. For the nontrivial case of members of the epicyclic subset, the equivalent linearization technique can be simply applied through the use of a complex variable. The existence of limit motions and the dependence of the type of motion on initial conditions can then be easily studied by means of a two-dimensional "amplitude plane."

6. *Nomographic solutions for position relations between a close earth satellite and its observer*, by Dr. R. H. Wilson, Jr., Naval Research Laboratory, Washington, D. C.

The ground range and azimuth of a close earth satellite are related to the observer's and satellite's latitudes and difference of longitude by the spherical trigonometrical solutions of the astronomi-

cal triangle, for which tables and nomograms are available. Additional alignment charts have been devised for the apparent angular elevation as related to the ground range and linear height of the satellite above the earth; also for its slant range as related to ground range and elevation. Such charts are especially desirable for routine use by nontechnical observers of artificial satellites.

D. B. LLOYD, *Secretary*

CONTINUATION OF THE PROGRAM OF VISITING LECTURERS

The recent grant of the National Science Foundation permits the continuation of this program through the academic year 1958–59.

The Association's Committee on Visiting Lecturers consists of Professors P. B. Johnson, B. W. Jones, D. E. Richmond, Rothwell Stephens and R. A. Rosenbaum, Chairman. The Committee is authorized to select lecturers and to arrange their itinerary. Any correspondence on either of these topics should be addressed to Professor Rosenbaum as Chairman of the Committee.

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, Ball State Teachers College, Muncie, May 3, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS, Kansas State Teachers College, Emporia, April 12, 1958.

KENTUCKY, University of Kentucky, Lexington, April 26, 1958.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Randolph-Macon Woman's College, Lynchburg, Virginia, April 26, 1958.

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April 19, 1958.

MICHIGAN, University of Michigan, Ann Arbor, March 22, 1958.

MINNESOTA, St. John's University, Collegeville, Minnesota, May 17, 1958.

MISSOURI, University of Missouri, Columbia, May 3, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, June 17, 1958 (joint meeting with ASEE, mathematics division).

OHIO, Denison University, Granville, April 26, 1958.

OKLAHOMA, Central State College, Edmond, April 18–19, 1958.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April 18–19, 1958.

UPPER NEW YORK STATE, École Polytechnique and University of Montreal, Montreal, Quebec, Canada, May 10, 1958.

WISCONSIN, Carroll College, Waukesha, May 3, 1958.

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SINGULARITY AND NEAR SINGULARITY IN NUMERICAL ANALYSIS*

GEORGE E. FORSYTHE, Stanford University

1. Introduction. In a number of areas of pure mathematics there is one formula or technique for the *regular* case, and a different formula or technique for the *singular* case. The author feels that from the common-sense standpoint of computing, such a sharp dichotomy between the regular and the singular is unlikely to make sense: there must be an important transition zone of the *nearly singular*. It is the purpose of this note to illustrate this idea with several examples. It turns out, moreover, that the nearly-singular case often cannot be solved accurately with the regular method. A sufficiently accurate treatment of nearly-singular problems may usually be presented in one of two ways: (i) by simply applying the method for a close-by singular problem, accepting the error caused by the change of problem; (ii) by recasting the method for the regular case in a form which converges to the method for the singular case, as the problem becomes singular. Examples of both (i) and (ii) will be given. Clearly (ii) is more desirable since it permits the nearly-singular problem to be solved directly and more accurately.

In addition to the rather special matrix problem of Section 7, one could profitably discuss singularity and near singularity of matrices themselves. Such a discussion would be too long and too technical for the present paper.

2. An elementary integral. For $\epsilon \neq 0$ one has the integration formula

$$\int_1^x \frac{dt}{t^{1+\epsilon}} = \frac{1 - x^{-\epsilon}}{\epsilon} \qquad \text{(REGULAR)}. \tag{1}$$

For the corresponding singular case $\epsilon = 0$,

$$\int_1^x \frac{dt}{t} = \log x \qquad \text{(SINGULAR)}. \tag{2}$$

The formulas (1, 2) represent a typical dichotomy between regular and singular cases. To understand the nearly-singular case (where ϵ is near 0), let us evaluate (1) for $x=2$ and $\epsilon=0.0001$. We first need $2^{-.0001}$. Using 6-place logarithms, we find

$$\log_{10} 2 = 0.301030,$$

$$\log_{10} 2^{-.0001} = -\ 0.00003\ 01030 = -\ 1 + .99996\ 98970.$$

* The author's work on this paper has been sponsored by various agencies of the United States: at The Ramo-Wooldridge Corp., by the Air Force; at the University of California, Los Angeles, by the Office of Naval Research; at Stanford University, by the National Science Foundation. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Hence $2^{-.0001}=0.99993$, and the integral in (1) becomes $(1-0.99993)(.0001)^{-1}=0.7$, a value correct to only one significant digit.

Every computing person would use some device to evaluate $x^{-\epsilon}$ for ϵ near 0. For example,

$$x^{-\epsilon} = e^{-\epsilon \log x} = 1 - \epsilon \log x + \tfrac{1}{2}(\epsilon \log x)^2 + \cdots,$$

whence

$$\epsilon^{-1}[1 - x^{-\epsilon}] = \log x - \tfrac{1}{2}\epsilon(\log x)^2 + \tfrac{1}{6}\epsilon^2(\log x)^3 - \cdots. \tag{3}$$

From (3) one readily finds for $\epsilon=.0001$, $x=2$, that the integral in (1) equals 0.693137.

We may consider (3) to consist of an expansion of (1) into a series consisting of the answer $\log x$ to the singular problem, plus correction terms. That is, the nearly-singular problem (1) is effectively solved by a formula which is continuous at $\epsilon=0$. On the other hand, formula (3) is useful only for comparatively small values of $|\epsilon \log x|$. Thus (3) illustrates method (ii) of Section 1 in part, but there remains the question of making a smooth transition from representation (3) to representation (1) of the integral, as $|\epsilon \log x|$ grows.

It should be noted that, lacking the series (3), one could solve (1) roughly in the nearly-singular case $\epsilon=.0001$ by simply setting $\epsilon=0$ and adopting the singular answer $\log 2=0.693149$. While correct only to 4 decimals, this value is far superior to 0.7. This paragraph illustrates method (i) of Section 1.

3. Another integral. Dr. Morris Weisfeld of The Ramo-Wooldridge Corp. dealt with an integration problem which illustrates the same principle. To simplify the problem, consider that we are to evaluate

$$\phi(\alpha) = \int_0^1 se^{-\alpha s}ds \qquad (0 \leqq \alpha \leqq 1) \tag{4}$$

for the purpose of later forming $\int_0^1\phi(\alpha)d\alpha$. We see from elementary calculus that for $\alpha\neq 0$

$$\phi(\alpha) = -\frac{e^{-\alpha}}{\alpha} - \frac{e^{-\alpha}}{\alpha^2} + \frac{1}{\alpha^2} \qquad \text{(REGULAR)}. \tag{5}$$

while for $\alpha=0$

$$\phi(0) = \tfrac{1}{2} \qquad \text{(SINGULAR)}.$$

The difficulty is that of evaluating $\phi(\alpha)$ from (5) for α near 0, the nearly-singular case. On casual inspection, (5) looks as though $\lim_{\alpha\to 0} \phi(\alpha)$ might not exist, but a brief study of (4) assures us of the continuity of $\phi(\alpha)$ at 0. But certainly (5) is not well suited to computing $\phi(\alpha)$ for α near 0.

The cure is to rewrite (5) as

$$\phi(\alpha) = e^{-\alpha}\left\{\frac{e^{\alpha} - 1 - \alpha}{\alpha^2}\right\}, \tag{6}$$

a form which shows the continuity of $\phi(\alpha)$ at 0. The use of (6) for computation involves the use of readily available tables of $e^{-\alpha}$, and also of some expansion of $(e^{\alpha}-1-\alpha)\alpha^{-2}$ near $\alpha=0$, in analogy with (3).

The point here is that the form (6), suggested by the knowledge that $\phi(0)=\frac{1}{2}$, is a far more useful formula for α near 0 than the formula (5), which is excellent for $\alpha \gg 0$.

4. A differential equation. As a slightly more advanced problem consider the linear differential equation with constant coefficients

$$y''(x) + c_1 y'(x) + c_2 y(x) = 0, \tag{7}$$

to be solved with initial conditions $y(0) = y_0$, $y'(0) = y_0'$.

The text book solution of (7) starts by finding the roots λ_1, λ_2 of the characteristic equation $r^2+c_1 r+c_2=0$. If $\lambda_1 \neq \lambda_2$, one has the basis Y_1, Y_2 of solutions of (7), where

$$Y_1(x) = e^{\lambda_1 x}, \qquad Y_2(x) = e^{\lambda_2 x} \qquad \text{(REGULAR)}.$$

However, if $\lambda_1=\lambda_2=\lambda$ one gets a totally different basis Z_1, Z_2:

$$Z_1(x) = e^{\lambda x}, \qquad Z_2(x) = xe^{\lambda x} \qquad \text{(SINGULAR)}.$$

The further solution of the problem in the regular case is to form the general solution $y(x)=A_1Y_1(x)+A_2Y_2(x)$. Bringing in the initial conditions, we must solve for A_1, A_2 the system

$$A_1 + A_2 = y_0, \qquad \lambda_1 A_1 + \lambda_2 A_2 = y_0', \tag{8}$$

whose determinant is $\lambda_2-\lambda_1$.

If $\lambda_1 \approx \lambda_2$, the solution of (8) is bound to be inaccurate. In such a nearly-singular case we are trying to resolve a function $y(x)$ in terms of two nearly-equal functions $Y_1(x)$, $Y_2(x)$.

On the other hand, when $\lambda_1=\lambda_2=\lambda$ we write $y(x)=B_1Z_1(x)+B_2Z_2(x)$, and it is easy to solve the problem. The equations analogous to (8) are

$$B_1 = y_0, \qquad \lambda B_1 + B_2 = y_0', \tag{9}$$

with determinant 1.

Consideration of the singular case suggests to us that when $\lambda_1 \approx \lambda_2$ the basis Y_1, Y_2 must be replaced by a basis which has the property, like Z_1, Z_2, of not consisting of nearly-equal functions. It is proposed that one use the basis

$$U_1(x) = \frac{1}{2}(e^{\lambda_1 x} + e^{\lambda_2 x}), \qquad U_2(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2}.$$

Introducing the notation $\frac{1}{2}(\lambda_1+\lambda_2)=\mu$, and $\frac{1}{2}(\lambda_1-\lambda_2)=\nu$, we can write

$$U_1(x) = e^{\mu x}\cosh \nu x, \qquad U_2(x) = e^{\mu x}\frac{\sinh \nu x}{\nu}\cdot$$

In the latter form it is clear that, as $\lambda_1\to\lambda$, $\lambda_2\to\lambda$ (and $\mu\to\lambda$ and $\nu\to 0$), we have $U_1(x)\to Z_1(x)$ and $U_2(x)\to Z_2(x)$. The basis U_1, U_2 is thus very well-behaved for the nearly-singular problems where $\lambda_1\approx\lambda_2$.

It may be noted that to deal with $U_1(x)$, $U_2(x)$ we need tables of $e^{\mu x}$, $\cosh \nu x$, and $\nu^{-1}\sinh \nu x$. Of these, only the last function is unusual, and it can be evaluated readily by a Taylor series.

5. Newton's root finder. Let $f(z)$ be an analytic function of the complex variable z, whose zeros are sought. The Newton root-solving process, taught in elementary calculus for real z, is immediately applicable to complex z. One picks some z_1, and for $i=1, 2, \cdots$ one computes

$$z_{i+1} = z_i - f(z_i)/f'(z_i). \tag{10}$$

If $f(a)=0$, if $f(z)$ is analytic at a, and if $|z_1-a|$ is small enough, then it can be shown that $z_i\to a$, as $i\to\infty$. (See Householder [4].)

We repeat an old definition stated, for example, in [5], where Schröder gave a most interesting treatment of Newton's and related algorithms.

DEFINITION. *Given any sequence* $\{z_i\}$ *with limit* a. *If*

$$\lim_{i\to\infty}\frac{z_{i+1}-a}{(z_i-a)^p} = c,$$

where $0<|c|<\infty$, *the convergence of* z_i *to* a *is said to be of order* p.

One usually speaks of *linear* convergence when $p=1$ (for any linear convergence we need $|c|\leqq 1$), *quadratic* convergence when $p=2$, *etc.*

If a is a simple zero of $f(z)$ (*i.e.*, if $f(a)=0$ but $f'(a)\neq 0$), then the convergence of (10), if it takes place, is at least quadratic. In fact, if $z_i\to a$, it will be shown below that

$$\frac{z_{i+1}-a}{(z_i-a)^2}\to\frac{f''(a)}{2f'(a)} = c, \qquad (\text{as } z_i\to a). \tag{11}$$

Thus if $f''(a)\neq 0$, we have precisely quadratic convergence. Quadratic convergence is desirable because it assures us that ultimately the number of correct decimal digits in z_i will approximately double at each step.

The proof of (11) depends on Taylor's expansion of $f(z)$ at z_i in powers of $z-z_i$:

$$f(a) = f(z_i) + f'(z_i)(a-z_i) + \tfrac{1}{2}f''(z_i)(a-z_i)^2 + o(a-z_i)^2 \qquad (\text{as } z_i\to a). \tag{12}$$

From (10), since $f(a)=0$, one finds that

$$\frac{z_{i+1} - a}{(z_i - a)^2} = \frac{f'(z_i)(z_i - a) - f(z_i) + f(a)}{(z_i - a)^2 f'(z_i)} . \tag{13}$$

By substituting for $f(a)$ in (13) from (12), and letting $z_i \to a$, one proves (11).

However, if a is a double zero, $f'(a) = 0$ but $f''(a) \neq 0$. It then follows from (12) that

$$\frac{z_{i+1} - a}{z_i - a} \to \frac{1}{2} \qquad (\text{as } z_i \to a), \tag{14}$$

representing only linear convergence, so that ultimately one gains only one binary digit per step.

From (14) Schröder proposed [5] that near a double zero a one should use a *doubly-relaxed* modification of Newton's process, whereby*

$$z_{i+1} = z_i - 2f(z_i)/f'(z_i). \tag{15}$$

From (15) it can be shown that for a double zero

$$\frac{z_{i+1} - a}{(z_i - a)^2} \to \frac{f'''(a)}{6f''(a)} \qquad (\text{as } z_i \to a). \tag{16}$$

Thus quadratic convergence has been restored to Newton's process for a double zero by the replacement of (10) by (15).

For an n-fold zero a, the corresponding formulas are

$$z_{i+1} = z_i - nf(z_i)/f'(z_i) \tag{17}$$

and

$$\frac{z_{i+1} - a}{(z_i - a)^2} \to \frac{f^{(n+1)}(a)}{n(n + 1)f^{(n)}(a)} \qquad (\text{as } z_i \to a), \tag{18}$$

but we shall here confine our attention to simple and double zeros. Formula (18) seems to be due to Ostrowski (unpublished).

In recapitulation, we have two forms of Newton's process:

$$z_{i+1} = z_i - f(z_i)/f'(z_i) \qquad (\text{REGULAR}), \tag{10}$$

and

$$z_{i+1} = z_i - 2f(z_i)/f'(z_i) \qquad (\text{SINGULAR}). \tag{15}$$

Once again we have a sharp distinction between the regular and the singular cases, and again intuition tells us that there should be cases in between. Is it then possible that there are zeros of analytic functions *intermediate between simple and double*?

---

* The author wishes to thank his former colleague Professor Henrici for mentioning the importance of double relaxation to him.

To study the situation, we shall consider that $f(z)$ is a quadratic polynomial in z with distinct zeros. Without loss of generality let the zeros be -1 and 1 and take $f(z)=z^2-1$. To simplify notation we shall denote z_i by z and z_{i+1} by z'. Then the classical Newton process (10) can be written in the following two equivalent forms:

$$z' = (1/2)(z + (1/z)), \tag{19}$$

$$z' - 1 = (1/2z)(z - 1)^2. \tag{20}$$

From (20) it is clear that for z near 1, we get the quadratic behavior of (11). But for $|z|$ very large, one sees from (19) that $z' \approx \frac{1}{2}z$, and we have at first the linear behavior represented in the limit by (14). We now understand that, speaking anthropomorphically, the pair of zeros looks like a double zero to Newton's process, when seen from a long distance away. The Newton formula is not powerful enough to be able to resolve two zeros until z gets into their general vicinity. This *resolving power* is the key to the transition between simple and double zeros.

One phenomenon appears in (19) which was impossible for a true double zero—that if $z \approx 0$, then $z' \approx \infty$; *i.e.*, the flat region where $z \approx 0$ has a strong repulsion in Newton's method.

The doubly-relaxed Newton process here yields

$$z' = 1/z. \tag{21}$$

Clearly (21) is able to bring us in one step from $z \approx \infty$ to $z \approx 0$, something quite impossible for (19). (That we are then likely to run into the repulsive region $z \approx 0$ is beside the point.)

We have demonstrated that the classical Newton process will have the behavior of linear convergence, whether the zeros are actually simple or multiple, until z is close enough to a zero for the algorithm to be able to "see" it as a separate entity. We also noted that one cure for the slow convergence is to use an overrelaxation suggested by (15) or (17).

To make a more detailed study it would be well to introduce an *overrelaxation factor* ω suggested by Young [7] in another setting; *i.e.*, let the *overrelaxed Newton process* be defined by

$$z_{i+1} = z_i - \omega f(z_i)/f'(z_i), \tag{22}$$

where in general we select $\omega > 0$ as best we can.

For the function $f(z) = z^2 - 1$, the algorithm (22) becomes

$$z' = (1 - (\omega/2))z + (\omega/2z).$$

Hence

$$z' - 1 = (1 - (\omega/2) - (\omega/2z))(z - 1). \tag{23}$$

Assume z is in the right half plane $\operatorname{Re} z \geqq 0$, and that we are aiming for the

root $z=1$. For each z we can define the (real) *optimum overrelaxation factor* $\omega_{\text{opt}}=W(z)$ by the requirement, taken from (23), that

$$|1-(\omega/2)-(\omega/2z)| = \text{minimum}.$$

This is equivalent to requiring that

$$|\omega(1+(1/z))-2| = \text{minimum}. \tag{24}$$

For $z=x$ (real), it is seen that $W(x)=2x/(1+x)$ (x real). Thus, for $1\leqq x<\infty$, one has $1\leqq W(x)<2$. We therefore get a continuous transition from double relaxation ($\omega=2$) to the classical Newton process ($\omega=1$). We might even say that the zero-pair $\{-1, 1\}$ forms a *zero of order* $W(x)$, as seen from the point $z=x$.

For complex z the computation of $W(z)$ is more difficult. If we admitted complex values of ω, then one would have precisely $W(z)=2z/(1+z)$. But in practice it would probably be sufficiently difficult to adjust ω over real values. Moreover, for application to a general function f probably the choice of ω should be a function only of the quantities $f(z)$, $f'(z)$, and z.

It seems to the author that some profitable research in numerical analysis might be directed towards taking the proper advantage of overrelaxation in Newton's process and in related iterations. The importance would be greatest when the computation of $f(z)$ or $f'(z)$ is relatively costly. This occurs, for example, when $f(z)=|A_0+A_1z+\cdots+A_mz^m|$, where the A_μ are large matrices.

6. Aitken's δ^2 process. In iterative computation of matrix eigenvalues by the "power method," Aitken [1] proposes the use of an acceleration scheme now usually known as the "δ^2 process." Let A be a matrix of order n, with eigenvalues λ_i so numbered that $|\lambda_1|\geqq|\lambda_2|\geqq=\cdots\geqq|\lambda_n|$. One selects an arbitrary vector v_0, and forms $v_1=c_1Av_0$, $v_2=c_2Av_1$, $\cdots$. Here one picks the normalizing scalars c_1, c_2, $\cdots$, so that, for example, the first component of each v_k is 1. Let y_k denote some component of v_k, say the rth.

If

$$|\lambda_1| > |\lambda_2|, \tag{25}$$

it is known that the sequence $\{y_k\}$ is convergent and that $\lim_{k\to\infty} y_k=y$ is the rth component of the unique eigenvector of A belonging to λ_1. The difficulty is that the convergence of y_k to y is very slow in many cases, so that too much machine time is consumed in estimating y. As a cure Aitken advocated use of the δ^2 acceleration process.

Suppose $v_0=\beta_1x_1+\beta_2x_2+\cdots$, and assume $\beta_1\neq0$, $\beta_2\neq0$, $|\lambda_2|>|\lambda_3|$. Then

$$\begin{aligned} A^k v_0 &= \beta_1\lambda_1^k x_1+\beta_2\lambda_2^k x_2+o(\lambda_2^k) \\ &= \beta_1\lambda_1^k[x_1+(\beta_2/\beta_1)(\lambda_2/\lambda_1)^k x_2]+o(\lambda_2^k) \qquad (\text{as } k\to\infty). \end{aligned} \tag{26}$$

It follows readily (see Wilkinson [6], for example) that

$$y_k = y + C\mu^k + o(\mu^k) \qquad (\text{as } k \to \infty), \tag{27}$$

where $\mu = \lambda_2/\lambda_1$, and where C is some constant depending on A and v_0. It is therefore usual to regard the sequence $\{y_k\}$ as having the asymptotic behavior of the sequence $\{\eta_k\}: \eta_k = \eta + C\mu^k$, where C is unknown, and to use an acceleration process which would yield the number η exactly from three iterates η_{k-1}, η_k, η_{k+1}. From the last formula it follows that η is equal to ζ_k, where

$$\zeta_k = \frac{\eta_{k-1}\eta_{k+1} - \eta_k^2}{\eta_{k-1} - 2\eta_k + \eta_{k+1}}.$$

Hence Aitken proposes that one take the value

$$z_k = \frac{y_{k-1}y_{k+1} - y_k^2}{y_{k-1} - 2y_k + y_{k+1}} \qquad (\text{REGULAR}) \tag{28}$$

as an approximation to the limit y of the sequence $\{y_k\}$.

It can be shown for the matrix application that asymptotically, as $k \to \infty$, $z_k - y = o(y_k - y)$, so that the δ^2 process is really useful for sufficiently large k, under the hypothesis (25). Nevertheless, for moderate values of k, even when (25) holds, the acceleration is not always very successful in matrix work.

In order to study a possible difficulty, let us examine the sequence y^k when $\lambda_1 = \lambda_2$ is a double eigenvalue with a non-diagonal Jordan canonical block

$$\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}.$$

This may be regarded as a limiting singular case of a close pair of distinct eigenvalues λ_1, λ_2. Define the *principal vectors* x_1, x_2 by the relations

$$Ax_1 = \lambda_1 x_1, \qquad Ax_2 = \lambda_1 x_2 + x_1. \tag{29}$$

Suppose $v_0 = \beta_1 x_1 + \beta_2 x_2 + \cdots$, and assume $\beta_1 \neq 0$, $\beta_2 \neq 0$, $|\lambda_1| > |\lambda_3|$. It is easy to show from the Jordan canonical form and (29) that

$$\begin{aligned} A^k v_0 &= \beta_1 \lambda_1^k x_1 + k\beta_2 \lambda_1^{k-1} x_1 + \beta_2 \lambda_1^k x_2 + o(\lambda_1^k) \\ &= \lambda_1^{k-1} [(\beta_1\lambda_1 + k\beta_2)x_1 + \beta_2\lambda_1 x_2] + o(\lambda_1^k) \\ &\simeq (\beta_1\lambda_1 + \beta_2 k)\lambda_1^{k-1}\left[x_1 + \frac{\beta_2/\beta_1}{1 + (k\beta_2/\lambda_1\beta_1)} x_2\right] + o(\lambda_1^k) \quad (\text{as } k \to \infty). \end{aligned} \tag{30}$$

Thus $A^k v_0$ converges in direction to x_1. But the asymptotic behavior of a component y_k of v_k is no longer that of (27). Instead, from (30) one can see that

$$y_k = y + \frac{c_1}{c_2 + k} + o\left(\frac{c_1}{c_2 + k}\right) \qquad (\text{as } k \to \infty), \tag{31}$$

where c_1, c_2 depend on A but not on v_0.

As before, let us therefore assume that $\{y_k\}$ has the asymptotic behavior of $\{\eta_k\}$, where

$$\eta_k = \eta + \frac{c_1}{c_2 + k}\,. \tag{32}$$

From (32) a straightforward calculation shows that

$$\zeta_k = \frac{\eta_{k-1}\eta_{k+1} - \eta_k^2}{\eta_{k-1} - 2\eta_k + \eta_{k+1}} = \eta + \frac{c_1}{2(c_2 + k)}\,. \tag{33}$$

Thus $\eta = \eta_k + 2(\zeta_k - \eta_k)$, so that to *obtain* η *from* $\{\eta_k\}$ *involves a double relaxation of the* δ^2 *process.*

Hence, returning to the $\{y_k\}$, it seems desirable to estimate y by the formula

$$w_k = y_k + 2(z_k - y_k) \qquad \text{(SINGULAR)}, \tag{34}$$

where z_k is given by (28).

Once more we have a discontinuity between the correct formula for the regular case of a single dominant eigenvalue, and for the singular case of a double dominant eigenvalue with a nondiagonal canonical block. (A double eigenvalue with a diagonal canonical block will behave like a single eigenvalue in the power method.)

As in Section 5, there are probably cases of distinct eigenvalues λ_1, λ_2 for which the singular formula (34) would be preferable to the regular formula (28). It would take a more detailed analysis of the machine representations of (26) and (30) to delimit such almost-singular cases precisely. But, whereas truly double eigenvalues are undoubtedly rare, relative closeness of a pair λ_1, λ_2 occurs often.

When the power method for computing eigenvalues converges slowly, it is normally a signal that $|\lambda_1| \approx |\lambda_2|$. If λ_1 and λ_2 are nevertheless very different, an appropriate small change of A to $A - pI$ will make $|\lambda_1 - p|$ and $|\lambda_2 - p|$ very different, and should improve the rate of convergence enough to permit use of the regular acceleration formula (28). However, if λ_1 is close to λ_2 the shift of origin will not improve the slow convergence. Such a continued slow convergence should suggest the possibility of using the singular acceleration formula (34).

In analogy with the discussion of Section 5, one might introduce an *over-relaxed* δ^2 *process* $w_k = y_k + \omega(z_k - y_k)$, where z_k is given by (28), and where $1 \leqq \omega \leqq 2$. Some numerical experiments might determine how to vary ω so as to speed the completion of the power method.

7. Eigenvectors of a tridiagonal matrix. Other examples of the singular and the nearly singular occur in matrix problems. We shall consider one example drawn from the work of Givens [3].

To compute the eigenvalues and eigenvectors of a real symmetric matrix A, Givens first reduces the problem to that for a tridiagonal matrix A, meaning that $a_{ij}=0$ for $|i-j|>1$. He then finds the eigenvalues λ_k of A by a process which need not be reproduced here. In the next stage, on which we focus our attention, we have an eigenvalue λ of A, and want to find a corresponding eigenvector x. The process considered is to solve the linear system

$$Bx = 0, \tag{35}$$

where B is the tridiagonal matrix $A-\lambda I$ with zero determinant. (Recall that A and B have the same eigenvector system.) Writing (35) out in detail, we have

$$\begin{aligned}
&\alpha_1x_1+\beta_1x_2 &&= 0,\\
&\beta_1x_1+\alpha_2x_2+\beta_2x_3 &&= 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &&\\
&\qquad\beta_{n-2}x_{n-2}+\alpha_{n-1}x_{n-1}+\beta_{n-1}x_n &&= 0,\\
&\qquad\qquad\beta_{n-1}x_{n-1}+\alpha_nx_n &&= 0.
\end{aligned} \tag{36}$$

The usual method of solving (36) is some variant of systematic elimination, and the difficulties to be mentioned below are found somewhere in any such variant. Although perhaps not the best form of elimination, one common method of solving the system (36) is the following.

We consider first the REGULAR case, where no $\beta_i=0$. Let $x_n=1$. Solve the last equation of (36) for x_{n-1}. Using these values of x_n and x_{n-1}, solve the next-to-last equation for x_{n-2}, *etc.* Finally, solve the second equation for x_1. The first equation should be satisfied, and serves to check the solution. The solution involves division by each of the β_i in turn.

In the SINGULAR case (meaning only that certain $\beta_i=0$; the determinant of B vanishes for both our regular and singular cases) we first decompose the matrix B into the direct sum of tridiagonal submatrices for which no $\beta_i=0$. Then at least one of the submatrices has a zero determinant, and in fact Givens' eigenvalue algorithm will have told us which ones. One then applies the above regular method to any such submatrix B_1 to find an eigenvector. Finally, one gets a corresponding eigenvector of A by setting equal to zero each component of x belonging to dimensions not in B_1.

As an example of the singular case, suppose that

$$B = \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 0 & & & \\ & 0 & 1 & 1 & & \\ & & 1 & 1 & 0 & \\ & & & 0 & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}. \tag{37}$$

Since the middle block ($i=3$, 4) has zero determinant, we solve it and find $x_4=1$, $x_3=-1$. Then the corresponding eigenvector of B is

$$x = (0, 0, -1, 1, 0, 0). \tag{38}$$

We thus have algorithms for both the regular case (no $\beta_i=0$) and the singular case (some $\beta_i=0$), and they are distinct. The thesis of this paper is that the boundary between these cases cannot be a sharp one. The question here takes the form: when should a small number be considered zero for practical purposes?

If some of the β_i are nearly zero, the regular method involves division by nearly-zero numbers, and Givens' experience (oral communication) is that often the resulting machine solution of (36) turns out to yield meaningless numbers. To illustrate the difficulty in a simple case, suppose that we alter B in (37) to read

$$B = \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & \epsilon & & & \\ & \epsilon & 1 & 1 & & \\ & & 1 & 1 & \epsilon & \\ & & & \epsilon & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}. \tag{39}$$

If ϵ is very close to 0, the determinant of B is practically 0, and there is an almost-zero eigenvalue of B. In solving the system (36) with the coefficients of (39), the divisions by ϵ cause an excessive magnification of the round-off error. Suppose, for example, that $\epsilon=0.0001$ and that we work to 6 significant figures with all numbers "rounded down" to a smaller absolute value. (We use downward rounding in this simple example to obtain some round-off error and thus to simulate the effect of ordinary round-off in more realistic examples.) One eigenvalue of B turns out to be 0 to more than 7 decimal places. To find the corresponding eigenvector x, we solve (36) recursively and find

$$\begin{aligned} x_6 &= 1.00000, & x_4 &= 30000.0, & x_2 &= -1000.00, \\ x_5 &= -2.00000, & x_3 &= -29999.9, & x_1 &= 2002.99. \end{aligned} \tag{40}$$

We suspect trouble when the values x_1, x_2 fail to satisfy the first equation of (36). When the numbers in (40) are normalized, we find

$$\begin{aligned} x_1 &= .066766, & x_3 &= -.999996, & x_5 &= -.000066, \\ x_2 &= -.033333, & x_4 &= .999999, & x_6 &= .000033. \end{aligned}$$

To 6 decimals, the true eigenvector is

$$\begin{aligned} x_1 &= -.000033, & x_3 &= -1.000000, & x_5 &= -.000067, \\ x_2 &= .000067, & x_4 &= 1.000000, & x_6 &= .000033. \end{aligned} \tag{41}$$

We thus see that for the nearly-singular case $\epsilon=.0001$, the values of x_1 and

x_2 are obtained far more accurately from (38) by putting $\epsilon=0$ than by attempting to apply the ordinary method of solving (36) with limited precision. This is a good example of method (i) of the Introduction.

For a more accurate determination of x in the almost-singular case we must study the singular method to see how to recast the solution of (36) in the regular case. For the present problem the recast method can look like this. We first examine the three minors into which B is *almost decomposed* by the smallness of ϵ, and find that the middle minor has the determinant nearest zero. We therefore solve the third and fourth equations of (36) for x_3 and x_4 (up to a scalar multiple), ignoring the coefficients ϵ in these equations only. Using the values of x_3, x_4, we then determine x_1, x_2 from the first two complete equations (including ϵ), and x_5, x_6 from the last two. We used this method to obtain the solution (41).

In this way one makes symmetrical use of all 6 equations, in contrast with our earlier method. The solution no longer consists of simple recurrences, however, and each block has to be solved as a simultaneous equation system with a tridiagonal matrix. But such systems are comparatively easy to solve in various ways.

If two or more blocks of (39) had both been nearly singular, enough error might have resulted from dropping the ϵ in the third and fourth equations to have warranted an iteration of the above process. That is, having found an approximate eigenvector x, we might have used the values of x_2 and x_5 with the previously ignored ϵ to modify the solution x_3, x_4 of the third and fourth equation obtained previously. With these improved values of x_3, x_4 one would once again have computed x_1, x_2 and x_5, x_6.

The author believes that a development of the above ideas might permit a more accurate solution of any nearly-singular tridiagonal system (36) than is possible by just replacing all small numbers by 0. The important idea is to look at the nearly singular ($\epsilon\approx 0$) in the light of the truly singular ($\epsilon=0$).

Givens has recommended the solution of systems (36) by triangularization with orthogonal transformations of the rows. See Dykaar and La Budde [2].

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UNDERGRADUATE RESEARCH IN MATHEMATICS

KENNETH MAY, Carleton College

A creative mathematician is the intersection of several unlikely events. For the most part we are ignorant of the nature of these events and of their probabilities. Some of them are at present quite beyond our control. An example is the probability of a genetic composition necessary for intensive and highly abstract thinking. Others are clearly subject to our influence. For example, the probability of an early acquaintance with living mathematics and with the joy of mathematical achievement is determined by educational practices. This article describes a program designed to increase this probability by providing unusually able undergraduates with appropriate mathematical experiences.

1. The program in outline. Selected freshmen are invited to enroll in an honors section in which they may remain for their first two years. Students are encouraged to begin original mathematical investigations either independently or in collaboration with other students and faculty members. During the junior and senior years research assistantships are available. A weekly faculty-student colloquium serves as a focus for activity at this level.

2. The context. Carleton College is a coeducational liberal arts college with less than 1,000 students. There is no college requirement in mathematics, but more than half the freshman class takes the elementary course, about half of these take more mathematics, and a good proportion of the most able students major in science. Intellectual interests are respectable, and there is considerable extra-curricular activity in mathematics.*

3. Selection and recruiting. During the summer, folders for incoming freshmen are studied with a view to identifying students with unusual promise in mathematics. Scores in *both* verbal and mathematical aptitude tests of the College Entrance Examination Board have served as one of the most useful criteria. A small group of students with fairly homogeneous background and ability is selected. Letters of invitation are sent so that incoming freshmen receive them when they arrive on campus. These letters explain the nature of the honors section. They make clear that the invitation is independent of the student's intended major and that he may drop out at any time without penalty. The number of students invited has been about ten percent of those taking freshman mathematics. About ten percent of these have declined, usually on the grounds that we have overestimated their mathematical abilities. A very few uninvited students request permission to register for the section. These include students on the borderline as judged by their records, upperclassmen who wish to take

* For further details on the program at Carleton see Report on a Conference on Undergraduate Mathematics Curricula held at Hunter College, New York City, October 12–13, 1956. This report has been circulated to mathematics departments, and additional copies may be obtained from the National Science Foundation, Washington 25, D.C.

elementary mathematics in a more stimulating class, and an occasional "crasher" who grossly overestimates his mathematical ability. It has proved satisfactory to admit a few students in the first two categories, but it is necessary to firmly exclude those in the third.

Research assistants are chosen informally by the entire staff on the basis of demonstrated interest and ability. Assistantships are awarded to nonmajors with the necessary interest and qualifications. We try to draw the student into research projects in a natural way, so that the appointment as a research assistant is a recognition of an already existing situation.

4. Grading. Students in the honors sections are graded in comparison with all the students taking elementary courses. This is achieved in part by including common questions on the examinations and in part by comparing the students with previous college generations in terms of their performance in attacking unusually difficult problems. The average grade in an honors section is expected to be somewhere between an A and a B, usually with more A's than B's being assigned. Students who do not maintain this standard are asked to transfer to another section. They may also transfer at their own request. These two categories have amounted to about 20 percent during or at the end of the freshman year.

5. Subject matter and texts. The usual freshman-sophomore sequence in mathematics consists of an introduction to logic, set theory, and other notions fundamental to modern mathematics, followed by a course in the calculus through manipulative differential equations and the calculus of several variables. The honors sections cover this material with considerable enrichment in three semesters instead of four. The second semester of the sophomore year is devoted to projects chosen by the class. The honors sections use the same textbooks as the regular sections and supplement them by outside reading. We found by experiment that students preferred to use the same textbooks in order to be assured that they were not missing any topic being covered by their classmates in the other sections.

6. Teaching procedures. Experience indicates the importance of keeping in mind that the honors students are unusually able and otherwise very much like their classmates. They have become accustomed in high school to being very obviously the outstanding students in a class. The experience of finding themselves to be merely one of many able students is sometimes traumatic. Most of them have not been accustomed to working hard in order to get top grades, and they may react negatively to the new challenge and the heavy competition. Typically, they are hard-working, careful, accurate, ambitious, and well-adjusted young people. They need to be challenged, but they also need plenty of encouragement and individual attention.

We found it undesirable to spend much class time on routine matters or on drill. Students of this capacity quickly mastered definitions and techniques. It

proved satisfactory to assign a moderate number of graded problems to be done by the student and kept in a loose-leaf notebook which was inspected occasionally. Frequent quizzes served to assure both the students and the teacher that routine matters were being mastered.

In order to encourage independent choice of problems, students were told that they would be *permitted* to hand in one problem a week of their own choice. This might be from the part of the book currently being discussed, from previous or future problem sets, or from any other source. In any case, a student was to select the problem which in his opinion represented his best work for the week. It was pointed out to the class that this manner of judging people was similar to the way in which scientists were judged by their colleagues, namely in terms of their best work. This device was very successful in stimulating work and in encouraging good exposition.

The class was conducted on an informal basis. Some time was taken for lectures introducing supplementary subject matter, but most of the class time was devoted to student exposition of their best work, to discussion of difficult points, and to cooperative effort in attacking difficult problems. Students were expected to meet their teacher frequently outside the class, either singly or in groups.

Examinations typically consisted of a few routine problems followed by a choice among quite difficult and unfamiliar ones. It was found desirable not to give too many choices, because of the time wasted trying to choose among many possibilities. We found that students of this ability could be expected to construct original proofs of unfamiliar statements of the level of difficulty of typical epsilon-delta proofs and of such theorems as the extended law of the mean.

Undoubtedly the most important thing in teaching the honors sections is to take the time to think about the individuals in the class and to work out individual measures to help them develop. This implies, of course, small classes and adequate time outside of class for meditation and student conferences. Probably fifteen is an upper bound to the size of a class in which one can do effective work of this kind.

7. Research projects. Freshmen and sophomores may be encouraged to think of mathematics in terms of creative work by giving them opportunities to develop some class topics on their own. For example, the theory of simplifying the general equation of a conic section by rotation and translation was given as a problem on a final examination. Another example is a term paper in which students were asked to develop the formula for the distance between a point and a straight line by several different methods. Sometimes students may become intrigued by an unsolved problem mentioned by the professor and, if given encouragement, may work on the problem over a long period of time. By mentioning research activities being carried on in the department, the teacher may provide the spark that starts the student studying and thinking about mathematics on his own. Once the student has taken this step, his mathematical

development accelerates sharply. Indeed, the taking of this step marks the transition toward which the program is aimed. The student who has begun to work on his own in this way, rather than as a mere doer of class assignments, still needs encouragement and guidance, but he is now very likely to develop his full capabilities in science. We know of students who continued to work during the entire undergraduate period on a problem presented to them in a freshman class, and who learned a great deal of mathematics in order to be able to attack the problem more effectively.

In addition to the casual mention of current research activities, the department has issued brief descriptions of current research projects and encouraged students to discuss them with the faculty members and students involved. A research assistant who completes a project or who is appointed without having any specific project in advance is expected to work out a program in consultation with members of the staff. His activities may vary from solving difficult problems, such as those given in the Putnam Competition, to carrying on a long-term project. We try to help the student choose problems suitable to his abilities and interests. We are not concerned as to whether results will be publishable. The student who obtains previously published results by doing work that is original for him, gains as valuable an experience as if the work had never been done before. Indeed, the fact that his results have previously been published is often very encouraging to the undergraduate because it assures him that his work is of a quality worthy of publication.

8. The colloquium. Approximately once a week the mathematics staff and research assistants meet informally in a room with comfortable chairs and a blackboard. The programs at the colloquia are varied: an introduction to some research problem, a report on research activities, an expository talk by a visiting mathematician, a joint discussion of a paper or book that has been read by the group, or a cooperative attack on a problem. Participation in the colloquium is open to all interested students and faculty at Carleton College and at neighboring St. Olaf College.

9. Examples of research activity. In a freshman class, the professor presented the famous Tower of Hanoi problem as an example of proof by induction. The number of moves required to transfer the discs from one of three needles to another is well known. The professor dropped the remark that very little was known about the cases in which the number of needles was greater than three. A girl in the class began working on this more general problem. She obtained some results by experimentation and continued to work on the problem off and on during the next year. In her junior year she made further progress on the project by making use of the calculus of finite differences. After her results were presented to the colloquium other students and faculty became interested and further results were obtained. The problem has not been completely solved to date.

Some sophomores visited the colloquium and decided that it would be a nice idea to organize a group of their own during the first semester of the junior year. This group met together and decided to study the calculus of finite differences. They met regularly and worked their way through an elementary textbook. No faculty member ever met with them, and they were not examined or given any credit. However, it was very evident that they had learned a great deal because they later frequently used finite difference methods in attacking problems.

After solving the familiar problem of the sailors on a desert island who divide up a pile of coconuts, one student conceived the idea of generalizing the problem to an arbitrary number of sailors and an arbitrary sequence of discards before each division of the pile. By using difference equations and some ideas from number theory he worked out an algorithm for solving any problem of this kind. His formulation of the problem and his method of attack were conceived without any faculty assistance. A faculty member helped him to polish up some details of the solution. He presented his results at a state meeting of the Mathematical Association.

In a course in number theory, the professor wrote every day on the side board some difficult problems which the students might attack if they became bored. Students who did such problems were invited to present their results to the class. One such presentation inspired a freshman student to generalize the problem and discover new results. His work stimulated a sophomore to discover that the number of representations of an integer as a sum of consecutive integers is equal to the number of odd divisors not including one. The professor collaborated with the students in polishing up the proofs of these results. The later discovery that the result had been published about thirty years ago, far from discouraging the students, confirmed the significance of their original work.

A professor had to calculate some rather large coefficients in connection with some work on the theory of equations. One of the research assistants did these computations for him.

A major in chemistry made a study of the mathematics and chemistry curricula in order to see how more applications to chemistry could be introduced in the mathematics courses and how more mathematics could be utilized effectively in teaching chemistry.

In describing research being carried on in the field of directed graphs, it was mentioned that the only treatise on the subject was in German. One of the research assistants conceived the idea of translating this treatise into English. In carrying out such a project he would, of course, have to learn a good deal about the subject. Another student became intrigued by some of the problems of graph theory and entered into collaboration with a faculty member working in this field.

Whenever students found themselves without a research project or momentarily against a blank wall in a project they had undertaken, it was suggested

that they attempt to solve Putnam Problems. A discussion of solutions in the colloquium usually proved stimulating. Satisfactory solutions were added to a collection of problems and solutions in the departmental library.

10. Staff and finance. A program of this kind is costly in time, since staff members must have time to think about their students, to confer with them, and to undertake the original work involved. If the honors sections are small, those teaching them do not need any special reduction in teaching load. However it is important that faculty members who are working with advanced students have a lighter teaching schedule. The program at Carleton is a four-year experiment, financed by a grant from the National Science Foundation. This grant permits the addition to the staff of one full-time person. This permits two extra sections of freshman and sophomore mathematics and a reduction of teaching load equal to one half a full-time load, which is distributed among the entire staff, since all staff members are expected to participate in the program. We have found that the size of the stipends to research assistants is not very important, because students of this calibre usually already have enough financial aid. We anticipated one or two research assistants per year and allowed about $500.00 per assistant per year. However, in the first year we had seven research assistants and gave them nominal stipends of $100.00 each.

Since research assistants are busy with a regular schedule of classes during the year and do not have ample time to pursue their mathematical interests, it would be desirable to arrange for student-faculty collaboration during the summer months. This was not provided for in our original plans, but we are trying to arrange for it.

As an alternative or supplement to reduced teaching loads, time for work with undergraduates may be made available by relieving faculty members of nonacademic chores. For this purpose adequate secretarial help is desirable.

11. Conclusions. Since the program is only in its second year of operation it is too early to draw conclusions with any certainty. We hope to follow these students' careers as undergraduates and in the future. We may also gain some insight into the subtle motivational factors involved. However, even our brief experience makes it very clear that a program of this kind is very stimulating both to faculty and to students, that it accelerates remarkably mathematical development of able undergraduates, and that the dividends are likely to be large compared to the cost. Of course any program of this kind has to be adapted to the particular institution, students, and faculty. Possibly the only universally necessary features are an enthusiastic staff and adequate time to translate enthusiasm into activity appropriate to individual students.

MÖBIUS TETRADS II*

SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India

5. Some results from Part I. We recall (Sec. 1.1) that $T_0 \equiv ABCD$ and $T_1 \equiv A_1B_1C_1D_1$ are a pair of Möbius tetrads and that AB_1, A_1B, CD_1, C_1D are a set of four skew lines intersecting another set AB, CD, A_1B_1, C_1D_1, forming two systems of generators of a quadric Q_1 that is inscribed as well as circumscribed to both T_0 and T_1, *i.e.*, T_0 and T_1 are m.i.† We denote by u and v the pair of generators of Q_1 that separate harmonically the two pairs of generators of the first system, AB_1, A_1B and CD_1, C_1D. A quadric Q may be constructed (Sec. 2.3) such that both T_0 and T_1 are self-polar for Q. Also, $T_2 \equiv A_2B_2C_2D_2$ is a third tetrad (Sec. 3.1) interlocked with both T_0 and T_1 such that A_2 lies in BCD and $B_1C_1D_1$, B_2 in ACD and $A_1C_1D_1$, C_2 in ABD and $A_1B_1D_1$, D_2 in ABC and $A_1B_1C_1$. It was shown (Sec. 3.3) that T_2 is also self-polar for Q.

6. Sets of tetrads and associated quadrics.

6.1. A fourth tetrad leading to two sets of four m.i. tetrads. Let l_1 and l_2 be lines in $B_2C_2D_2$ common to BCD and $B_1C_1D_1$, respectively; and l_1' and l_2', in $A_2C_2D_2$ common to ACD and $A_1C_1D_1$, respectively. Through the pole of AA_1A_2 for Q, say A_3, where the three planes BCD, $B_1C_1D_1$, and $B_2C_2D_2$ meet (*i.e.*, where l, l_1, and l_2 meet) draw a secant to u and v that will meet l', l_1', and l_2' in their common point where the three planes ACD, $A_1C_1D_1$, and $A_2C_2D_2$ meet, *i.e.*, the pole of BB_1B_2, say B_3. By an argument of Section 3.4 according to which A_3B_3 meets CC_1, DD_1, C_1C_2, D_1D_2, C_2C, and D_2D, it follows that A_3B_3 is the common line of the planes CC_1C_2 and DD_1D_2. Similarly, C_3D_3 is the common line of the planes AA_1A_2, BB_1B_2, and secant to u and v, where C_3 is the pole of CC_1C_2, and D_3, of DD_1D_2 for Q. Now, A_3, B_3 and C_3, D_3 as located are pairs of conjugate points for Q, proving that $A_3B_3C_3D_3$ is a tetrad T_3, self-polar for Q and interlocked (Secs. 3.3, 3.4) with T_i ($i=0, 1, 2$) such that T_i ($i=0, 1, 2, 3$) *form a set of four m.i. tetrads giving rise to another set of four m.i. tetrads, viz.*, $T_A=AA_1A_2A_3$, $T_B=BB_1B_2B_3$, $T_C=CC_1C_2C_3$, $T_D=DD_1D_2D_3$, *having the same vertices and faces as those of the first set.*

The two sets may be called conjugate to each other with the property that each tetrad of either set has one vertex and its opposite face common with each tetrad of the other set. The eight tetrads of the two sets are evidently self-polar for Q.

6.2. Observations. (a) We now observe that the sixteen vertices of the four tetrads of the preceding section lie in the sixteen faces of the same tetrads by sixes that form two self-conjugate triangles for Q, *i.e.*, for the conic section of Q by the face in which they lie. For example, A_1, A_2, A_3, and B, C, D, lie in a face and form self-polar triads for Q. Here we are reminded of a well-known property

* Part I of this paper appeared in this MONTHLY, vol. 64, 1957, pp. 471–478. The sections of Part II are numbered to follow those of the first.

† m.i. denotes mutually interlocked.

(**B-2**, p. 35, ex. 12)* of a pair of self-conjugate triangles for a conic, that is, their vertices will lie on a conic and their sides will touch a conic. Hence, we may infer that *the sixteen vertices of four m.i. tetrads lie by sixes on sixteen conics, one in each face of the same tetrads, having a pair of vertices common with one another* (**B-4**, p. 131). We shall speak of *the sixteen conics as associated with the two conjugate sets of four m.i. tetrads.*

(b) We also observe (reciprocally) that *the sixteen faces of four m.i. tetrads pass through, by sixes, sixteen vertices of the same tetrads and envelope, by sixes, sixteen cones whose vertices are the vertices of these tetrads having a pair of tangent faces common with one another*. We shall speak of *the sixteen cones as associated with the two conjugate sets of four m.i. tetrads.*

This reciprocation with respect to the quadric Q for which the four m.i. tetrads are self-polar brings us back to the same set of tetrads. We may, however, reciprocate with respect to any quadric whatever and observe that the reciprocal set of four tetrads will also be m.i. and that what we have said (reciprocally) above will also hold for this set of four m.i. tetrads. Moreover, whatever property holds for a set of four m.i. tetrads must also hold for any set of four such tetrads.

6.3. Associated quadrics. (a) Let a quadric Q_{1AB} be constructed to pass through the four vertices of T_0, three of T_1, and two, A_2, B_2, of T_2, where T_0, T_1, and T_2 belong to a set of four m.i. tetrads (Sec. 6.1). Since Q_{1AB} is out-polar (**B-3**, p. 52) to Q, it will pass through the fourth vertex of T_1 and two vertices, A_3, B_3, of the fourth tetrad of the set, these being the fourth vertices of the two tetrads T_A and T_B of the conjugate set, self-polar for Q; the other vertices of T_A and T_B are already there. Hence Q_{1AB} passes through four of the sixteen conics of the preceding section *viz.*, $BCDA_1A_2A_3$, $B_1C_1D_1AA_2A_3$, $ACDB_1B_2B_3$, $A_1C_1D_1BB_2B_3$.

We can choose the first two tetrads of a set in six ways and two vertices of a third (implying two vertices of the fourth also) in six ways, proving that *there are thirty-six quadrics out-polar to Q, each passing through four of the sixteen associated conics and twelve vertices of two pairs of tetrads* (*e.g.*, T_0, T_1 *and* T_A, T_B *as above*), *one pair from each of the two conjugate sets of four m.i. tetrads each.*

The symmetry of the result shows that we may choose either set and still have the same thirty-six quadrics. A little calculation shows that nine quadrics of the thirty-six pass through each conic.

(b) We may also state a reciprocal result in the same manner in which (b) was obtained from (a) in Section 6.2. In case reciprocation is made with respect to the quadric Q, the tetrads under consideration reciprocate into themselves, being self-polar for Q. Hence, *the latter thirty-six quadrics will be the reciprocals of the former thirty-six with respect to Q and, therefore, will be inpolar to Q.* In case reciprocation is made with respect to any other quadric, say Q^*, then Q

* **B-i** denotes H. F. Baker, Principles of Geometry, vol. i ($i=1, 2, 3, 4$).

will be replaced in what we have just said by its reciprocal quadric, say Q', with respect to Q^*.

7. Sets of quadrics.

7.1. Enumeration. We made a passing reference at the end of Section 1.1 that there are three quadrics of the type Q_1 for a pair of Möbius tetrads. Let us name the other two, for T_0 and T_1, as Q_1' and Q_1'' generated by BC, AD, B_1C_1, A_1D_1 and CA, BD, C_1A_1, B_1D_1, respectively, Q_1 being the one already defined (Sec. 1.3). *For a set of four m.i. tetrads T_i $(i=0, 1, 2, 3)$, there will therefore be eighteen such quadrics, three for each pair of tetrads and all self-reciprocal for Q* (Secs. 2.3, 2.5). We designate the sets of three quadrics as Q_2, Q_2', Q_2'' : Q_3, Q_3', Q_3'' ; Q_{12}, Q_{12}', Q_{12}''; Q_{23}, Q_{23}', Q_{23}''; Q_{31}, Q_{31}', Q_{31}''; for the five pairs of tetrads T_0, T_2; T_0, T_3; T_1, T_2; T_2, T_3; T_3, T_1; respectively, in the same manner as Q_1, Q_1', Q_1'' were defined above for the sixth pair T_0, T_1.

(b) *These eighteen quadrics of the preceding paragraph may be divided into three sets of six each, viz., Q_{ij}, Q_{ij}', Q_{ij}'', where i, $j=1, 2, 3$, $ij=ji$ for $i\neq j$, and $ij=ji=i$ for $i=j$. Each set of six quadrics has a pair of generators of the same system in common with Q.*

It may be noted here that *these three pairs of generators separate each other harmonically and the three couples of the pairs of four m.i. tetrads of the conjugate set (Sec. 6.1), viz.,*

(1) T_A, T_B; T_C, T_D; (2) T_B, T_C; T_A, T_D; (3) T_C, T_A; T_B, T_D;

are harmonically inverse in pairs with respect to them, respectively (**B-4**, p. 133).

This follows from the fact that if A inverts harmonically (with respect to a pair of generators of a quadric) into B, if B inverts (with respect to another pair of generators of the same system) into C and if C inverts (with respect to a third pair of generators of the same system) into A, then the three pairs of generators separate each other harmonically. For the three pairs of points on the conic section in the plane ABC of the quadric, as intersections of the plane with the above three pairs of generators of the quadric, separate each other harmonically by reason of the behavior of the three pairs of intersections of a conic with the sides of a triangle self-conjugate for it (**B-2**, pp. 27–28, ex. 5).

(c) *For the conjugate set of m.i. tetrads T_i $(i=A, B, C, D)$ we shall similarly have eighteen other quadrics, three for each pair of tetrads, all self-reciprocal for Q.* In this case, the quadrics may be designated by Q_{ij}, Q_{ij}', Q_{ij}'', where $i, j=A, B, C, D$, $i\neq j$, $ij=ji$.

7.2. Further relations. (a) We may observe from their definitions of the quadrics of a set Q_{ij}, Q_{ij}', or Q_{ij}'' of the preceding section that the three of them circumscribing the same tetrad form a pencil of quadrics having two pairs of generators in common and that there are four such pencils.

Hence, if each quadric be represented by a point, the quadrics of the set can be represented by the vertices of a quadrilateral assuming the pencil of quadrics

equivalent to a range of collinear points, the relation being linear in either case.

(b) We notice further that the three diagonal points of each quadrilateral will lead to three distinct quadrics which confirm the harmonic property of the quadrilateral. This is clear from consideration of the nine new quadrics represented by the nine diagonal points of the three quadrilaterals representing the three sets of quadrics Q_{ij}, Q'_{ij}, Q''_{ij}, for either of the two conjugate sets of four m.i. tetrads each. Let us designate the quadrics represented by the diagonal points where the pairs of lines joining the opposite vertices Q_{ij}, Q_k; Q_{ki}, Q_j meet by $Q_{i,jk}$, $i, j, k=1, 2, 3$, $ij=ji$, $ki=ik$, $jk=kj$, $i\neq j\neq k$; where Q_{ij}, Q_{kl}; Q_{jk}, Q_{li} meet by $Q_{ki,jl}$, $i, j, k, l=A, B, C, D$, $ij=ji$, $kl=lk$, $jk=kj$, $li=il$, $ki=ik$, $jl=lj$, $i\neq j\neq k\neq l$.

We should like to note here that *the nine quadrics $Q_{i,jk}$, $Q'_{i,jk}$, $Q''_{i,jk}$ representing the diagonal points of the three quadrilaterals Q_{ij}, Q'_{ij}, Q''_{ij} arising out of a set of four m.i. tetrads are the same as those arising out of the conjugate set, viz., $Q_{ij,kl}$, $Q'_{ij,kl}$, $Q''_{ij,kl}$. The order is such that the three planes represented by the quadrilaterals Q_{ij}, Q'_{ij}, Q''_{ij} of either set, being skew to each other, have each one point in common with each of the three such planes representing the quadrics of the conjugate set. Each plane consists of nine points and seven lines only, viz., the six vertices, the three diagonal points, the four sides, and the three diagonals of the quadrilateral it represents. These nine new quadrics are observed to have two pairs of generators, each common with Q, one pair from either of the three pairs of generators of the same system associated with the eighteen quadrics* (Sec. 7.1) *related to either of the two conjugate sets of four m.i. tetrads each* (*Fig.* 5).

Quadric Q

$Q_{1,23}$	(1) $\equiv$	$Q_{AB,CD}$; $Q'_{1,23}$	(1′) $\equiv$	$Q_{BC,AD}$; $Q''_{1,23}$	(1″) $\equiv$	$Q_{CA,BD}$
$Q_{2,31}$	(2) $\equiv$	$Q'_{AB,CD}$; $Q'_{2,31}$	(2′) $\equiv$	$Q'_{BC\,AD}$; $Q''_{2,31}$	(2″) $\equiv$	$Q'_{CA,BD}$
$Q_{3,12}$	(3) $\equiv$	$Q''_{AB,CD}$; $Q'_{3,12}$	(3′) $\equiv$	$Q''_{BC,AD}$; $Q''_{3,12}$	(3″) $\equiv$	$Q''_{CA,BD}$

FIG. 5

(c) Again, we observe that *the thirty-six quadrics associated with the two conjugate sets of four m.i. tetrads* (Sec. 6.3) *also arrange themselves* (*on their representation by points*) *into two sets of six planes each. Each plane is determined by a quadrilateral of six quadrics represented by its vertices such that the planes of a set are skew to each other and have each point common with each plane of the other set, the sets of planes being formed according as we count the quadrics passing through the six pairs of tetrads of either of the two conjugate sets of four m.i. tetrads.*

(d) To be more explicit we may put down the scheme of the two sets of six planes as Q_{1ij}, Q_{2ij}, Q_{3ij}, Q_{12ij}, Q_{23ij}, Q_{31ij}, where $i, j=A, B, C, D$, $i\neq j$, $ij=ji$.

Each subscript ij represents the six quadrics of a plane of one set and each numeral subscript represents those of a plane of the second set for the six values of ij. Figure 6 illustrates the scheme of a plane of either set.

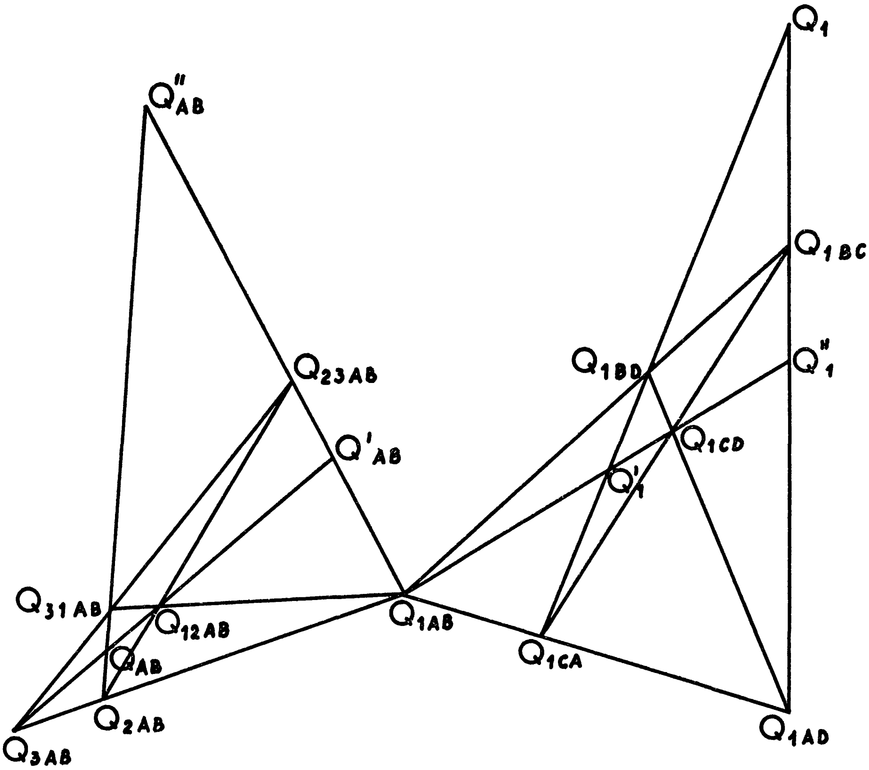

FIG. 6

The diagonal points of the six quadrilaterals of either set are observed to represent the eighteen quadrics Q_{ij}, Q'_{ij}, Q''_{ij}, for the corresponding set of four m.i. tetrads. This suggests that we may add to each set of six planes, three more planes, viz., those determined by $Q_{i,jk}$, $Q'_{i,jk}$, $Q''_{i,jk}$, respectively, to one set and those determined by $Q_{ij,kl}$, $Q'_{ij,kl}$, $Q''_{ij,kl}$, to the other. We then have two sets of nine planes such that the planes of each set, being skew to each other, have each one point common with each plane of the other set.

For the net of quadrics Q_1, Q_1', Q_1'' of Section 7.1 and the four pairs of Möbius tetrads of Section 1.2, reference should be made to two papers by W. L. Edge.*

My thanks are due to the referee for the present form of the paper and to Professor B. R. Seth for his continual encouragement to carry on with my research.

* The net of quadric surfaces associated with a pair of Möbius tetrads, Proc. London Math. Soc. (2), vol. 41, 1936, pp. 337–360; The contact net of quadrics, *ibid.*, vol. 48, 1943, pp. 112–121.

ON A THEOREM OF HÖLDER

DONALD W. MILLER, University of Wisconsin and University of Nebraska

1. Introduction. A well-known result, due to Hölder [1], is the following: The symmetric group S_n has outer automorphisms if and only if $n=6$. The classical proof of the existence of a class of outer automorphisms of S_6, as formulated by Burnside [2], rests in part on the theory of primitive groups and entails extensive computation. In this note we offer a direct method for constructing such automorphisms.

The author is grateful to Professor R. H. Bruck for raising this problem and for subsequent helpful remarks.

2. Construction of an outer automorphism of S_6. Let S_6 be defined on the set $M=\{1, 2, 3, 4, 5, 6\}$; let I denote the identity of S_6. Call two elements of S_6 *disjoint* if no element of M is displaced by both of them.

Define the mapping ψ by: $(1\ 2)\psi=(1\ 2)(3\ 6)(4\ 5)=P_2$, $(1\ 3)\psi=(1\ 3)(2\ 4)(5\ 6)=P_3$, $(1\ 4)\psi=(1\ 4)(2\ 6)(3\ 5)=P_4$, $(1\ 5)\psi=(1\ 5)(2\ 3)(4\ 6)=P_5$, $(1\ 6)\psi=(1\ 6)(2\ 5)(3\ 4)=P_6$. Write $N=\{2, 3, 4, 5, 6\}$, $\mathcal{P}=\{P_i \mid i\in N\}$. Note that the elements of $\mathcal{P}$ include as factors the 15 distinct transpositions of S_6; consequently $\mathcal{P}$ is transitive on M. Moreover, for $i, j, k\in M$, $i\neq j$,

$$P_i^2 = I, \qquad kP_i \neq kP_j, \qquad iP_j \neq i.$$

Note that $iP_j=jP_i$ implies $i=j$. For if $iP_j=jP_i=k$ then $P_i=(1\ i)(j\ k)(r\ s)$, $P_j=(1\ j)(i\ k)(r\ s)$, so $i=j$. Also, $P_iP_j=(i\ \ j\ \ jP_iP_j \cdots (1\ \ iP_j\ \ iP_jP_iP_j \cdots$. Hence $(jP_iP_j)P_iP_j$ equals i or 1. But in the latter case $jP_iP_j=1P_jP_i=jP_i$, whereas P_j fixes no element of M. Thus P_iP_j has order three, so $P_iP_jP_i=P_jP_iP_j$, all $i, j\in N$.

If i, j, k are distinct elements of N, then

$$iP_j = jP_k = kP_i \tag{1}$$

cannot hold. For, if so, write $iP_j=q$ and $N=\{i, j, k, q, r\}$. Now $q=fP_r$ for some f in M. Certainly f is not one of i, j, k, or q. But if $f=r$ then $q=rP_r=1$, contradicting $i\neq j$.

If P_i, P_j, P_k are distinct elements of $\mathcal{P}$, then

$$(P_iP_kP_j)P_i = P_j(P_iP_kP_j). \tag{2}$$

It is sufficient to prove that P_k commutes with $P_iP_jP_i$, for then $P_kP_iP_jP_i = P_iP_jP_iP_k$, $P_iP_kP_iP_jP_i = P_jP_iP_k$, $P_iP_kP_jP_iP_j = P_jP_iP_kP_jP_j$, $P_iP_kP_jP_i = P_jP_iP_kP_j$. Now

$$Q = P_iP_jP_i = P_jP_iP_j = (1\ \ iP_jP_i)(i\ \ jP_i)(j\ \ iP_j).$$

Each of the three transpositions of Q is a factor of some P_k, $k\neq i, j$. If Q should have two cycles in common with some P_t then $Q=P_t$. But in that case the dis-

played representation of Q would yield $iP_j=jP_t$, $iP_jP_i=t$ (so $iP_j=tP_i$), whence $tP_i=iP_j=jP_t$, contradicting (1). (Thus we can write $Q=(a\ b)(c\ d)(e\ f)$, $P_k=(a\ b)(c\ f)(d\ e)$. But then $QP_k=(c\ e)(d\ f)=P_kQ$.

If $A_1, \cdots, A_n, B, C$ are distinct elements of $\mathcal{P}$, then

$$B(CA_1 \cdots A_nB) = (CA_1 \cdots A_nB)C. \tag{3}$$

If $n=1$, (3) follows from (2). Assume inductively that (3) holds for n; then

$$\begin{aligned} &B(CA_1 \cdots A_nA_{n+1}B) \\ &= B(CA_1 \cdots A_nB)(BA_{n+1}B) = (CA_1 \cdots A_nBC)(A_{n+1}BA_{n+1}) \\ &= (CA_1 \cdots A_nA_{n+1})(A_{n+1}BCA_{n+1})BA_{n+1} = (CA_1 \cdots A_nA_{n+1})(BCA_{n+1}B)BA_{n+1} \\ &= (CA_1 \cdots A_nA_{n+1}B)C. \end{aligned}$$

Further, if $A_1, \cdots, A_n, B, C$ are distinct elements of $\mathcal{P}$, then

$$CB(A_1 \cdots A_n)B = B(A_1 \cdots A_n)BC. \tag{4}$$

For by (3), $CBA_1 \cdots A_nB = BC(BCA_1 \cdots A_nB) = BC(CA_1 \cdots A_nBC)$ $=B(A_1 \cdots A_nBC)$.

Define the mapping θ as follows. Let $a_1, \cdots, a_n$ be distinct elements of N and write $(1\ a_i)\psi=A_i$. Then set

$$\begin{aligned} I\theta = I, \qquad (1a_1 \cdots a_n)\theta = A_1 \cdots A_n, \\ (a_1a_2 \cdots a_n)\theta = A_nA_1A_2 \cdots A_n, \qquad (QR)\theta = (Q\theta)(R\theta), \end{aligned} \tag{5}$$

where Q, R are arbitrary disjoint cycles of S_6. By (3),

$$(a_1a_2 \cdots a_n)\theta = A_1A_2 \cdots A_nA_1.$$

Clearly θ maps S_6 into itself.

To show that θ is single-valued it will be sufficient to establish that if $Q=(a_1 \cdots a_m)$, $R=(b_1 \cdots b_n)$ are arbitrary disjoint cycles in S_6, then

(i) $(QR)\theta=(RQ)\theta$;
(ii) $(a_1a_2 \cdots a_m)\theta=(a_2a_3 \cdots a_ma_1)\theta$.

If Q displaces 1 then $Q\theta$ is uniquely defined; if not, (ii) follows from (3). As to (i), suppose without loss of generality that R does not displace 1; then $R\theta$ is of the form $BA_1 \cdots A_nB$, so by successive applications of (4), $(QR)\theta=(Q\theta)(R\theta)$ $=(R\theta)(Q\theta)=(RQ)\theta$.

For arbitrary elements Q, R of S_6, $(QR)\theta=(Q\theta)(R\theta)$. To prove this it is sufficient to consider the case where R is a transposition (since every element of S_6 is a product of transpositions). If Q and R are disjoint the asserted relation is trivial. Hence we write Q as a product of disjoint cycles and let Q' denote the product of those factors of Q which are not disjoint from R. We need to show that $(Q'R)\theta=(Q'\theta)(R\theta)$.

Let $1, e, f, a_1, \cdots, a_m, b_1, \cdots, b_n$ denote distinct elements of M.

(i) If $Q' = (1\ a_1 \cdots a_m)$, $R = (1\ b_1)$, then $(Q'\theta)(R\theta) = A_1 \cdots A_m B_1 = (1\ a_1 \cdots a_m\ b_1)\theta = (Q'R)\theta$.

(ii) If $Q' = (e\ a_1 \cdots a_m)$, $V = (e\ b_1 \cdots b_n)$, then $(Q'\theta)(V\theta) = (EA_1 \cdots A_m E) \cdot (EB_1 \cdots B_n E) = EA_1 \cdots A_m B_1 \cdots B_n E = (e\ a_1 \cdots a_m\ b_1 \cdots b_n)\theta = (Q'V)\theta$.

(iii) If $Q' = (1\ a_1 \cdots a_m\ e\ b_1 \cdots b_n)$, $R = (1\ e)$, with $m, n \geqq 0$, then $(Q'\theta)(R\theta) = A_1 \cdots A_m(EB_1 \cdots B_n E) = A_1 \cdots A_m B_n E B_1 \cdots B_n = [(1\ a_1 \cdots a_m) \cdot (e\ b_1 \cdots b_n)]\theta = (Q'R)\theta$.

(iv) If $Q' = (1\ a_1 \cdots a_m)(e\ b_1 \cdots b_n)$, $R = (1\ e)$, then $(Q'\theta)(R\theta) = A_1 \cdots A_m E B_1 \cdots B_n E E = A_1 \cdots A_m E B_1 \cdots B_n = (1\ a_1 \cdots a_m\ e\ b_1 \cdots b_n)\theta = (Q'R)\theta$.

(v) If $Q' = (e\ a_1 \cdots a_m\ f\ b_1 \cdots b_n)$, $R = (e\ f)$, with $m, n \geqq 0$, then by (4), $(Q'\theta)(R\theta) = (EA_1 \cdots A_m F B_1 \cdots B_n E)(EFE) = (EA_1 \cdots A_m)(FB_1 \cdots B_n FE) = (EA_1 \cdots A_m)(EFB_1 \cdots B_n F) = [(e\ a_1 \cdots a_m)(f\ b_1 \cdots b_n)]\theta = (Q'R)\theta$.

(vi) If $Q' = (e\ a_1 \cdots a_m)(f\ b_1 \cdots b_n) = Q_1' Q_2'$, $R = (e\ f)$, then, by (ii), $(Q'\theta)(R\theta) = (Q_1'\theta)(Q_2'\theta)(R\theta) = (Q_1' Q_2' R)\theta = (Q'R)\theta$.

θ is an automorphism of S_6. Indeed, the kernel, K, of θ is a normal subgroup of S_6, so K is one of S_6, A_6, $\{I\}$, where A_6 denotes the alternating group of degree 6. But $[(3\ 6)(4\ 5)]\theta = (3\ 6)(4\ 5)$, so $K \neq S_6$, $K \neq A_6$. Therefore $K = \{I\}$ so θ is 1-1 and hence an automorphism.

Finally, θ is outer since $(1\ 3\ 5)\theta = (1\ 2\ 6)(3\ 5\ 4)$, whereas if θ were inner it would map every conjugate class of S_6 onto itself. This completes the proof.

We observe in conclusion that *all* outer automorphisms of S_6 are obtainable with the aid of the above construction. Indeed, as shown by Hölder [1], the automorphism group of S_6 has order $1440 = 2(6!)$; thus the group, $\mathfrak{J}$, of inner automorphisms is of index 2 in the full automorphism group. Hence if θ is any outer automorphism of S_6 then the right coset $\mathfrak{J}\theta$ includes all outer automorphisms of S_6.

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THE MOTION OF AN ELECTRIC BELL

HAJIMA AKASHI, Ritsumeikan University, Kyoto, Japan AND SHELDON LEVY, Midwest Research Institute, Kansas City, Missouri

1. Introduction. One of the oldest electromechanical devices is the electric bell; however, very little reference is made to its mathematics. The simplicity of the analysis involved in determining the motion of the bell's clapper provides an easily understood example of the use of the phase-plane in solving nonlinear ordinary differential equations. Emphasis is placed on the exposition of phase-plane techniques to students of calculus, differential equations and physics.

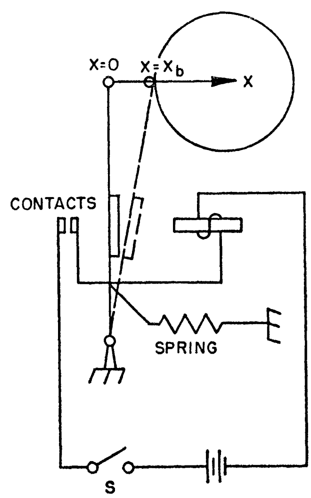

FIG. 1

The equations of motion of the bell are simplified by neglecting damping, by assuming the force due to the magnet has only two possible values, O and F, and by treating the rod carrying the clapper as being weightless and rigid, Figure 1. The equation of motion of the bell is:

$$(1) \qquad m\ddot{x} + kx = \begin{cases} 0 & \text{when} \quad i = 0, \\ F & \text{when} \quad i = i_0 \neq 0 \end{cases}$$

where i is the current flowing through the electromagnet.

Furthermore, the restriction that the clapper cannot penetrate the body gives

$$x \leqq x_b. \tag{2}$$

The bell is designed so that the current is interrupted by a contact mounted on the clapper arm. We shall denote by x_c and x_0 the values of x for which the contacts make and break, respectively.

2. Analysis of the motion. The motion governed by (1) is best analyzed by introducing the nondimensional coordinates u and t defined by:

$$u(\tau) = kx(t)/F \qquad \tau = \omega\tau, \qquad \omega = \sqrt{k/m}. \tag{3}$$

The equation of motion now has the form:

$$u'' + u = 0, \qquad i = 0, \tag{4a}$$

$$u'' + u = 1, \qquad i = i_0, \tag{4b}$$

where $' = d/d\tau$; with the restriction

$$u \leqq u_b = kx_b/F. \tag{5}$$

Multiplying by u' and integrating gives

$$(u')^2 + u^2 = \text{const}, \qquad i = 0 \tag{6a}$$

$$u'^2 + (u - 1)^2 = \text{const}, \qquad i = i_0. \tag{6b}$$

These are equations of circles with centers at (0, 0) and (0, 1) in the u', u-plane and are called the *trajectories of the motion*, Figure 2. The u', u-plane is called the phase-plane. The radius of the circles is proportional to the kinetic energy, and the dimensionless time is measured by the arc length of the trajectory in radians.

Consider the clapper at rest, where $u' = u = 0$, and at $t = 0$, the switch S is closed completing the circuit, Figure 1. The initial motion is described by the trajectory $O-A$ in Figure 2. When the clapper reaches x_0 or $u = u_0$, the current is interrupted and the motion of the bell is described by the homogeneous equation (4a). The trajectories are now circles about the origin with the motion following the trajectory $A-B$ striking the bell. The collision must be inelastic if any sound is to be emitted and the velocity, u_c', with which the clapper leaves the bell, will be given by:

$$u_c' = -eu_b', \qquad 0 \leqq e < 1 \tag{7}$$

where u_b' is the velocity at B, and e is the coefficient of restitution. The motion continues from C to D, $u = u_c$, at which point the circuit is completed and again the motion is described by circles about the point (0, 1). The motion has completed a "cycle" when it has traveled the trajectory $OABCDO'$. A cycle can be completely described by its initial point (0, u) or the radius of the initial arc of the motion. If the motion starts at the point (0, u_m) with initial radius r_m, then, after a complete cycle, the radius has the value r_{m+1} given by:

$$r_{m+1}^2 = e^2 r_m^2 + (1 - e^2)(1 + u_b^2 - 2u_0) + 2(u_0 - u_c). \tag{8}$$

The limit cycle of the motion is defined as the cycle which the motion follows after infinite time and is characterized by $r_{m+1}/r_m \to 1$ as $m \to \infty$.*

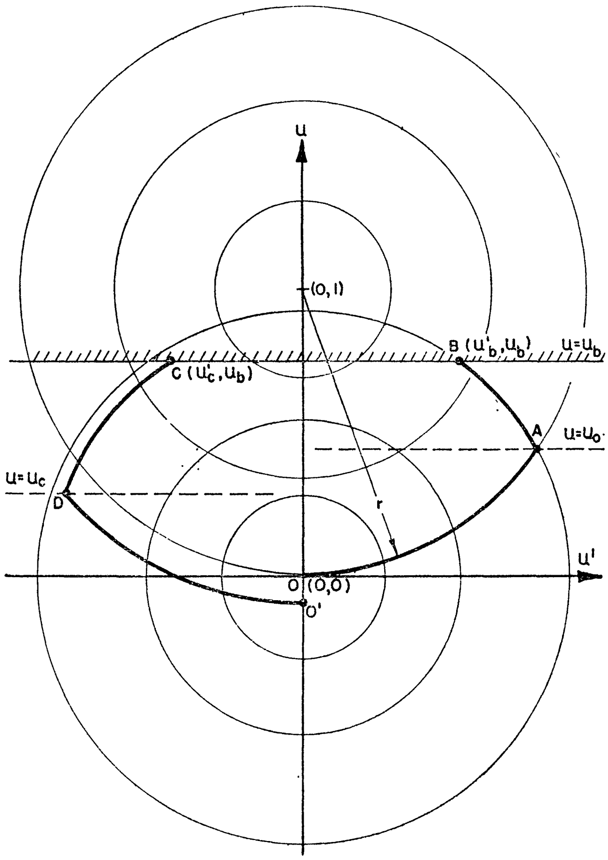

Fig. 2

In the limit cycle the radius is

$$r^2 = (1 + u_b^2 - 2u_0) + 2(u_0 - u_c)/(1 - e^2). \tag{9}$$

The motion of the bell shall now be analyzed for several special cases.

---

* As the referee comments, the solution of (8) can be written

$$r_m^2 = (1 + u_b^2 - 2u_0) + 2(u_0 - u_c)/(1 - e^2) + ce^{2m},$$

which also shows the limit cycle exists.

Case I. Ideal switch on clapper arm, $u_0 = u_c$. In this case the motion "spirals" inward until the motion is bounded above by $u = u_b$ for which position $u' = 0$. This is shown graphically in Figure 3. In this case the bell starts clanging loudly, the intensity decreases, motion and the time interval between clangs is shortened. In the final state the bell does not sound, although the clapper continues to vibrate. Furthermore, energy is supplied to the system only during the *O—A* portion of the first cycle, and when this energy has been converted to sound, there is no longer a "ringing bell."*

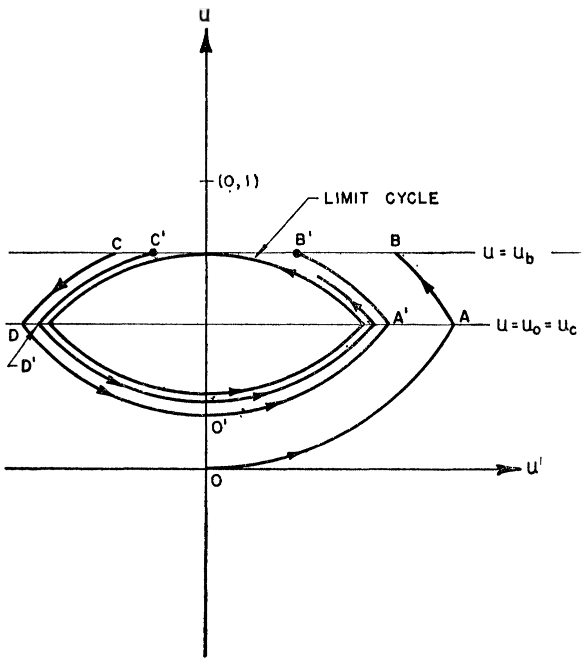

FIG. 3

Case II. Imperfect switch on clapper arm, $u_0 > u_c$. The motion of the bell when $u_0 > u_c$ yields a limit cycle as is shown in Figure 4*. Electrical energy supplied during the time interval 1, 2 of Figure 4 amounts to $E = \text{force} \times \text{distance} = u_0 - u_c$. This is precisely the energy converted into sound per cycle $E = \frac{1}{2}(u_c'^2 - u_b'^2)$ in the limit cycle.

Many more very basic physical facts can be computed from this simple example. Introduction of air damping, friction damping and nonlinear force functions, while distracting in this note, are easily treated.

* It is possible to supply insufficient energy to have any sound produced.

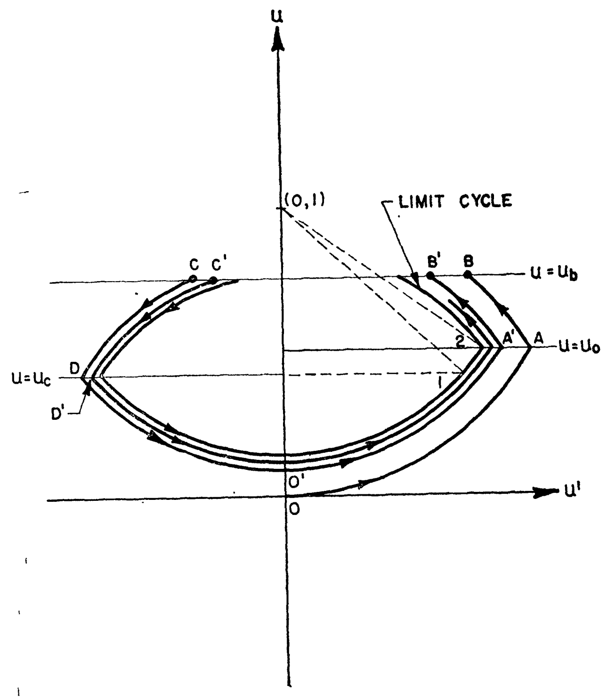

FIG. 4

NOTE ON SOME SEMIMODULI OF A RECTANGULAR MATRIX

C. R. MARATHE, Muslim University, Aligarh, India

DEFINITION. If $\|A\|$ is a nonnegative function of the elements of a rectangular complex matrix $A=(a_{ij})$ such that (i) $\|A\|=0$ if and only if $A=0$; (ii) $\|sA\|=|s|\cdot\|A\|$ for all scalars s; (iii) $\|A+B\|\leqq\|A\|+\|B\|$ where A and B are of the same size; (iv) $\|AB\|\leqq\|A\|\cdot\|B\|$ where A and B are conformable; then we shall say that $\|A\|$ is a *semimodulus* of A.

NOTATION. Let $[N(A)]^2=\sum_i\sum_j|a_{ij}|^2$. Write $R_i(A)=\sum_s|a_{is}|$, $R(A)=\max_i R_i(A)$; $C_j(A)=\sum_s|a_{sj}|$, $C(A)=\max_j C_j(A)$. Denote by $c_{\min}(H)$ and $c_{\max}(H)$ the minimum and the maximum ch. value (read characteristic value) of the hermitian matrix H. Write $|A|^0=[c_{\max}(A^*A)]^{1/2}$, $|A|_0=[c_{\min}(A^*A)]^{1/2}$ where A^* is the conjugate transpose of A. By $c(A)$ we shall denote *any* ch. value of the (square) matrix A. For a complex number α, $\operatorname{Re}\alpha$ and $\operatorname{Im}\alpha$ will stand for the real and the imaginary part of α. Every square complex matrix P can be expressed as $P=P_1+iP_2$ where P_1 and P_2 are hermitian. We shall say that P_1

and P_2 are the *hermitian components* of P. This decomposition of P into its hermitian components is unique. Indeed, $P=\frac{1}{2}(P+P^*)$, $P=\frac{1}{2}i(P-P^*)$.

THEOREM 1. *$N(A)$, $R(A)$, $C(A)$ and $|A|^0$ are semimoduli of A.*

Proof. It is known that $N(A)$ satisfies (i), (ii), (iii) and (iv). So $N(A)$ is a semimodulus. Consider $R(A)$. Conditions (i), (ii) and (iii) clearly hold. To prove (iv):

$$
(1)\qquad \begin{aligned} R_i(AB) &= \sum_s \left| \sum_k a_{ik}b_{ks} \right| \leqq \sum_k \sum_s | a_{ik} | \cdot | b_{ks} | \\ &= \sum_k | a_{ik} | \cdot R_k(B) \leqq R(B)R_i(A), \quad \text{where} \quad B = (b_{ij}). \end{aligned}
$$

Since (1) is true for all i, we get $R(AB) \leqq R(A)R(B)$. Thus $R(A)$ is a semimodulus. Conditions (i), (ii) and (iii) also hold for $C(A)$. Also $C(A)=R(A')$, where A' is the transpose of A. Now $C(AB)=R(B'A')\leqq R(A')R(B')=C(A)C(B)$. So $C(A)$ is a semimodulus of A. Consider $|A|^0$. For a $(n\times n)$ hermitian matrix H and a unit-length column vector $u=\{u_1, \cdots, u_n\}$ it is known that $c_{\min}(H) \leqq u^*Hu \leqq c_{\max}(H)$. Since A^*A is hermitian we get for $|u|=1$,

$$
(2)\qquad | A |_0^2 \leqq u^*A^*Au = | Au|^2 \leqq | A |^{0^2} \quad \text{or} \quad | A |_0 \leqq | Au| \leqq | A |^0.
$$

Also for two column vectors $u=\{u_1, \cdots, u_n\}$ and $v=\{v_1, \cdots, v_n\}$

$$
(3)\qquad | u + v| \leqq | u| + | v| \quad \text{and} \quad | u^*v| \leqq | u| \cdot | v| .
$$

Further, we note that there exist unit-length ch. vectors x, y of A^*A such that the lower and upper bounds in (2) are attained, *i.e.*, $|A|_0^2=x^*A^*Ax$ and $|A|^{0^2}=y^*A^*Ay$. Now conditions (i) and (ii) hold for $|A|^0$. To prove (iii): Let x be a unit-length ch. vector of $(A+B)^*(A+B)$ such that $|A+B|^0=|(A+B)x|$. Such a vector exists as we observe above. Hence we have $|A+B|^0=|(A+B)x| = |Ax+Bx| \leqq |Ax|+|Bx| \leqq |A|^0+|B|^0$ by (2) and (3). This proves (iii). To prove (iv): Consider the unit-length ch. vector y of $(AB)^*AB$ such that $|AB|^0 = |ABy|$. Two cases arise. Case (1): $By=0$. Then $|B|^0=0$. Also $ABy=0$ and so $|ABy|=0=|AB|^0$. Hence (iv) holds. Case (2): $By\neq 0$. Let $By=z=k\bar{z}$ where $\bar{z}$ is of unit length. Now $0<k=|z|=|By|\leqq |B|^0$ by (2). Hence $|AB|^0=|ABy| = |Az|=k|A\bar{z}|\leqq k|A|^0\leqq |B|^0.|A|^0$. Thus $|A|^0$ is a semimodulus and the theorem is proved.

THEOREM 2. *If X, Y, A are matrices and α a scalar such that $\alpha X=AY$ and $\|X\|=\|Y\|\neq 0$, then $|\alpha| \leqq \|A\|$.*

Proof. $\|\alpha X\| = |\alpha| \cdot \|X\| = \|AY\| \leqq \|A\| \cdot \|Y\|$. But $\|X\|=\|Y\|$ and is nonzero. So $|\alpha| \leqq \|A\|$.

COROLLARY 2.1. $|c(A)| \leqq \|A\|$.

Proof. There exists a nonnull ch. vector x such that $\alpha x=Ax$ where $\alpha=c(A)$.

Take $X=Y=x$ in Theorem 1 and observe that $\|x\|\neq 0$ and the result follows.

COROLLARY 2.2. *If* $S=\Sigma_k A_k$, $|c(S)|\leqq \Sigma_k\|A_k\|$.

Proof. $\|S\|=\|\Sigma_k A_k\|\leqq\Sigma_k\|A_k\|$ by (iii) while by Corollary 2.1, $|c(S)|\leqq\|S\|$, which proves the result.

COROLLARY 2.3. *If* $P=\prod_k A_k$, *then* $|c(P)|\leqq\prod_k\|A_k\|$.

Proof. By (iv) $\|P\|\leqq\prod_k\|A_k\|$. So $|c(P)|\leqq\prod_k\|A_k\|$.

In this corollary the matrices A_k need not all be square. Their product P need only be a square matrix.

In Theorem 1 we have mentioned some instances of semimoduli of a matrix. The above corollaries together with Theorem 1 bring out a number of known results. We mention a few by way of illustration. Take $\|A\|$ to be $R(A)$ and $C(A)$. Then Corollary 2.3 gives, when $P=\prod_k A_k$,

$$|c(P)|\leqq\min\left[\prod R(A_k),\ \prod C(A_k)\right]. \tag{4}$$

Now take $\|A\|$ to be $|A|^0$. Then

$$|c(P)|^2\leqq\prod\left[c_{\max}(\overset{*}{A}_k A_k)\right]. \tag{5}$$

It is possible to obtain a lower bound for $|c(P)|$ in (5). It can be seen that $|A|_0$ is not a semimodulus of A. For, although $|A|_0$ satisfies (i) and (ii), it may not always satisfy (iii) and (iv). But $|A|_0$ satisfies (iii)$'$: $|A+B|_0\leqq\min\left[|A|^0+|B|_0,\ |A|_0+|B|^0\right]$ and (iv)$'$: $|AB|^0\geqq|AB|_0\geqq|A|_0\cdot|B|_0$. To see this, let x be unit-length ch. vector of B^*B such that $|B|_0=|Bx|$. Now $|A+B|_0\leqq|(A+B)x|=|Ax+Bx|\leqq|A|^0+|B|_0$ by (2). Similarly, $|A+B|_0\leqq|A|_0+|B|^0$. This proves (iii)$'$. To prove (iv)$'$: Let y be a unit-length ch. vector of $(AB)^*AB$ such that $|AB|_0=|ABy|$. Hence $|AB|_0=|Az|=k|A\bar{z}|\geqq k|A|_0\geqq|B|_0\cdot|A|_0$, where $k=|By|=|z|=k|\bar{z}|$, $\bar{z}$ is a unit-length vector, k being zero if $z=0$. So (iv)$'$ is satisfied. Hence the following theorem is proved.

THEOREM 3. *If* $P=\prod_k A_k$ *is a square matrix then*

$$\prod_k c_{\min}(\overset{*}{A}_k A_k)]\leqq|c(P)|^2\leqq\prod_k\left[c_{\max}(\overset{*}{A}_k A_k)\right].$$

The matrices A_k *need not be all square.*

The above result is a generalization of some results due to Roy [5], Nagy [4], Afriat [1] and others. The following particular cases deserve attention.

COROLLARY 3.1. *When* AB *is a square matrix,*

I $c_{\min}(A^*A)c_{\min}(B^*B)\leqq|c(AB)|^2\leqq c_{\max}(A^*A)c_{\max}(B^*B)$.

II *When* A *and* B *are normal of the same order,*

$$|c(A)|_{\min}|c(B)|_{\min} \leqq |c(AB)| \leqq |c(A)|_{\max}|c(B)|_{\max}.$$

Proof. Part I follows from Theorem 3. To get the second part we note that when P is normal the ch. values of P^*P are the square of the moduli of the ch. values of P, *i.e.*, $|P|^0=|c(P)|_{\max}$ and $|P|_0=|c(P)|_{\min}$.

Inequality (4) may be improved by a result due to Barankin [2]. We shall only consider the case when P is a product of two matrices.

COROLLARY 3.2. *If AB is a square matrix then*

$$|c(AB)|^2 \leqq \min \left[C(A)R(B) \max_i R_i(A)C_i(B),\ C(B)R(A) \max_j R_j(B)C_j(A)\right].$$

Proof. From (1) we have $R_i(AB) \leqq R_i(A)R(B)$. Similarly it can be shown that $C_i(AB) \leqq C(A)C_i(B)$. By Barankin's result

$$|c(AB)|^2 \leqq \max_i R_i(AB)\cdot C_i(AB). \tag{6}$$

Hence

$$|c(AB)|^2 \leqq C(A)R(B) \max_i R_i(A)C_i(B).$$

Also,

$$|c(BA)|^2 \leqq C(B)R(A) \max_j R_j(B)C_j(A).$$

Since the ch. values of AB and BA are the same the result is proved.

LEMMA 1. Re $c(A) \leqq \|A_1\|$; Im $c(A) \leqq \|A_2\|$, *where A_1 and A_2 are the hermitian components of A.*

Proof. By a result due to Bromwich [3] we get Re $c(A) \leqq c_{\max}(A_1)$; Im $c(A) \leqq c_{\max}(A_2)$. But by Corollary 2.1, $|c_{\max}(A_1)| \leqq \|A_1\|$; $|c_{\max}(A_2)| \leqq \|A_2\|$.

THEOREM 4. *If A and B are complex square matrices of the same order then*

$$\operatorname{Re} c(A \pm B) \leqq \|A_1\| + \|B_1\|, \qquad \operatorname{Im} c(A \pm B) \leqq \|A_2\| + \|B_2\|.$$

Proof. $A \pm B = A_1 \pm B_1 + i(A_2 \pm B_2) = S = S_1 + iS_2$, say. Thus S_1 and S_2 are the hermitian components of S. So

$$\operatorname{Re} c(S) = \operatorname{Re} c(A \pm B) \leqq \|S_1\| = \|A_1 \pm B_1\| \leqq \|A_1\| + \|B_1\|,$$
$$\operatorname{Im} c(S) = \operatorname{Im} c(A \pm B) \leqq \|S_2\| = \|A_2 \pm B_2\| \leqq \|A_2\| + \|B_2\|.$$

COROLLARY 4.1.

(a) $\operatorname{Re} c(A + B) \leqq \frac{1}{2}[R(A + A^*) + R(B + B^*)]$,

(b) $\operatorname{Im} c(A + B) \leqq \frac{1}{2}[R(A - A^*) + R(B - B^*)]$,

(c) $\operatorname{Re} c(A + B) \leqq \frac{1}{2}[|c(A + A^*)|_{\max} + |c(B + B^*)|_{\max}]$,

(d) $\operatorname{Im} c(A + B) \leqq \frac{1}{2}[|c(A - A^*)|_{\max} + |c(B - B^*)|_{\max}]$.

Proof. (a) and (b) follow since $R(A)$ is a semimodulus. A similar result will also hold for $C(A)$. Results (c) and (d) follow from the fact that $|A|^0$ is a semimodulus and, when P is normal, $|P|^0 = |c(P)|_{\max}$.

THEOREM 5. *If $AB=BA$ then*

$$\operatorname{Re} c(AB) \leqq \|A_1\|\cdot\|B_1\| + \|A_2\|\cdot\|B_2\|, \qquad \operatorname{Im} c(AB) \leqq \|A_1\|\cdot\|B_2\| + \|A_2\|\cdot\|B_1\|.$$

Proof. $AB = A_1B_1 - A_2B_2 + i(A_1B_2 + A_2B_1)$ while we have $BA = B_1A_1 - B_2A_2 + i(B_2A_1 + B_1A_2)$. Since $AB = BA$ we have $A_1B_1 - A_2B_2 = B_1A_1 - B_2A_2 = (A_1B_1 - A_2B_2)^*$. So $P_1 = A_1B_1 - A_2B_2$ is hermitian. Similarly, $P_2 = A_1B_2 + A_2B_1$ is hermitian. Hence if $P = AB$ then P_1 and P_2 are the hermitian components of P and the result follows by Lemma 1.

COROLLARY 5.1. *When A and B commute*

$$\operatorname{Re} c(AB) + \operatorname{Im} c(AB) \leqq [\|A_1\| + \|A_2\|]\cdot[\|B_1\| + \|B_2\|].$$

COROLLARY 5.2. *When A and B commute*

(a) $\operatorname{Re} c(AB) \leqq \frac{1}{4}[R(A + A^*)R(B + B^*) + R(A - A^*)R(B - B^*)]$,

(b) $\operatorname{Im} c(AB) \leqq \frac{1}{4}[R(A + A^*)R(B - B^*) + R(A - A^*)R(B + B^*)]$,

(c) $\operatorname{Re} c(AB) \leqq \frac{1}{4}[\,|c(A + A^*)|_{\max}|c(B + B^*)|_{\max} + |c(A - A^*)|_{\max}|c(B - B^*)|_{\max}]$,

(d) $\operatorname{Im} c(AB) \leqq \frac{1}{4}[\,|c(A + A^*)|_{\max}|c(B - B^*)|_{\max} + |c(A - A^*)|_{\max}|c(B + B^*)|_{\max}]$.

Proof. The proof is similar to that of Corollary 4.1.

My thanks are due to Professor S. M. Shah for his suggestions.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

L'HOSPITAL'S RULE FOR COMPLEX-VALUED FUNCTIONS

D. S. CARTER, University of California, Los Alamos Scientific Laboratory

L'Hospital's rule for real functions may be stated in the form:

Let $f(x)$ and $g(x)$, and their derivatives $f'(x)$, $g'(x)$, be continuous on an open interval $(0, a)$, and let

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$$

*If the ratio f'/g' is defined on $(0, a)$ (in the sense that g' does not vanish) and has a finite limit at $x=0$, then the ratio of functions, f/g, is defined and has the same limit.**

A simple example shows that this rule is not generally valid when f and g are complex-valued functions of a real variable. Taking $f=x$, $g=xe^{-i/x}$, we have

$$f'/g' = \frac{xe^{i/x}}{x+i},$$

which vanishes as $x\to 0$, while $f/g=e^{i/x}$ has no limit. The more pathological example $f=x$, $g=x(e^{-i/x}-1)$ shows that f/g need not even be defined.

However, by placing additional restrictions on the functions, we can obtain generalizations of the rule which cover many cases. For example, the rule is valid provided

I. *The ratio $|g'|/|g|'$ is defined (in the sense that the derivative $|g|'$ of $|g|$ exists and does not vanish) and bounded on $(0, a)$.*

This is the simplest of a class of conditions obtained by introducing the difference, $\delta(x)$, between f'/g' and its limit

$$L = \lim_{x\to 0} f'(x)/g'(x).$$

Multiplying the equation $\delta(x)=f'(x)/g'(x)-L$ by $g'(x)$ and integrating, we have

$$f(x) - Lg(x) = \int_0^x \delta(y)g'(y)dy,$$

in which the integral may be improper. On dividing by $g(x)$, it is clear that f/g has the limit L provided

* This rule is equally valid if f'/g' has an infinite limit at $x=0$, since the conditions on f and g are the same. This is not true of the conditions derived below for complex-valued functions. Here the roles of f and g must be interchanged explicitly to cover cases of infinite limits.

II. *The ratio*

$$R(x) \equiv \left[\int_0^x \delta(y)g'(y)dy\right] \Big/ g(x)$$

is defined on $(0, a)$ *and vanishes as* $x \to 0$. We will show that each of the two following conditions, as well as I, implies Condition II and is therefore sufficient for validity:

III. *The ratio* $r(x) \equiv |\delta(x)||g'(x)|/|g(x)|'$ *is defined on* $(0, a)$ *and vanishes as* $x \to 0$, *where* $|g|'$ *is the derivative of* $|g|$.

IV. $|g(x)|$ *is monotone and the real and imaginary parts of* $\delta(x)$ *are of bounded variation on* $(0, a)$.

The first step in the proof is to show that g cannot vanish on $(0, a)$. For Conditions I and III this follows immediately from Rolle's theorem and the fact that $|g|$ is continuous and vanishes at $x=0$, while $|g|'$ is defined and does not vanish. For Condition IV it follows from the monotonicity of $|g|$: If $g(\xi)=0$, $0<\xi<a$, then g vanishes identically on $(0, \xi)$, which contradicts the hypothesis that $g' \neq 0$.

To show that III implies II, we note that $|g|'$ is given by $|g|'=(g\bar{g}'+g'\bar{g})/2|g|$, where the complex conjugate $\bar{g}$ enjoys the same continuity properties as g. Hence $|g|'$ is continuous; and if we define $r(0)=0$, it follows that $r(x)$ is continuous on the closed interval $[0, a/2]$. Let ρ be an upper bound for r on that interval. Then

$$\int_\epsilon^x |\delta(y)| \, |g'(y)| \, dy \leqq \rho \int_\epsilon^x |g(y)|' dy < \rho \, |g(x)| \, ,$$

and on taking the limit $\epsilon \to 0$ we see that

$$\int_0^x |\delta(y)| \, |g'(y)| \, dy$$

is defined, continuous, and has the continuous derivative $|\delta||g'|$ on $(0, a/2)$. Hence the ratio

$$R_1(x) = \left[\int_0^x |\delta(y)| \, |g'(y)| \, dy\right] \Big/ |g(x)|$$

satisfies the conditions of l'Hospital's rule for real functions. Applying the rule to R_1, we find that Condition II is satisfied, since $|R| \leqq |R_1|$.

Condition I clearly implies III, since $\delta(x)$ vanishes as $x \to 0$. Hence I implies II.

To show that IV implies II, we integrate the numerator of $R(x)$ by parts and use the monotonicity of $|g|$ to obtain the estimate

$$|R(x)| \leqq |\delta(x)| + \int_0^x |d\delta(y)|.$$

Hence R vanishes with x and II is satisfied.

Similar arguments show that Conditions I, III, and IV are also valid for the Stolz extension of l'Hospital's rule, in which the conditions

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$$

are replaced by

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} |g(x)| = \infty.$$

ON A DETERMINANTAL INEQUALITY

MARVIN MARCUS,* University of British Columbia

Recently L. K. Hua [2] proved the following interesting inequality:

Let A and B be n-square complex matrices and assume

$$I - A^*A \quad \text{and} \quad I - B^*B \tag{1}$$

are both positive semidefinite. Then

$$|d(I - A^*B)|^2 \geqq d(I - A^*A)d(I - B^*B) \tag{2}$$

where d is the determinant and A^* is the conjugate transpose of A.

We prove here an extension of the inequality (2). Let λ_j, α_j, β_j be respectively the eigenvalues of $I-A^*B$, A^*A and B^*B so indexed that

$$|\lambda_j| \geqq |\lambda_{j+1}|, \qquad \alpha_j \geqq \alpha_{j+1}, \qquad \beta_j \geqq \beta_{j+1} \qquad \text{for } j = 1, \cdots, n-1.$$

THEOREM. *If $I-A^*A$ and $I-B^*B$ are both positive semidefinite then for each k satisfying $1 \leqq k \leqq n$,*

$$\prod_{j=1}^{k} |\lambda_{n-j+1}|^2 \geqq \prod_{j=1}^{k} (1 - \alpha_j)(1 - \beta_j). \tag{3}$$

We first establish an inequality.

LEMMA. *If u and v are complex n vectors and*

$$\|u + v\| \leqq 2 \tag{4}$$

then

$$|1 - (u, v)|^2 \geqq (1 - \|u\|^2)(1 - \|v\|^2). \tag{5}$$

* This work was completed under a National Research Council-National Bureau of Standards Postdoctoral Research Associateship. The author also wishes to acknowledge the helpful suggestions of Dr. Olga Taussky-Todd.

Proof. Let $u \wedge v$ denote the Grassmann exterior product of u and v. Then it is known ([4], p. 63) that

$$\|u \wedge v\|^2 = \|u\|^2\|v\|^2 - |(u, v)|^2. \tag{6}$$

Now we assert that in general

$$\|u \wedge v\|^2 \leqq \tfrac{1}{4}\|u - v\|^2\|u + v\|^2. \tag{7}$$

For set $x = u + v$, $y = u - v$ and note that

$$u \wedge v = \left(\frac{x + y}{2}\right) \wedge \left(\frac{x - y}{2}\right) = (1/4)(x \wedge x - x \wedge y + y \wedge x - y \wedge y)$$
$$= (1/2)(y \wedge x).$$

Hence by (6)

$$\|u \wedge v\|^2 = (1/4)(y \wedge x, y \wedge x) = (1/4)(\|y\|^2\|x\|^2 - |(x, y)|^2)$$
$$\leqq (1/4)\|x\|^2\|y\|^2 = (1/4)\|u - v\|^2\|u + v\|^2.$$

It follows from (4) and (7) that

$$\|u\|^2\|v\|^2 - |(u, v)|^2 \leqq \|u - v\|^2 = \|u\|^2 + \|v\|^2 - [(u, v) + (v, u)].$$

Hence

$$1 - [(u, v) + (v, u)] + |(u, v)|^2 \geqq 1 - (\|u\|^2 + \|v\|^2) + \|u\|^2\|v\|^2$$

and this is precisely the inequality (5). Proceeding to the proof of (3) we next note that if x is a vector with $\|x\| = 1$ then by (1)

$$\|Ax + Bx\| \leqq \|Ax\| + \|Bx\|$$
$$= (A^*Ax, x)^{1/2} + (B^*Bx, x)^{1/2} \leqq 2.$$

Hence by (5) with $u = Ax$ and $v = Bx$

$$\begin{aligned} |([I - A^*B]x, x|^2 &= |1 - (Bx, Ax)|^2 \geqq (1 - \|Ax\|^2)(1 - \|Bx\|^2) \\ &= (1 - (A^*Ax, x))(1 - (B^*Bx, x)) = ([I - A^*A]x, x)([I - B^*B]x, x). \end{aligned} \tag{8}$$

By a theorem of Schur [3] there is an orthonormal set of vectors $x_1, \cdots, x_k$ for which $([I - A^*B]x_j, x_j) = \lambda_{n-j+1}$ for $j = 1, \cdots, k$. Then from (8) we have

$$\begin{aligned} \prod_{j=1}^{k} |\lambda_{n-j+1}|^2 &= \prod_{j=1}^{k} |(I - A^*B]x_j, x_j)|^2 \\ &\geqq \prod_{j=1}^{k} ([I - A^*A]x_j, x_j)([I - B^*B]x_j, x_j). \end{aligned} \tag{9}$$

A theorem of Ky Fan [1] states that *if H is positive semidefinite Hermitian then*

$$\prod_{j=1}^{k} (Hx_j, x_j) \geqq \prod_{j=1}^{k} h_{n-j+1} \tag{10}$$

where $h_1 \geqq \cdots \geqq h_n$ *are the eigenvalues of* H *and* $x_1, \cdots, x_k$ *is a set of* k *orthonormal vectors.* We see then that (3) follows immediately from (9) and (1), and the proof is complete.

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APPROXIMATING A CIRCULAR SEGMENT BY USE OF DIOPHANTINE EQUATIONS

J. S. Frame, Michigan State University

The purpose of this note is to obtain a simple rational approximation for the ratio $G=S/T$ of the area S of a segment of a circle to the area T of the inscribed isosceles triangle, as a function of the ratio $h=H/R$ of the height H of the segment to the radius R of the circle. This approximation is uniformly more accurate for $0<h\leqq 1$ than the approximation $22/7$ for π. Yet it involves few terms or factors and only relatively small integral coefficients.

Let $2\theta(\leqq\pi)$ be the radian measure of the central angle that subtends the bounding arc and bounding chord of the given segment. Then the arc length and chord length are $2R\theta$ and $2R\sin\theta$, respectively, and the arc-chord ratio is the function $F=\theta/\sin\theta$. The areas S of the segment and T of the inscribed isosceles triangle, and their common height Rh are

$$S = R^2(\theta - \sin\theta\cos\theta), \qquad T = R^2\sin\theta(1-\cos\theta), \qquad Rh = R(1-\cos\theta). \tag{1}$$

The segment-triangle ratio G and the arc-chord ratio F are the following functions of h:

$$S/T = G(h) = 1 + (F-1)/h, \tag{2}$$

$$F(h) = \theta/\sin\theta = 2\arcsin(h/2)^{1/2}/(2h-h^2)^{1/2}. \tag{3}$$

To obtain a series for F in powers of h we may differentiate $(\sin\theta)F-\theta$ with respect to θ and replace $\cos\theta$ by $1-h$. We find

$$(\sin^2\theta)(dF/dh) + (\cos\theta)F - 1 = 0, \qquad (2h-h^2)F' + (1-h)F - 1 = 0, \tag{4}$$

$$2hF' + F - 1 = h^2F' + hF. \tag{5}$$

We substitute in (5) a power series of the form

$$F = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \cdots, \tag{6}$$

and obtain the recurrence relations

$$(2n+1)a_n = n a_{n-1}, \qquad a_0 = 1. \tag{7}$$

Hence the functions $F(h)$ and $G(h)$ have the power series expansions

$$F(h) = \theta/\sin\theta = 1 + h/3 + 1\cdot 2h^2/3\cdot 5 + 1\cdot 2\cdot 3h^3/3\cdot 5\cdot 7 + \cdots, \tag{8}$$

$$G(h) = 1 + (F-1)/h = 1 + 1/3 + 1\cdot 2h/3\cdot 5 + 1\cdot 2\cdot 3h^2/3\cdot 5\cdot 7 + \cdots, \tag{9}$$

which for $h=1$ give the same series for $\pi/2$:

$$\pi/2 = F(1) = 1 + 1/3 + 1\cdot 2/3\cdot 5 + 1\cdot 2\cdot 3/3\cdot 5\cdot 7 + \cdots. \tag{10}$$

Since ten terms of the series (10) must be used to reduce the remainder to about 0.0005, the series (9) does not of itself yield a polynomial approximation of low degree with a uniform error less than 0.0005.

We require, if possible, a simple rational function $f(h)$ such that the function $G(h) = S/T$ in (2) is approximated uniformly for $0 < h \leq 1$, with a maximum error of 0.0005, by

$$g(h) = 1 + (f(h) - 1)/h. \tag{11}$$

Clearly we must require that $f(0) = 1$, in order for (11) to be finite at $h=0$. Although the approximation

$$f_1(h) = 1/(1 - h/3), \quad \text{or} \quad \theta \sim 3\sin\theta/(2 + \cos\theta), \tag{12}$$

gives the angle θ to within one minute for angles less than 37°, it is always too small and differs from $F(h)$ by 0.071 for $h=1$. It can be shown that no approximation of the form $1/(1-bh)$ can differ from $F(h)$ by less than 0.021 for all h between 0 and 1. The more complicated approximation

$$f_2(h) = 1 + h/(3 - h - h/4), \text{ or } \theta \sim \sin\theta(11 + \cos\theta)/(7 + 5\cos\theta), \tag{13}$$

is considerably closer than (12), since it gives 11/7 for $\pi/2$ at $h=1$, and has a maximum error of about 0.003 near $h=2/3$.

But to attain the desired accuracy, we try a three parameter approximation of the form

$$f(h) = a/(1 - bh) + (1-a)/(1 - ch), \qquad a, b, c \text{ rational}, \tag{14}$$

and we first demand that its power series about $h=0$ agree with (8) to terms in h^2. Thus we are led to the pair of Diophantine equations

$$ab + (1-a)c = 1/3, \tag{15}$$

$$ab^2 + (1-a)c^2 = 2/15. \tag{16}$$

Squaring (15) and subtracting from (16) we derive the relations:

$$a(1-a)(b-c)^2 = 2/15 - 1/9 = 1/45, \tag{17}$$

$$[5(1-a)/a][3a(b-c)]^2 = 1. \tag{18}$$

Denoting $3a(b-c)$ by $1/u$, and solving (18) and (15) for a, b, c we obtain the parametric solution

$$a = 5/(5+u^2), \qquad 3b = 1 + u/5, \qquad 3c = 1 - 1/u. \tag{19}$$

Next we demand that u be chosen in (19) so that $f(1)$ in (14) is a close approximation to $\pi/2$. Setting

$$f(1) = \frac{\pi}{2} = \frac{5/(5+u^2)}{2/3 - u/15} + \frac{u^2/(5+u^2)}{2/3 + 1/3u}, \tag{20}$$

we find on simplification that

$$\frac{\pi}{3} - 1 = \frac{1}{(10-u)(2+1/u)}, \tag{21}$$

$$(10-u)(2+1/u) = 3/(\pi - 3) = 21.1876, \cdots, \quad u = 1.755 \cdots \tag{22}$$

Finally we pick the simple rational value $u = 7/4$ near $1.755 \cdots$ and obtain from (19) and (20) the rational solution

$$a = 80/129, \quad b = 9/20, \quad c = 1/7, \quad f(1) = 311/198 = 1.5707 \cdots. \tag{23}$$

Thus we obtain the desired approximation formulas

$$\begin{aligned} \frac{\theta}{\sin\theta} &= \frac{1}{129}\left(\frac{80}{1 - 9h/20} + \frac{49}{1 - h/7}\right) = \frac{420 - 109h}{(20-9h)(21-3h)} \\ &= \frac{311 + 109\cos\theta}{(11 + 9\cos\theta)(18 + 3\cos\theta)}, \end{aligned} \tag{24}$$

$$\frac{\textbf{Segment}}{\textbf{Triangle}} = 1 + \frac{1}{129}\left(\frac{36}{1 - 9h/20} + \frac{7}{1 - h/7}\right) = 1 + \frac{140 - 27h}{(20-9h)(18-3h)}, \tag{25}$$

$$0 < h = 1 - \cos\theta \leqq 1.$$

For small h these formulas give values too large by about $h^3/2100$ and $h^2/2100$ respectively. The maximum errors of 0.00017 and 0.00023 respectively are found near $\theta = 75°$. Both errors vanish near $\theta = 88°$ and become about -0.00009 at $\theta = 90°$, $h = 1$.

In conclusion we state without proof that $G(h)$ has the continued fraction expansion

$$G(h) - 1 = (F(h) - 1)/h = \cfrac{1}{3 - h - \cfrac{h}{5 - \cfrac{6h}{7 - R}}}, \quad \text{where} \quad R = \cfrac{6h}{9 - \cfrac{15h}{11 - \cfrac{15h}{13 - \cdots}}}, \tag{26}$$

and that formulas (24) and (25) result from (26) when R is replaced by 7/9.

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A FIXED POINT THEOREM

L. E. WARD, JR., U. S. Naval Ordnance Test Station, China Lake, California

By the term *continuum* we mean a compact connected metric space. A continuum M is *chainable* if, for each $\epsilon > 0$, there is a finite collection of open sets, $U_1, \cdots, U_{n(\epsilon)}$, with the following properties.

(i) *diam* $U_i < \epsilon$ *for each* i,
(ii) $M \subset \bigcup \{ U_i : i = 1, \cdots, n(\epsilon) \}$,
(iii) $\overline{U}_i$ *meets* $\overline{U}_j$ *if and only if* $|i-j| \leqq 1$,
(iv) *for each* i, $U_i - \bigcup \{ \overline{U}_j : j \neq i \}$ *is not empty.*

The simplest nontrivial example of a chainable continuum is the closed unit interval of real numbers. However, Bing [2] has exhibited a very pathological chainable continuum, the so-called pseudo-arc.

A space M has the f.p.p. (=fixed point property) if, for each continuous function $f: M \to M$ there exists some $x \in M$ such that $x = f(x)$. Hamilton [1] has proved that the chainable continua have the f.p.p.

Let $F: M \to M$ be a multivalued function, *i.e.*, $F(x)$ is a subset of M for each $x \in M$. Strother [3] has defined such a mapping to be continuous if each $F(x)$ is a closed set and, whenever $x_n \to x_0$ in M, the following two conditions are satisfied:

(a) *if* $y_n \in F(x_n)$ *then* $\{y_n\}$ *has a limit point in* $F(x_0)$,
(b) *if* $y_0 \in F(x_0)$ *then there exists* $y_n \in F(x_n)$ *such that* $y_n \to y_0$.

The space M is said to have the F.p.p. if each continuous multivalued function $F: M \to M$ has a fixed point, *i.e.*, $x \in F(x)$ for some $x \in M$. It is the purpose of this note to generalize Hamilton's theorem to the multivalued case.

THEOREM. *Each chainable continuum has the* F.p.p.

Proof. Let M be a chainable continuum and let $F: M \to M$ be a continuous multivalued function. In view of (a) it is clearly sufficient to prove that, for $\epsilon > 0$, there exists $\bar{x} \in M$ such that $d(\bar{x}, y) < \epsilon$ for some $y \in F(\bar{x})$. Accordingly, we select a collection of open sets, $U_1, \cdots, U_{n(\epsilon)}$, satisfying conditions (i)–(iv)

above. Let A be the set of all $x \in M$ such that for some i, $x \in \overline{U}_i$ and $F(x) \cap U_j$ is empty for $j < i$; we define B to be the set of all $x \in M$ such that $x \in \overline{U}_i$ implies that $F(x)$ meets $\overline{U}_j$ for some $j < i$. It is obvious that $A \cup B = M$ and that $\overline{U}_1 \subset A$. If $A = M$, then the $\bar{x}$ we seek may be any point in $U_{n(\epsilon)}$. Otherwise, B is non-empty and from continuity we infer that A and B are closed sets. Since M is connected, there is a point, $\bar{x}$, common to A and B. It follows that, for some i, $\bar{x} \in \overline{U}_i \cap \overline{U}_{i+1}$ and there exists $y \in F(\bar{x}) \cap U_i$. Since $d(\bar{x}, y) < \epsilon$, the proof is complete.

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CURVE-FITTING MATRICES

HERBERT S. WILF, Nuclear Development Corporation of America, White Plains, N. Y.

1. Introduction. Suppose we are given N function values $y_1, \cdots, y_N$ at N equally spaced points $x_1, \cdots, x_N$, where $x_{n+1} - x_n = h$. Let it be desired to find the coefficients of the polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1} \tag{1}$$

such that

$$p(x_i) = y_i \qquad (i = 1, \cdots, N). \tag{2}$$

The conditions (2) are a set of N equations in the unknowns a_j of the form

$$V_N A = \pi, \tag{3}$$

where $A = \{a_0, a_1, \cdots, a_{N-1}\}$, $\pi = \{y_1, \cdots, y_N\}$ and V_N is the Vandermonde matrix of order N

$$(V_N)_{ij} = x_i^{j-1} \qquad (i, j = 1, \cdots, N). \tag{4}$$

A change of origin and scale

$$s = (x - x_1)/(x_N - x_1) \tag{5}$$

reduces (4) to the form

$$(V_N)_{ij} = s_i^{j-1} = \left[\frac{i-1}{N-1}\right]^{j-1} \qquad (i, j = 1, \cdots, N). \tag{6}$$

The unknown coefficients A may then be written

$$A = V_N^{-1} \pi \tag{7}$$

with V_N given by (6).

The matrices $\bar{V}_N^{-1}$ may be calculated once and for all, so that curve fitting becomes simply a matrix by vector multiplication. We tabulate below the V_N^{-1} for $N=2, \cdots, 6$.

2. The Matrices $\bar{V}_N^{-1}$.

$$V_2^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad V_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{bmatrix},$$

$$V_4^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ -11 & 18 & -9 & 2 \\ 18 & -45 & 36 & -9 \\ -9 & 27 & -27 & 9 \end{bmatrix}, \quad V_5^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -25 & 48 & -36 & 16 & -3 \\ 70 & -208 & 228 & -112 & 22 \\ -80 & 288 & -384 & 224 & -48 \\ 32 & -128 & 192 & -128 & 32 \end{bmatrix},$$

$$V_6^{-1} = \frac{1}{24}\begin{bmatrix} 24 & 0 & 0 & 0 & 0 & 0 \\ -274 & 600 & -600 & 400 & -150 & 24 \\ 1125 & -3850 & 5350 & -3900 & 1525 & -250 \\ -2125 & 8875 & -14750 & 12250 & -5125 & 875 \\ 1875 & -8750 & 16250 & -15000 & 6875 & -1250 \\ -625 & 3125 & -6250 & 6250 & -3125 & 625 \end{bmatrix}.$$

3. An example. We consider the problem of fitting a curve of fifth degree to a table of values of $k=c_p/c_v$ versus temperature in air at low pressure. The table of values is [1]:

T (F Abs.)	k
600°	1.399
900°	1.387
1200°	1.368
1500°	1.350
1800°	1.336
2100°	1.325

We find easily that

$$V_6^{-1}\begin{pmatrix}1.399\\1.387\\1.368\\1.350\\1.336\\1.325\end{pmatrix}=\begin{pmatrix}1.399000\\-.021916\\-.255208\\.359375\\-.182292\\.026042\end{pmatrix}$$

so that

$$k = f(u) = 1.399 - .021916u - .255208u^2 + .359375u^3 - .182292u^4 + .026042u^5$$

where $u=(T-600°)/1500°$ is the desired fit.

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CLASSROOM NOTES

Edited by C. O. Oakley, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

A UNIFYING TECHNIQUE FOR THE SOLUTION OF THE QUADRATIC, CUBIC, AND QUARTIC

Morton J. Hellman, Rutgers University, Newark, N.J.

In this paper, starting with Gauss's fundamental theorem of algebra and its immediate extension that every algebraic equation of degree n has n roots, the solutions of the quadratic, cubic, and quartic are derived. The only other results used are the relations between the roots and coefficients of the equations. The solution in the cubic case is identical with Cardan's and, in the quartic case, the result is the same as that obtained by the method of Descartes. In each case, the motivation for the solution is clearer than in conventional methods. The quadratic case, which is very simple, is included to show the unifying technique and, in order to obtain the familiar formula, the leading coefficient will be taken to be a, but in the case of the cubic and the quartic, the equations will be in reduced form with leading coefficients unity.

Case I ($n=2$). The general quadratic equation is $ax^2+bx+c=0$, $a\neq 0$. Let r_1 and r_2 be its two roots so that

$$r_1 + r_2 = -b/a, \qquad r_1r_2 = c/a. \tag{1}$$

Then

$$(r_1 - r_2)^2 = \frac{b^2}{a^2} - \frac{4c}{a} \cdot \tag{2}$$

Extracting the square root of (2) and solving with the first equation of (1) results in

$$r_1 = (-b \pm \sqrt{b^2 - 4ac})/(2a), \qquad r_2 = (-b \mp \sqrt{b^2 - 4ac})/(2a).$$

These give permuted values for r_1 and r_2, and the two roots are given by $x = (-b \pm \sqrt{b^2-4ac})/(2a)$.

Case II ($n=3$). It is sufficient to consider the reduced cubic

$$x^3 + ax + b = 0. \tag{3}$$

Let r_1, r_2, and r_3 be its three roots so that

$$r_1 + r_2 + r_3 = 0, \tag{4}$$

$$r_1r_2 + r_1r_3 + r_2r_3 = a, \tag{5}$$

$$r_1r_2r_3 = -b. \tag{6}$$

Substituting r_1, r_2, and r_3 successively in (3), adding the resulting equations, and using (4) and (6) yields $r_1^3+r_2^3+r_3^3-3r_1r_2r_3=0$, which factors into

$$(r_1 + r_2 + r_3)(r_1 + \omega r_2 + \omega^2 r_3)(r_1 + \omega^2 r_2 + \omega r_3) = 0, \tag{7}$$

where ω and ω^2 are the imaginary cube roots of unity. From (7) we have the three possibilities $r_1 = -(r_2+r_3)$, $r_1 = -(\omega r_2+\omega^2 r_3)$, $r_1 = -(\omega^2 r_2 + \omega r_3)$. These results suggest that the three roots of (3) can be found by setting

$$r_1 = -(s + t), \qquad r_2 = -(\omega s + \omega^2 t), \qquad r_3 = -(\omega^2 s + \omega t) \tag{8}$$

and determining s and t so that (8) satisfies (4), (5), and (6).

Now (4) is satisfied for all s and t since $\omega^2+\omega+1=0$; therefore, (5) and (6) become, respectively,

$$st = -a/3, \tag{9}$$

$$s^3 + t^3 = b. \tag{10}$$

Raising (9) to the third power and solving simultaneously with (10), using the method of Case I, yields

$$s = \left\{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{1/3}, \qquad t = \left\{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{1/3}.$$

The first equation of (8) then becomes

$$r_1 = \left\{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{1/3} + \left\{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{1/3}.$$

which is Cardan's result. The other two roots are given by the second and third equations of (8).

Case III ($n=4$). Here it is sufficient to consider $x^4+ax^2+bx+c=0$ whose roots will be denoted by r_1, r_2, r_3, and r_4. Then

$$\begin{aligned} r_1 + r_2 + r_3 + r_4 &= 0, \\ r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 &= a, \\ r_1r_2r_3 + r_1r_2r_4 + r_2r_3r_4 + r_1r_3r_4 &= -b, \\ r_1r_2r_3r_4 &= c. \end{aligned} \tag{11}$$

Making the substitutions

$$u = r_1 + r_2, \qquad v = r_3 + r_4, \qquad s = r_1r_2, \qquad t = r_3r_4 \tag{12}$$

in equations (11) yields

$$u + v = 0, \qquad s + t + uv = a, \qquad sv + tu = -b, \qquad st = c.$$

These reduce to

$$s + t = a + u^2, \tag{13}$$

$$u(s - t) = b, \tag{14}$$

$$st = c. \tag{15}$$

Solving (13) and (14) for s, t in terms of u, a, and b results in

$$s = \frac{1}{2}\left\{a + u^2 + \frac{b}{u}\right\}, \qquad t = \frac{1}{2}\left\{a + u^2 - \frac{b}{u}\right\}. \tag{16}$$

Substituting s, t from (16) in (15) and simplifying leads to

$$u^6 + 2au^4 + (a^2 - 4c)u^2 - b^2 = 0. \tag{17}$$

This is a cubic equation in u^2 and can be solved by the method of Case II; from this u is found and hence v. Using the values of u, s and v, t in pairs according to equation (12) and the technique of Case I determines r_1, r_2, r_3, and r_4.

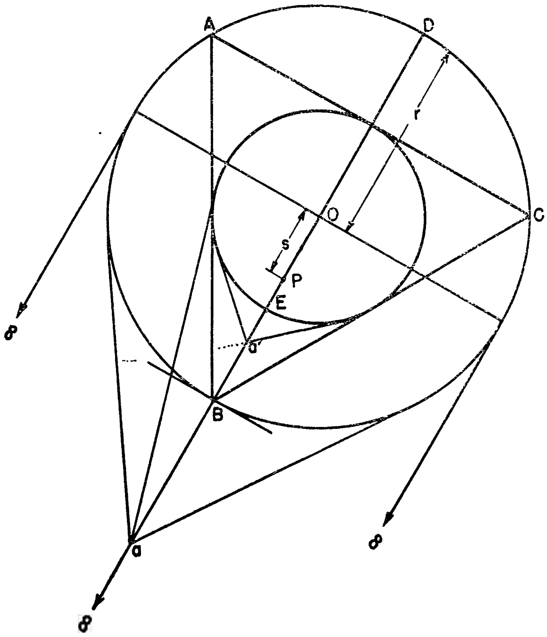
Equation (17) is identical with that obtained by the method of Descartes for the solution of the quartic.

A NOTE ON BERTRAND'S PROBLEM*

Oliver L. Lacey, University of Alabama

Bertrand's problem has long been known as an example of the fact that a seemingly unambiguous question may turn out upon analysis to have multiple answers according to the particular probability model applied to the problem.

The paradox arises in simple form when we ask the probability of obtaining at random a chord of a circle greater in length than the length of the side of the inscribed equilateral triangle. Two common answers to this problem are evident from the figure. It is clear that if we choose all the chords which pass through a



point on the circumference, say B, 1/3 of them will lie in the angle ABC, and the probability is thus intuitively 1/3. On the other hand, the problem may be approached by considering all chords having their midpoints upon the radius OD and running perpendicular to this line. Here it is intuitively evident that half of the chords will be greater in length than AC, and the probability is thus 1/2.

Of course, both of these answers are correct. It is simply that our random mechanisms differ. The first approach considers the angle at B between the tangent and the chord uniformly distributed between 0 and 180°. The second considers the distance of the midpoint of the chord from the center to be uni-

* Preparation of the manuscript was supported in part by funds from the Research Committee of the University of Alabama.

formly distributed from 0 to r. Other random mechanisms are possible which will yield still other results, pointing up even more sharply the ambiguity of the first statement of the problem. It is interesting, however, to observe that these two apparently quite different mechanisms can be derived as special results of a more general case. The figure also illustrates an approach wherein the chord is drawn from a starting point a distance, s, from the center of the circle. It is intuitively obvious (and easy to prove) that if s equals r the probability in question is 1/3 and that the limit of the probability as s approaches infinity is 1/2.

It is of some interest to examine the general situation and probability distributions as s ranges from 0 to ∞. There are three separate cases.

For $s \leqq r/2$, it is clear that the probability is 1 that a chord so drawn shall be greater than $\sqrt{3}r$ (the length of the side of an inscribed equilateral triangle).

For $r/2 \leqq s \leqq r$, θ may be assumed uniformly distributed from 0 to 90°. The probability is then easily found to be $(2/\pi)$ arcsin $(r/2s)$ and this is equal to 1/3 for $s=r$.

For $s \geqq r$, θ may be assumed uniformly distributed between 0 and arcsin (r/s) (the angle which yields the tangent). The probability is then found to be

$$\frac{\arcsin (r/2s)}{\arcsin (r/s)};$$

this is equal to 1/3 for $r=s$ and increases monotonically to approach 1/2 as a limit as $s \to \infty$.

A NEW METHOD OF SOLVING A QUARTIC

PAVAN K. MALHOTRA, South Dakota School of Mines and Technology

Let the quartic equation be written in the form:

$$x^4 + px^3 + qx^2 + rx + s = 0. \tag{1}$$

Break the coefficient on the x^2 term so that $qx^2 = (q-k)x^2 + kx^2$, where k is some constant whose value can be determined. Therefore (1) becomes

$$x^4 + px^3 + kx^2 + (q - k)x^2 + rx + s = 0. \tag{2}$$

Regroup the terms in (2) in the following manner:

$$x^4 + (px + k)x^2 + [(q - k)x^2 + rx + s] = 0. \tag{3}$$

Solve the above equation as a quadratic in x^2, treating $(px+k)$ and $[(q-k)x^2 + rx+s]$ as constant factors:

$$x^2 = \frac{-(px + k) \pm \sqrt{x^2(p^2 + 4k - 4q) + x(2pk - 4r) + k^2 - 4s}}{2}. \tag{4}$$

The value of k shall now be determined so that the quantity under the radical

in (4) is a perfect square, *i.e.*,

$$(5) \qquad 2(p^2 + 4k - 4q)^{1/2}(k^2 - 4s)^{1/2} = 2pk - 4r.$$

Squaring and simplifying (5) results in the following cubic equation in k with roots k_1, k_2, and k_3:

$$(6) \qquad k^3 - qk^2 + (pr - 4s)k + 4qs - p^2s - r^2 = 0. \qquad (k = k_1, k_2, k_3).$$

Putting any one of these values (k_1, k_2, k_3), *i.e.* k_i, into (4) results in

$$(7) \qquad x^2 = [-px - k_i \pm (Lx + M)]/2,$$

where $L = (p^2 + 4k_i - 4q)^{1/2}$ and $M = (k_i^2 - 4s)^{1/2}$. But (7) is simply two separate quadratics in x, namely

$$(8) \qquad x^2 + x(p - L)/2 + (k_i - M)/2 = 0,$$

$$(9) \qquad x^2 + x(p + L)/2 + (k_i + M)/2 = 0.$$

Each of these quadratics can be solved giving the following four values for x which are obviously the roots of (1):

$$(10) \qquad \begin{aligned} x_1 &= \frac{(L - p)/2 + \sqrt{(L - p)^2/4 - 4(k_i - M)/2}}{2}, \\ x_2 &= \frac{(L - p)/2 - \sqrt{(L - p)^2/4 - 4(k_i - M)/2}}{2}, \\ x_3 &= \frac{-(L + p)/2 + \sqrt{(L + p)^2/4 - 4(k_i + M)/2}}{2}, \\ x_4 &= \frac{-(L + p)/2 - \sqrt{(L + p)^2/4 - 4(k_i + M)/2}}{2}. \end{aligned}$$

Example. (All the roots are imaginary.)

$$x^4 - 10x^3 + 57x^2 - 148x + 200 = 0,$$

$$x^4 - 10x^3 + kx^2 + (57 - k)x^2 - 148x + 200 = 0,$$

$$x^4 - x^2(10x - k) + (57 - k)x^2 - 148x + 200 = 0.$$

$$\text{(i)} \quad x^2 = \frac{(10x - k) \pm \sqrt{100x^2 + k^2 - 20kx - 228x^2 + 4kx^2 + 592x - 800}}{2}$$

$$= \frac{(10x - k) \pm \sqrt{x^2(100 - 228 + 4k) - x(20k - 592) + k^2 - 800}}{2}$$

$$2\sqrt{4k - 128}\sqrt{k^2 - 800} = 20k - 592.$$

From the above we get the resolvent equation

$$k^3 - 57k^2 + 680k + 3696 = 0, \quad k_1 = 33, \quad k_2 = 28, \quad k_3 = -4.$$

Putting k_1 in (i)

$$x^2 = \frac{(10x - 33) \pm \sqrt{4x^2 - 68x + 289}}{2}$$

$$= \frac{(10x - 33) \pm (2x - 17)}{2} = \begin{cases} 6x - 25, \\ 4x - 8. \end{cases}$$

$$\begin{aligned} &x^2 - 6x + 25 = 0, \qquad && x^2 - 4x + 8 = 0, \\ &x_1 = 3 + 4i, && x_3 = 2 + 2i, \\ &x_2 = 3 - 4i. && x_4 = 2 - 2i. \end{aligned}$$

INTEGRATION BY MATRIX INVERSION

WILLIAM SWARTZ, Montana State College

The integration of several functions using differential operators was considered by Osborn.* The integration of these and certain other functions by matrix inversion can furnish an application of several aspects of matrix theory of interest to the student of matrix algebra.

Let V be the vector space of differentiable functions. Let the n-tuple f be a basis spanning a subspace S of V which is closed under differentiation. Then differentiation comprises a linear transformation T of S into itself. If the matrix A represents T relative to f, then when A is nonsingular the elements of fA^{-1} yield antiderivatives of the elements of f.

To integrate $e^{ax} \sin bx$ and $e^{ax} \cos bx$ consider $f = (e^{ax} \sin bx,\ e^{ax} \cos bx)$. Then

$$fT = (ae^{ax} \sin bx + be^{ax} \cos bx,\ -be^{ax} \sin bx + ae^{ax} \cos bx)$$

and

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Furthermore

$$A^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and then

$$fA^{-1} = \left(\frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx),\ \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) \right)$$

yields antiderivatives of the elements of f.

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* Roger Osborn, Note on integration by operators, this MONTHLY, vol. 64, 1957, p. 431.

To derive the formula

$$\int x^n e^x dx = e^x[x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^n n!] \tag{1}$$

for positive integers n consider $f=(e^x, xe^x, x^2e^x, \cdots, x^ne^x)$. Then

$$fT = (e^x, e^x + xe^x, \cdots, nx^{n-1}e^x + x^ne^x)$$

and there follows the interesting matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 1 & 2 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & n-1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & n \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 \end{bmatrix}$$

Since only an antiderivative of the last element of f is required, one is only interested in the last column of A^{-1}. Due to the peculiar form of A the inverse is easily deduced. One surmises that the last column of A^{-1} is the transpose of the row

$$\left((-1)^n n!, (-1)^{n-1}n!, (-1)^{n-2}\frac{n!}{2!}, \cdots, n(n-1), -n, 1\right). \tag{2}$$

That this supposition is correct may be verified by induction on n. In this connection it is useful to consider the $(n+2)$ rowed matrix corresponding to A as a partitioned matrix containing A as a principal submatrix. Finally one notes that multiplication of (2) by f yields the required formula (1).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1311. *Proposed by G. K. Wenceslas, Santa Monica, Calif.*

The dissection of a unit square into one $1/8\times 1$ and two $1/2\times 7/8$ rectangular pieces shows that it is possible to cover the unit square with three sets each having diameter $d=\sqrt{65}/8$. Prove that the unit square cannot be covered by three sets all of which have diameter less than d.

E 1312. *Proposed by C. F. Pinzka, University of Cincinnati*

"What kind of doodling is that?" I asked, noticing that Willie had the following steps on his paper:

(0) x_1y_2

(1) $x_3y_4y_5x_6$

(2) $x_7y_8y_9x_{10}y_{11}x_{12}x_{13}y_{14}$

"It's sequential doodling," replied Willie. "I get each new step from the last step by inserting an x after each y, a y after each x, and continuing the subscripts in order."

"Does your doodling have any interesting properties?" I asked in jest.

"Oh, yes," said Willie. "In step n you will find that the sum of the kth powers of the x subscripts equals the sum of the kth powers of the y subscripts for $k=0, 1, \cdots, n$. In fact, you can use any numbers in arithmetic progression for the subscripts and this property will still hold."

Prove that Willie's assertions hold for all (nonnegative integers) n.

E 1313. *Proposed by Roger Pinkham, Princeton University*

If $pz^2-qz+1=0$, then what values of p and q yield roots exterior to the unit circle?

E 1314. *Proposed by J. M. Gandhi, Jain Engineering College, Gurukul, Panchkoola, India*

Let $S(n, i)=\sum a_1\cdots a_i$, where the sum is taken over all possible integral choices of the a's such that $1\leqq a_1\leqq a_2\leqq\cdots\leqq a_i\leqq n$, and put $S(0, i)=0$, $S(n, 0)=1$. Prove that for all positive integral n

$$\sum_{k=0}^{n-1}(-1)^k(n-k)!S(n-k, k) = 1.$$

E 1315. *Proposed by R. L. Caskey, Oklahoma State University*

Let M represent a magic square of order n, the elements of which are the positive integers from 1 to n^2. Let M_i represent the magic square obtained by replacing each element k of M by $k+(i-1)n^2$. Prove that the square array of elements obtained by replacing each element i of M by M_i is a magic square of order n^2.

SOLUTIONS

An Insolvable Diophantine Equation

E 1281 [1957, 592]. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Prove that no perfect square is 7 more than a perfect cube.

Solution by M. S. Klamkin, AVCO Research Division, Lawrence, Mass. This is a problem due to V. A. Lebesgue in 1869. A reference to this and the following proof are given in H. Davenport, *The Higher Arithmetic* (1952), p. 160.

If $y^2=x^3+7$, then x must be odd, since a number of the form $8k+7$ cannot be a square. Now

$$y^2 + 1 = (x + 2)(x^2 - 2x + 4) = (x + 2)[(x - 1)^2 + 3],$$

and the final factor is of the form $4n+3$, and hence must have a prime factor of this same form. But it is well known that y^2+1 cannot have a prime factor of this form.

Also solved by Merrill Barnebey, W. J. Blundon, Leonard Carlitz, P. L. Chessin, I. A. Dodes, N. J. Fine, Jose Gallego-Diaz, Joseph Hershenov and Joseph Muskat (jointly), Sam Kravitz, D. C. B. Marsh, C. S. Ogilvy, Anatol Rapoport, G. J. Simmons, Art Steger, Dmitri Thoro, R. M. Warten, and the proposer. Late solutions by D. R. Breach, D. A. Breault, Joe Lipman, and E. P. Starke.

L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 534, gives the Lebesgue reference.

Triangle of Minimum Perimeter

E 1282 [1957, 592]. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Tangents PQ, PR are drawn from a point P to a circle (O). Another tangent touches the circle at a point B on the minor arc QR and cuts PQ, PR at A, C. Show that QR is equal to the sum of the sides of the triangle of minimum perimeter that can be inscribed in triangle OCA.

Solution by C. F. Pinzka, University of Cincinnati. Let QR cut OA, OC at A', C'. Then $QA'=BA'$, $RC'=BC'$, and QR equals the perimeter of triangle $A'BC'$. But $A'BC'$ is the triangle of minimum perimeter, since it makes equal angles with the sides of acute angled triangle OCA.

Also solved by J. W. Armstrong, J. W. Clawson, A. L. Epstein, Jose Gallego-Diaz, J. D. E. Konhauser, D. C. B. Marsh, William Moser, C. S. Ogilvy, L. A. Ringenberg, Roscoe Woods, and the proposer. Late solution by Geoffrey Kandall.

A Sufficient Condition for a Real Polynomial to Have Some Imaginary Zeros

E 1283 [1957, 592]. *Proposed by J. W. Andrushkiw, Seton Hall University*

If three consecutive coefficients of the real polynomial

$$P(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

are equal and different from zero, show that the zeros of $P(x)$ are not all real.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* We assume, without loss of generality, that $P(0)\neq 0$. If the real polynomial $P(x)$ has at least three consecutive coefficients equal, then $Q(x)=(x-1)P(x)$ has at least two consecutive zero coefficients, and Descartes' rule of signs shows that the sequences of coefficients of $Q(x)$ and of $Q(-x)$ possess a total of no more than $n-1$ variations in sign. It follows that $Q(x)$ has no more than $n-1$ real zeros, and consequently $P(x)$ has no more than $n-2$ real zeros.

II. *Solution by R. E. Greenwood, University of Texas.* Oliver D. Kellogg, *Annals of Mathematics*, ser. 2, vol. 9 (1908), p. 97, proved that if all the coefficients of

$$f(x) = c_0 + c_1x + \cdots + c_nx^n$$

are real, and if all the zeros of $f(x)$ are real, then all the determinants

$$D_j = \begin{vmatrix} c_{j-1} & c_j \\ c_j & c_{j+1} \end{vmatrix}$$

are negative for $k<j<n$, where c_k is the first nonzero coefficient in $f(x)$.

The given problem is an immediate consequence of this result, for, since one $D_j=0$, the polynomial must have some imaginary zeros.

Also solved by W. E. Deskins, A. L. Epstein, Jose Gallego-Diaz, F. D. Parker, J. J. Price, J. E. Scroggs, Marlow Sholander, and the proposer. Late solutions by Juris Hartmanis, E. P. Starke, and D. E. Thoro.

Editorial Note. If we assume $P(0)\neq 0$, then the restriction that the three equal coefficients be different from zero is not necessary.

Radius of Curvature of a Conic

E 1284 [1957, 592]. *Proposed by Charles Fox, McGill University*

Let P be any point on a central conic C, and let the normal to C at P and the polar of P with respect to the director circle of C meet at Q. Show that PQ is equal to the radius of curvature of C at P. What form does this result take for a parabola?

I. *Solution by Roscoe Woods, State University of Iowa.* Denote the center of C by O', the director circle of C by C', the polar of P with respect to C' by p, and the line through O' parallel to the tangent to C at P by L. Let d denote the distance from P to p, r the distance from L to P, R' the distance PQ, and d' the distance $O'P$. Each distance may be considered positive. From similar triangles it follows that $R'=dd'/r$. It is to be proved that $R'=R$, where R is the radius of curvature of C at P.

In rectangular coordinates (X, Y), let the equation of C be taken in the form $Y^2=aX^2+2bX$, $b\neq 0$, and a an arbitrary constant. The radius of curvature R of C at $P(x, y)$ is given by $R=z^{3/2}/b^2$, where

$$z = x^2(a^2 + a) + 2bx(a + 1) + b^2.$$

The following equations and relations may now be readily found:

$$C'\colon a(X^2 + Y^2) + 2bX + b^2 = 0,$$
$$p\colon a(xX + yY) + b(X + x) + b^2 = 0,$$
$$L\colon (aX + b)(ax + b) - ayY = 0,$$
$$d = z/u, \quad \text{where} \quad u = [(ax + b)^2 + a^2y^2]^{1/2},$$
$$r = (b^2/a)/z^{1/2} \quad \text{and} \quad d' = u/a.$$

By substitution we find $R'=dd'/r=R$.

If we let a approach 0, the conic C approaches the parabola $Y^2=2bX$ and R' approaches $(2x+b)^{3/2}/b^{1/2}$, the radius of curvature of the parabola at P. We also note that the director circle C' approaches the line $X=-b/2$, the directrix of the parabola, and p approaches the line $X=-b-x$. This line has the property that if Q is any point on it, the line segment PQ is bisected by the directrix. For a parabola, then, the result may be stated thus: If P is any point of a parabola and if the normal to the parabola at P cuts the directrix in S, PS is equal in length to half the radius of curvature of the parabola at P.

Remarks. (1) This problem is discussed in Jacob Steiner's *Gesammelte Werke*, vol. 2, pp. 341–342. (2) The result of this problem furnishes a simple construction for the center of curvature of a conic section. Numerous constructions for the center of curvature of a central conic may be found in a paper by F. Balitrand in *L'Enseignement Mathématique*, vol. 18, pp. 260–268.

II. *Solution by Jose Gallego-Diaz, Vanderbilt University.* Serret has shown (*Nouvelles Annales*, 1861, p. 80) that if a triangle T which is self-conjugate for a conic C approaches as a limit a point P on C, then the circumcircle of T approaches as a limit the circle Γ externally tangent to C at P and having diameter equal to the radius of curvature of C at P. But, by a theorem of Faure (see, *e.g.*, Salmon, *A Treatise on Conic Sections*, Ex. 2, p. 341), the circumcircle of a triangle T which is self-conjugate for a central conic (parabola) C is orthogonal to the director circle (directrix) of C. Consequently, the circle Γ is orthogonal to C'. But the circle on PQ as a diameter is orthogonal to C'. It follows that PQ is equal to the radius of curvature of C at P.

An interesting case is that in which C is a rectangular hyperbola, for it is known that in this case the circumcircle of a triangle T which is self-conjugate for C passes through the center O' of C (see Salmon, Ex. 5, p. 215). Point Q is thus easily located as the intersection of the normal at P with the line through O' perpendicular to $O'P$.

Also solved by Roland Deaux, Jose Gallego-Diaz (analytically), D. C. B. Marsh, Sister M. Stephanie, P. D. Thomas, and the proposer.

Bazin's Theorem

E 1285 [1957, 592]. *Proposed by James Ax and Lawrence Shepp, Polytechnic Institute of Brooklyn*

Let A and B be two nth order determinants, and let C be a third determinant whose (i, j)th element is the determinant A with its ith column replaced by the jth column of B. Show that $C=A^{n-1}B$.

Solution by D. A. Robinson, University of Wisconsin. Let a, b be the matrices whose determinants are A, B respectively, and let c be the matrix with determinant C described above. Let A_{ij} be the cofactor of the (i, j)th element of a and let adj a be the matrix whose (i, j)th element is A_{ji}. Then $c=(\text{adj } a)b$. Taking determinants and noting Cauchy's result that the determinant of adj a is A^{n-1}, the desired result follows.

It is interesting to note the following generalization. If k distinct columns of B should be substituted for k distinct columns of A in all possible ways and the resulting determinants arranged lexicographically to form C, then

$$C = A^{\binom{n-1}{k}} B^{\binom{n-1}{k-1}}.$$

This result and several others appear in Aitken, *Determinants and Matrices* (1956), pp. 101–103.

Also solved by J. F. Burke, A. L. Epstein, M. S. Klamkin, D. C. B. Marsh, Chih-yi Wang, David Zeitlin, and the proposers. Late solutions by D. R. Breach and Joe Lipman.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4783. *Proposed by Leonard Carlitz, Duke University*

Find the polynomial solution of the system

$$xf'(x) - g'(x) = \mu f(x),$$
$$xg'(x) - f'(x) = \nu g(x),$$

where μ, ν are assigned constants.

4784. *Proposed by D. W. Jonah, Purdue University*

In John L. Kelley, *General Topology*, p. 18, a definition of a ring is given in which the left and right distributive laws are replaced by the composite distributive law: $(u+v)(x+y)=ux+uy+vx+vy$.

(a) Show by an example that such a system is not necessarily a ring.

(b) Show that if such a system contains an element a such that $a0=0$ (in particular, if the system has a multiplicative identity), then the system is a ring.

4785. *Proposed by Paul Erdös, University of British Columbia*

Let $n_1<n_2< \cdots$ be a sequence of integers. Denote by $f(N)$ the number of the n's not exceeding N. Assume that $f(2N)-f(N)<c_1$. Then there is a constant c_2, depending only on c_1, such that as $x\to 1$

$$\left|\sum_{k=1}^{\infty} x^{n_k} - f\left(\frac{1}{1-x}\right)\right| < c_2.$$

4786. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let A, B be two positive definite Hermitian matrices of order n, and let $C=A+B$. For any positive integer $p<n$, let A_p denote the principal submatrix of A formed by the first p rows and columns, and let B_p, C_p have similar meaning. Prove

$$\left(\frac{\det C}{\det C_p}\right)^{1/(n-p)} \geqq \left(\frac{\det A}{\det A_p}\right)^{1/(n-p)} + \left(\frac{\det B}{\det B_p}\right)^{1/(n-p)}$$

[The case $p=n-1$ is known and due to H. Bergström (See R. Bellman, this MONTHLY, vol. 62, 1955, pp. 172–173). Also the inequality of Minkowski (Hardy, Littlewood and Polya, *Inequalities*, 1934, p. 35) may be regarded as the case $p=0$.]

4787. *Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.*

Express as a single definite integral

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(m+n+r)!\, m^n n^r}{m!n!r!(r+1)^{m+n+r+1}}.$$

SOLUTIONS

Generators of Ideals

4732 [1957, 202]. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Show that, in the ring of polynomials with integral coefficients, $(J(x))$, there are ideals which require arbitrarily many generators.

Solution by the proposer. We prove $(2^n, 2^{n-1}x, \cdots, x^n) = I_n$ requires $n+1$ generators. For, consider I_n modulo I_{n+1}. We claim that this contains at least 2^{n+1} elements; that is, there are 2^{n+1} elements of I_n whose differences do not lie in I_{n+1}. If we form the 2^{n+1} elements

$$\sum_{i=0}^{n} a_i 2^{n-1} x^i, \qquad a_i = 0 \text{ or } 1,$$

then these are all incongruent. For if

$$\sum_{i=0}^{n} (a_i - b_i) 2^{n-i} x^i \in I_{n+1},$$

then, setting $x = 2y$,

$$\sum_{i=0}^{n} (a_i - b_i) y^i \in (2, 2y, 2y^2, \cdots, 2y^{n+1})$$

and so all the $a_i - b_i$ are even, whence $a_i = b_i$.

Now suppose $I_n = (P_1, P_2, \cdots, P_r)$. If $Q(x)$ is any polynomial such that $Q(0)$ is even, then $Q \cdot P_i \in I_{n+1}$, hence every element in $(P_1, \cdots, P_r)$ is congruent to

$$\sum_{i=1}^{r} \lambda_i P_i, \qquad \lambda_i = 0 \text{ or } 1,$$

but therefore the total number of elements in $(P_1, \cdots, P_r)$ modulo I_{n+1} is at most 2^r. Hence, finally, $2^r \geqq 2^{n+1}$, $r \geqq n+1$.

Dirichlet Series

4734 [1957, 276]. *Proposed by George Brauer, University of Minnesota*

Construct an ordinary Dirichlet series $F(s) = \sum_{n+1}^{\infty} c_n n^{-s}$ such that

$$\sum_{n=1}^{\infty} |c_n| < \infty \quad \text{and} \quad \lim_{\sigma \to 0+} |F'(\sigma + i\tau)| = \infty$$

for almost all points on the τ-axis $(s = \sigma + i\tau)$. (A Taylor series with analogous properties is given in Rudin's paper, *On a problem of Bloch and Nevanlinna,* Proc. Amer. Math. Soc., vol. 6, 1955, pp. 202–204.)

Solution by the proposer. Take $n_1 = 1$; and having chosen $n_2, n_3, \cdots, n_{k-1}$, take a positive integer n_k such that

$$n_k > k^2 n_{k-1}, \tag{1}$$

$$\sum_{s=1}^{k-1} n_s [1 - (1 - n_k^{-1/2})^n < 1. \tag{2}$$

Also let

$$f(z) = \sum_{k=1}^{\infty} k^{-2} z^{n_k}. \tag{3}$$

Rudin (in the paper already cited) shows that the series (3) converges absolutely on $|z| = 1$ while $f'(re^{i\theta}) \to \infty$ as $r \to 1-0$ for almost all values of θ. Let N_k denote the least integer greater than e^{n_k}, and let

$$F(s) = \sum_{k=1}^{\infty} k^{-2} N_k^{-s}, \qquad s = \sigma + i\tau. \tag{4}$$

The series (4) converges absolutely for $\sigma = 0$. We show that $F'(\sigma+i\tau) \to \infty$ as $\sigma \to 0^+$, for almost all values of τ.

Let $r = e^{-\sigma}$, $s = \sigma + i\tau$, then (differentiation is with respect to s)

$$\begin{aligned}
&\left| -re^{-i\tau} f'(re^{-i\tau}) - F'(\sigma + i\tau) \right| \\
&\qquad = \left| -\sum_{k=1}^{\infty} k^{-2} n_k e^{-n_k s} + \sum_{k=1}^{\infty} k^{-2} \log N_k e^{-s \log N_k} \right| \\
&\qquad = \sum_{k=1}^{\infty} k^{-2} \{ \log N_k e^{-s \log N_k} - \log N_k e^{-n_k s} + \log N_k e^{-n_k s} - n_k e^{-n_k s} \} \\
&\qquad \leqq \sum_{k=1}^{\infty} k^{-2} \{ \log N_k \left| e^{-s \log N_k} - e^{-n_k s} \right| + \left| \log N_k - n_k \right| \}.
\end{aligned}$$

Since $\log(N_k - 1) \leqq n_k < \log N_k$, $0 < \log N_k - n_k \leqq 1/(N_k - 1)$ and

$$\left| e^{-s \log N_k} - e^{-n_k s} \right| \leqq |s| (\log N_k - n_k)$$

we have, for $0 < \sigma < 1$,

$$\begin{aligned}
&\left| -re^{-i\tau} f'(re^{-i\tau}) - F'(\sigma + i\tau) \right| \\
&\qquad \leqq (|s| + 1) \sum_{k=1}^{\infty} \log N_k / k^2 (N_k - 1) \\
&\qquad \leqq (|\tau| + 2) \sum_{k=1}^{\infty} \log N_k / k^2 (N_k - 1) \leqq M(|\tau| + 2).
\end{aligned}$$

As $\sigma \to 0^+$, $r \to 1-0$; hence $\lim_{(\sigma \to 0^+)} F'(\sigma+i\tau) = \infty$ for all values of τ such that $\lim_{(r \to 1-0)} re^{-i\tau} f'(re^{-i\tau}) = \infty$; that is $\lim_{(\sigma \to 0^+)} F'(\sigma+i\tau) = \infty$ for almost all values of τ.

The Encyclopaedias with Missing Volumes

4736 [1957, 277]. *Proposed by M. v. Ments, Jerusalem, Israel*

An encyclopaedia consists of N volumes, and in a certain town p persons originally possessed complete sets. But by some event, a lot of volumes of the various sets were destroyed at random, so that only n_i volumes remain in the possession of the ith person. If all the townspeople now pool their books, what

is the probability that M volumes (out of the total N) are available somewhere in town?

*Solution by the proposer.** Let A_i be the event that the ith volume is missing from the collection of all the volumes remaining, and let $p_i = pr(A_i)$ be the probability of the event A_i. Let $p_{ij} = pr(A_iA_j)$ be the probability of both A_i and A_j, $p_{ijk} = pr(A_iA_jA_k)$, and so on, $i, j, k = 1, 2, \cdots, N$ with $i<j<k< \cdots$. It is immediately evident that

$$p_i = \prod_{k=1}^{p} \frac{\binom{N-1}{n_k}}{\binom{N}{n_k}}, \qquad p_{ij} = \prod_{k=1}^{p} \frac{\binom{N-2}{n_k}}{\binom{N}{n_k}}, \qquad p_{ijk} = \prod_{k=1}^{p} \frac{\binom{N-3}{n_k}}{\binom{N}{n_k}}, \cdots .$$

Now define $S_1 = \sum p_i$, $S_2 = \sum p_{ij}$, $S_3 = \sum p_{ijk}$, *etc.* We will have

$$S_t = \binom{N}{t} \prod_{k=1}^{p} \binom{N-t}{n_k} \Big/ \binom{N}{n_k}, \qquad t = 1, 2, \cdots, N.$$

Fréchet has proved that, for any integer r with $1 \leqq r \leqq N$, the probability $P_{[r]}$ that exactly r among the N events $A_1, \cdots, A_N$ occur simultaneously is given by

$$(1) \qquad P_{[r]} = S_r - \binom{r+1}{r} S_{r+1} + \binom{r+2}{r} S_{r+2} - + \cdots + \binom{N}{r} S_N.$$

(See Feller, *Probability Theory and its Applications*, 1950, pp. 64–65.)

Now the probability that exactly M of the volumes remain is the probability that exactly $r = N - M$ of the events $A_1, A_2, \cdots$ occur. According to (1) the desired probability is then

$$P_{[N-M]} = S_{N-M} - \binom{N-M+1}{N-M} S_{N-M+1} + \binom{N-M+2}{N-M} S_{N-M+2} - + \cdots \pm \binom{N}{N-M} S_N.$$

Upon substituting the value of S_t from above and employing the easy identity

$$\binom{N-M+v}{N-M}\binom{N}{N-M+v} = \binom{N}{M}\binom{M}{v},$$

this becomes

* Abridged through the use of a theorem of Fréchet.—*Ed.*

$$\frac{\binom{N}{M}}{\prod_{k=1}^{p}\binom{N}{n_k}} \sum_{v=0}^{\infty} (-1)^v \binom{M}{v} \prod_{k=1}^{p} \binom{M-v}{n_k}.$$

Cesáro First Order Mean

4742 [1957, 370]. *Proposed by Joshua Barlaz, Rutgers University*

Evaluate the Cesáro first order mean for the series $\sum_{n=2}^{\infty} (-1)^n \log n$.

Solution by Robert Breusch, Amherst College. The partial sums are

$$S_{2n} = \log \frac{2\cdot 4 \cdots (2n)}{3\cdot 5 \cdots (2n-1)}, \qquad S_{2n+1} = \log \frac{2\cdot 4 \cdots (2n)}{3\cdot 5 \cdots (2n+1)}.$$

Thus

$$S_{2n} + S_{2n+1} = \log \frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5} \cdots \frac{(2n)(2n)}{(2n-1)(2n+1)} = \text{lo } (\pi/2) + O(1/n).$$

Therefore

$$\sum_{n=2}^{N} S_n = \frac{N}{2}\log(\pi/2) + O(\log N), \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n=2}^{N} S_n = \frac{1}{2}\log(\pi/2).$$

Also solved by J. L. Alperin, R. G. Buschman, N. J. Fine, and Chih-yi Wang.

Editorial Note. As Buschman notes, the result follows directly from G. H. Hardy, *Divergent Series*, p. 347, lines 6 to 10.

A Sequence Developed from a Recurrence Relation

4743 [1957, 436]. *Proposed by Aaron Herschfeld, Canisius College, Buffalo, N. Y.*

Given an arbitrary positive number u_0, consider the sequence u_n, where $u_n = u_{n-1} + 1/u_{n-1}$. Prove the relation $u_n = \sqrt{2n} + (\log n)/\sqrt{32n} + O(1/\sqrt{n})$.

Solution by Leopold Flatto, Reeves Instrument Corporation, New York City. From the given recurrence relation we get $u_n^2 = u_{n-1}^2 + 1/u_{n-1}^2 + 2$ and hence

$$u_n^2 = u_0^2 + \left(\frac{1}{u_0^2} + \frac{1}{u_1^2} + \cdots + \frac{1}{u_{n-1}^2}\right) + 2n, \tag{1}$$

$$u_n^2 > 2n. \tag{2}$$

Using (1) and (2) we have

$$u_n^2 - 2n < (u_0^2 + 1/u_0^2) + \frac{1}{2}\sum_{k=1}^{n-1} 1/k < \left(u_0^2 + 1/u_0^2 + \frac{1}{2}\right) + \frac{1}{2}\log n,$$

and so

$$(3)\qquad \sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{u_n^2}\right)=\sum_{n=1}^{\infty}\frac{u_n^2-2n}{2nu_n^2}<\sum_{n=1}^{\infty}\frac{u_0^2+1/u_0^2+\frac{1}{2}+\frac{1}{4}\log n}{4n^2}.$$

Hence (1) may be rewritten as

$$u_n^2=2n+\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2(n-1)}\right)+O(1)=2n+\frac{1}{2}\log n+O(1).$$

Then finally we have $u_n=(2n+\frac{1}{2}\log n+O(1))^{1/2}=\sqrt{2n}(1+(\log n)/4n+O(1/n))^{1/2}=\sqrt{2n}(1+(\log n)/8n+O(1/n))$, from which the desired relation follows immediately. The positive square root is chosen since $u_0>0$ implies all $u_n>0$.

Also solved by L. R. Bragg, Robert Breusch, Emil Grosswald, Ya'akov Shima, Chih-yi Wang, G. S. Wells, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The Theory of Lie Derivatives and Its Applications. By Kentaro Yano. Interscience, New York, 1957. x+299 pp. $8.75.

One of the remarkable results in classical differential geometry is the fact that a surface admits either no or a one-parameter or a three-parameter group of transformations; in the last case it is of constant curvature. In the present book one finds far-reaching generalizations of this result to Riemannian, affinely-connected, projectively-connected, conformally-connected spaces by Egorov, Muto, Vranceanu, Wang, and the author.

Naturally, a basic analytic tool is the Lie derivative. It is an operator defined by a vector field and acts on an arbitrary tensor field, or, more generally, on a geometric object. The vector field generates a one-parameter group of transformations and the Lie derivative is, roughly speaking, the derivative along its orbits. The notion is extremely natural and was used by Lie with success in various special cases. Its general formulation was due to W. Slebodzinski, according to the author (who is, incidentally, one of the leading experts on modern literature on differential geometry).

The book has ten chapters and an appendix. The first three chapters deal with generalities, the next five treat mostly the local theory of transformation groups in spaces which are of interest in differential geometry, and the last two chapters and the appendix study some global aspects of the problem. The latter

forms an introduction to a new field which should be full of promise. In fact, spaces with a transformation group present, in a sense, a certain amount of symmetry and are naturally more important than the general ones. The differential-geometric theory should be a worthy sequel to the theory of topological transformation groups.

The book is well written and is undoubtedly a valuable addition to mathematical literature.

S. CHERN
University of Chicago

Introduction to Abstract Algebra. By C. Racine. Viswanathan, Madras, 1957. xii+194 pp. 6 rupees.

The aim of this book, according to the author's preface, is "to make it easy for students to familiarize themselves with the essential elements of a fundamental structure of Modern Mathematics—to make it cheap too!" The result is a fairly tasteful selection of basic topics from modern algebra. The first three chapters of the book, dealing with groups, rings and fields, cover about the same ground as Volume 1 of van der Waerden's *Moderne Algebra.* Chapter four—the final chapter—treats matrices and their canonical forms.

The author has done a remarkably poor job of proofreading his manuscript. In the errata 65 items are listed, but there are perhaps a hundred more slips of a typographical or careless nature. For example, on p. 70, one "learns" that a sufficient condition for a ring to be a unique factorization domain is that it be Noetherian. But this is followed by the contradictory example $Z[\sqrt{-5}]$.

In the reviewer's opinion, Professor Racine has not achieved his difficult objective of writing a useful book for students. This "introduction" to algebra, written as it is in the unpalatable "definition-theorem-proof" jargon of a research paper, demands more mathematical maturity than can be expected of even a good graduate student. At best, the book can serve as a convenient set of lecture notes, to be used by graduate students cramming for an examination, or by an overburdened teacher following the geodesic of least resistance.

R. S. PIERCE
University of Washington

The Number-System. By H. A. Thurston. Interscience, New York, 1956. vii+134 pp. $2.50.

Every serious student of mathematics is exposed at some point in his education to the systematic construction of the complex numbers starting from the Peano postulates. This exposure may take place piecemeal. Construction of the integers and rational numbers is usually done in courses in abstract algebra; the passage from the rationals to the reals in courses in analysis; and the introduction of the complex numbers in courses in college algebra, or complex variables. For students who prefer to see the entire construction of the number-system in a single consecutive development the author has provided a most

acceptable alternative to the classical treatment by Landau in *Grundlagen der Analysis.* (Leipzig, 1930. English Translation, New York, 1951.)

The purpose and scope of the present book is the same as that of Landau's *Grundlagen* but the flavor is different. In the first place it has been made more palatable to the inexperienced reader by the inclusion of a preliminary, informal, heuristic discussion designed to motivate the postulates and definitions and generally to explain the formal treatment that follows. This informal discussion occupies a little more than one-third of the book and its style is both lucid and stimulating.

The formal treatment in Part 2 is, fittingly enough, in the severe mathematical style of "Definition:—Theorem:—Proof:" but again it is lucid and easily read. The order of presentation differs from that of Landau. The natural numbers are first imbedded in the integers which are then imbedded in the rational numbers. This order, though it no doubt departs from the historical, makes it possible to use the techniques of abstract algebra to greater advantage in the successive constructions. It also serves to separate as far as possible the algebraic part of the construction from the analytic, so that the first eight chapters of Part 2 contain the complete construction of the integers and of the rational numbers, constructions that never appear, as such, in Landau. The real numbers are defined by means of Cauchy sequences, the results being obtained for any ordered field. A discussion of the exponential function and the decimal representation of real numbers are included. Exercises are provided at the end of each chapter and these include several important results not formally stated in the text. "A Key to the Exercises" outlines the solutions. There is a short but adequate bibliography and an index.

D. C. MURDOCH
University of British Columbia

Understanding Arithmetic. By Robert L. Swain. Rinehart, New York, 1957. xxi+264 pp. $4.75.

This book deals with arithmetic on a fairly mature level and is addressed primarily to future teachers of arithmetic in the elementary schools. The following summary of its contents will indicate its scope.

The first chapter is historical in nature and discusses the Babylonian, Egyptian, Greek and Roman systems of numerals and compares them with the Hindu-Arabic system. Chapter 2 deals with sets, one to one correspondence and (finite) cardinal numbers. In Chapter 3 addition and multiplication are defined in terms of the cardinal numbers of the union and Cartesian product of disjoint sets. The "laws of arithmetic" are then derived and the inverse operations, subtraction and division, are introduced. The integers are defined by a comparatively concrete and colorful (no pun intended) version of the usual ordered pairs of natural numbers. The next three chapters are devoted to the algorisms for addition, subtraction, multiplication and division in the Hindu-Arabic system, first with base 10 and then in the duodecimal and binary scales.

Chapter 7 contains a little elementary number theory, including the proof of the existence of an infinite number of primes and of the unique factorization theorem. Fractions, decimals and percent are discussed in Chapters 8 and 9. Chapter 10 treats the rational number system, the existence of irrationals, the real numbers (informally, as infinite decimals), and the rational numbers as repeating decimals. The last two chapters discuss measurement and units, accuracy of approximations, significant digits, *etc.*

The book is written for the student with little or no background of college mathematics. The exposition is clear and the style is pleasing and likely to maintain the student's interest. There are frequent historical anecdotes and many useful references to books where the student can find more complete discussion of various mathematical topics. It is a pity that these references were not all listed again at the end of the book as a composite list of suggestions for further reading.

The author's remark that the book could well have been titled *What Every Teacher Ought to Know about Arithmetic* represents a modest enough aim and yet one that, if attained, would probably mean a significant educational advance.

D. C. MURDOCH
University of British Columbia

Vector Analysis. By Louis Brand. Wiley, New York, 1957. xiv+282 pp. $6.00.

This short work is a contribution to the field of elementary vectors. The treatment is concise, definite and vigorous. The first five chapters are concerned with the algebra and calculus of vectors. Vectors need something to develop with. For them geometry is natural and these chapters abound with numerous applications to this field. The other four chapters are brief treatments of Dynamics, Fluid Mechanics, Electrodynamics and Vector Spaces.

Chapter I. Vector Algebra. The topics are standard. The treatment is clear. The applications are on analytics, mechanics, plane and spherical trigonometry, and several of the theorems from projective geometry and college geometry.

Chapter II. Line Vectors. Applications to Statics.

Chapter III. Vector Functions of One Variable. Differentiation of vectors with applications to differential geometry and kinematics.

Chapter IV. Differential Invariants. Here one finds applications to surfaces and fields. Divergence and rotation are presented as invariants of a dyadic. The chapter ends with curvilinear coordinates.

Chapter V. Integral Theorems. The topics are the usual ones, line integrals, Green's Theorem, Stokes' Theorem, Divergence Theorem. The treatment is rather extensive. The best applications are in later chapters.

Chapter VI, . . . , IX. These are brief compact courses on the topics mentioned above. Perhaps they should be left for full courses in these topics.

The book would be suitable as a text for students past the calculus. The

average student would probably find the problems interesting, but difficult, at least until he acquires some skill in applying vector techniques.

Earl La Fon
University of Oklahoma

Models of Man. By Herbert A. Simon. Wiley, New York, 1957. xiv+287 pp. $5.00.

How would you describe the social behavior of man? The many aspects of his rational behavior? The author has selected from journal publications during the past decade sixteen of his essays (two co-authored) which analyze various facets of man's behavior. The papers are grouped into four parts; an introduction to each part links together the essays of that group and discusses their interconnection with the remaining parts. Some of the topics treated are: causal relations between variables; measurement of political power or influence; effect of predicting an election on changing its outcome; friendliness in small groups; pressures toward uniformity in social groups; why people join organizations, remain in or leave organizations; servomechanisms controlling production; the nature of rational choice. In the latter area the principle of bounded rationality is enunciated and other explanations of the choice process are attacked. Man's goal in decision making is not necessarily to maximize his return or benefit but rather to satisfice, to find a course of action that is good enough. The statement of the principle of bounded rationality is: "The capacity of the human mind for formulating and solving complex problems is very small compared with the size of the problems whose solution is required for objectively rational behavior in the real world—or even for a reasonable approximation to such objective rationality."

The physical sciences have long utilized the tools of mathematics. Mathematical formulation often assists in understanding scientific principles, makes further discovery easier and, in return, the sciences have challenged mathematics to develop more techniques. We are experiencing the early days of the era in which the social sciences will similarly be joining forces with mathematics for mutual benefit. "Models of Man" shows some of the mathematical paths which may be taken in economics, sociology, political science, psychology. Traveling the routes is pioneering; the future will tell which lead to significant progress, which are merely gropings. But always our sights must be forward. The reader must not expect to find this book a definitive treatise. Many of the models need broader and deeper development. Much new mathematics and strange approaches to old mathematics may help to handle the analyses. With an open mind, the reader can glimpse in this book fascinating vistas for future mathematical applications.

Some of the present mathematical topics which are utilized in the essays are set theory, probability theory, systems of differential equations, difference equations, and even the Brouwer fixed point theorem. Sometimes the down-to-earth usage of mathematics seems rather superficial; other times the author contends

that the mathematical formulation permitted the discovery of results not otherwise recognized. The reader interested in the role of mathematics in social science should certainly contemplate the first paragraph on page 89, the two paragraphs at the bottom of page 97, and most of page 142. Two sentences may be quoted here: "We must not expect to find the models we need ready-made in a mathematical textbook. If we are lucky, we shall not have to invent new mathematics, but we are very likely to have to assemble our model from a variety of raw materials." After you have decided how you would describe man's behavior in social and rational situations, then critically read these essays and compare their models with yours.

R. A. Good
University of Maryland

Wahrscheinlichkeits-Theorie. By Hans Richter. Springer (Die Grundlehren Der Mathematischen Wissenschaften in Einzeldarstellungen, Band LXXXVI). 435 pp. DM 69.60.

The book under review is a lucid and timely introduction to the modern theory of mathematical probability, and is warmly recommended to graduate students and all others who would like to attain the research level in this vital and constantly expanding field.

Recognizing that not only a rigorous development of the theory, but also the very working tools, depend upon the language and techniques of set theory, the author begins the volume with a very readable description of the fundamentals of set theory and measure theory.

He then turns to an interesting discussion of the meaning of probability, as a philosophic concept, as a working scientific concept, and finally as a precise mathematical concept.

The next section of the book is devoted to developing the basic results of probability theory from the fundamental axioms. This discussion prepares the way for the more advanced results which follow, and affords an opportunity for the treatment of many interesting and important problems.

Since the subsequent results require the Lebesgue-Stieltjes integral, the author once again detours to furnish an excellent introduction to integration theory. With this disposed of, there follow two long chapters devoted to the anatomy of random variables, characteristic functions, moments, and so forth.

The last chapter is dedicated to the study of convergence of sequences of random variables, the weak and strong versions of the law of large numbers and the central limit theorem.

An excellent feature of the book is a collection of problems at the ends of chapters with answers at the back of the book. The typography is attractive and readable, as is to be expected from Springer.

Richard Bellman
The RAND Corporation

Vorlesungen ueber Inhalt, Oberflaeche und Isoperimetrie. By H. Hadwiger. Springer Verlag, Berlin-Goettingen-Heidelberg, 1957. 312 pp. $11.95.

In this monograph the well-known author presents a formal development of the theory of measure. The early chapters are devoted to the discussion of an elementary measure space (P, Φ) consisting of the class of polyhedra P and a functional Φ defined on polyhedra in the usual way except that only translation-invariance is required. This concept is generalized in chapter III insofar as a measure space (F, φ) is defined as follows: F is a class of bounded point sets in the E_k which is closed under translation, additive, and contains the unit cube and the measure φ is translation-invariant, additive, nonnegative definite, and normalized. The elementary measure space as introduced in the beginning appears as embedded in (F, φ) if the latter has the measure theoretic field property. Jordan's measure space appears as the smallest measure space with the field property whereby "smallest" shall mean that it belongs to the class and is embraced by all other elements of the class. Lebesgue's measure space is characterized as the smallest complete measure space whereby "complete" has the usual meaning, *i.e.*, every set which can be enclosed by two measurable sets with arbitrarily small volume difference shall be measurable. Due to the author's approach in working with translation-invariance only and not with invariance with respect to the euclidean group, Tarski's measure space can be introduced for all $k \geqq 1$.

It is not possible to go into any details in discussing the rich content of the following chapters in the limited available space. The reader, however, may get a rough idea of what to expect by reading the remaining chapter headings in free translation: IV. Selected Studies in Set Geometry, V. Volume, Surface Area and Isoperimetry, VI. Convex Bodies and General Integral Geometry.

It should be stated that the present book is carefully planned and logically designed; it is written in an extremely clear style. It is of great advantage that every chapter is followed by an extensive bibliography.

The usual care and high quality of Springer Verlag is again manifested in this book.

HANS SAGAN
University of Idaho

Linear Algebra for Undergraduates. By D. C. Murdoch. Wiley, New York, 1957. xi+239 pp. $5.50.

The style of presentation here is well above average—direct, consistent, well-organized, succinct. The clarity of writing and publication should be welcome to instructors and especially to self-help students.

The matter of content presents some questions. With the author, I consider a course in vectors and matrices an appropriate undergraduate study. As he points out, it has in addition to ready application the advantage of a logical deductive development. I think the title "linear algebra" a bit ambitious for

such a course, especially when " . . . every effort has been made to keep the treatment of the various topics elementary and concrete and to introduce as few abstract ideas as possible." As to the avoidance of abstraction it may indeed make the "how" simple, but sooner or later the student is going to be stumped by the "why," and a book of this type will give him little help. Not only is there no help here from the point of view of fitting matrix algebra into the general picture as "*an* algebra" and the study of invariants under certain sets of transformations as "geometries," but there isn't even any practical motivation in the problems, which for the most part serve to round out the theory in the brief text. Such "applied" problems as appear are given in the sifted collected general form derived long after the construction of the mathematical model.

Considerable energy is going now into the development of freshman courses that will include some abstract ideas to organize burgeoning mathematical thought into a manageable form. In this light a deliberately nonabstract book, intended as a sequel to college algebra, seems a step backward.

MARGARET W. MAXFIELD
U. S. Naval Ordnance Test Station

BRIEF MENTION

The Tao of Science. By R. G. H. Siu. The Technology Press (MIT), and Wiley, New York, 1958. xvi+180 pp. $4.25.

This interesting account of the difference in basic philosophy between the Oriental and occidental teachings, written by a scientist (Ph.D. in Bio-organic Chemistry) who has received part of his training in the Orient and part in the United States, contains many points of interest.

Analytical Conics. By Barry Spain. Pergamon, New York, 1957. ix+145 pp. $5.00.

It is interesting to note a new book on the analytical geometry of the conic sections appearing at a time when most American universities are de-emphasizing conics. In addition to the usual analytic geometry of conic sections, the invariant theory classification of the general second degree equation and the concept of the line at infinity are introduced along with the basic ideas of cross ratio, homographic correspondence and line coordinates.

Probability Statistics and Truth. By Richard von Mises. Macmillan, New York, 1957. viii+244 pp. $5.00.

Whereas the original English translation of this book was based on the second German edition, this translation is based on the third (1951) German edition which includes work by Tornier, Doob, and others. The present translation by von Mises's widow, Professor Hilda Geiringer, is based on the earlier (1939) translation by Neyman, Scholl, and Rabinowitsch, supplemented by the new material in the 1951 edition. This translation will be doubly welcomed by mathe-

maticians since the earlier translation has been out of print for several years.

The Michigan Mathematical Journal. Vol. 4, No. 2. November 22, 1957. The University of Michigan Press, Ann Arbor. $1.00 per issue on direct orders from individuals, and $3.00 on orders from institutions.

It is a pleasure to call the *Michigan Mathematical Journal* to the attention of our readers. Its current editorial board consists of: Lipman Bers, Wilfred Kaplan, Irving Kaplansky, Edwin E. Moise, George Y. Rainich, Raymond L. Wilder; under the managing editorship of George Piranian.

Economic Models. By E. F. Beach. Wiley, New York, 1957. xi+227 pp. $7.50.

This book, written by an economist, contains what he considers "an elementary exposition of the mathematical and statistical implications of multiple relations in economic theory." It is intended to "fill the gap . . . between workers (in economics) who have a good grounding in advanced mathematics, and those whose mathematical background is sketchy." This reviewer is pleased to see a book with this purpose, but feels that the gap is considerably wider than the contents of this book can bridge. Only very elementary algebra is assumed, and the exposition falls considerably short of the calculus usually taught in integrated freshman mathematics courses. Current economic theory, as illustrated for example by the work of Ashby and Spivey, makes much use of linear difference equations, matric algebra and partial differential equations. However, Professor Beach's book should be welcomed as a firm step in the right direction.

Yano's Tables of Calculation. By Tsuneta Yano. The Tsuneta Yano Memorial Society, Tokyo, Japan, 1957. vi+162 pp.

The tables in this book are not as extensive or as carefully done as those of the standard American handbooks, but this is the only table, to my knowledge, which gives extensive conversion tables of weights and measures including the Oriental terms. It may prove a valuable reference for this reason.

Thermal Stresses. By B. E. Gatewood. McGraw-Hill, New York, 1957. xv+232 pp. $7.50.

"The mathematical procedures used in the first eight chapters do not go beyond elementary partial differential equations. Chapter 9 requires some knowledge of the theory of complex variables and of energy methods as used in structural analysis." Since the problems of high temperature are becoming more important in aircraft and reactor design, mathematicians may wish an up-to-date reference.

Approximations for Digital Computers. By Cecil Hastings, Jr. Princeton University Press, 1955. viii+201 pp. $4.00.

Although not a new book, it has apparently been overlooked in these re-

views. This monograph will prove of considerable use to the many mathematicians currently interested in high speed computing. Many of the approximation methods discussed here are not to be found conveniently in other sources.

CORRECTION

The publishers of *Introduction à l'Étude de l'Analyse Symbolique* (reviewed in this MONTHLY, vol. 65, 1958, pp. 215–216) have asked that the following addition and correction be made to the review.

The book contains 49 figures and the catalog price is 3,500 frs. plus postage.

NEWS AND NOTICES

EDITED BY L. J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NATIONAL SCIENCE FOUNDATION SUMMER INSTITUTES

The National Science Foundation has announced the following Summer Institutes for college and high school teachers of mathematics. Inquiries about a particular institute should be sent to the Director listed below. Unless otherwise indicated, the Director is located at the same institution as the institute:

University of Kansas, June 9–August 2: Senior high school and college teachers. G. B. Price.

High school teachers only:

Carleton College, June 16–July 25: K. O. May.

Oberlin College, June 16–August 8: Wade Ellis

Rutgers University, June 29–August 8: E. P. Starke.

University of Buffalo, July 7–August 1: Harriet F. Montague.

University of Chicago, June 23–August 1: A. L. Putnam.

University of Notre Dame, June 19–August 6: A. E. Ross.

University of Vermont, June 23–August 8: N. J. Schoonmaker.

University of Wisconsin, June 30–August 22: For teachers of eleventh and twelfth grade mathematics. H. Van Engen.

University of Wyoming, June 16–August 8: W. Norman Smith.

Western Michigan University, June 23–August 1: C. H. Butler.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1958.

Boston University, July 23 to August 14: Professor Scheid, seminar in applied mathematics.

Columbia University, Teachers College, July 7 to August 15: Professor Fehr, arithmetic in elementary school, history of mathematics; Professor Rosskopf, logic for teachers, current problems in the teaching of mathematics, departmental seminar in teaching mathematics; Dr. J. Hlavaty, probability and statistical inference for teachers, profes-

sionalized subject matter and advanced secondary school mathematics; Dr. Max Peters, the teaching of algebra, teaching of mathematics to exceptional children.

DePaul University, June 23 to August 1: Professor DeCicco, infinite series, linear algebras; Professor Merkes, college geometry; June 12 to August 1 (evenings): Professor Caton, mathematical statistics.

Duke University, June 10 to July 16: Professor Warner, finite mathematics; Professor Carlitz, advanced calculus. June 30 to August 8: Professor McGavock, fundamental concepts in algebra and geometry, finite mathematics. NSF Institute, June 9 to June 28: Professors Dressel, Elliott and McGavock, mathematics for teachers.

Kent State University, June 16 to July 19: Professor Brooks, selected topics for classroom teachers, Boolean algebra; Professor Olson, theory of numbers. July 21 to August 23: Professor Johnson, college geometry; Professor Kaiser, functions of a complex variable.

Michigan State University, June 24 to August 1: Professor Ewald, theory of numbers; Professor Grove, projective geometry; Professor Herzog, advanced calculus, theory of sets; Professor L. Kelly, concepts in geometry, mathematical problems; Professor Powell, concepts in calculus; Professor Sheedy, differential equations; Professor Weeg, computer coding; Visiting Professor H. J. Zassenhaus, representations of groups and Lie algebras. June 24 to August 22: Professor Campbell, matrices and groups, advanced mathematics for engineers I; Professor Hall, boundary value problems; Professor Helms, vector analysis, partial differential equations, advanced mathematics for engineers II; Professor Hill, functions of a real variable III; Professor Oehmke, advanced calculus III, functions of a complex variable I; Professor Reid, fluid dynamics, advanced mathematics for engineers III; Professor Wasserman, differential equations, advanced calculus II, numerical analysis I. NSF Institute for Community-Junior College Science and Mathematics Teachers, June 23 to August 1.

New York University, June 16 to July 24: Professor Shapiro, advanced calculus and applications I; Professor A. Peters, singular integral equations; Professor Magnus, special functions of mathematical physics; Professor Schwartz, introduction to Banach algebra. July 28 to September 5: Professor Rubin, advanced calculus and applications II; Professor Isaacson, elementary numerical methods and applications; Professor Shapiro, special topics in number theory (all half courses).

Northwestern University, June 21 to August 16: Engineering mathematics I; engineering mathematics II; numerical methods in mathematics; probability; history of mathematics; complex variables for applications; topics in modern mathematics for teachers; introduction to topology.

Syracuse University, June 30 to August 8: Research-oriented courses: Professor Kibbey, functions of a complex variable; Professor Gilbert, introduction to topology; Professor Gilchrist, programming digital computers. Courses for college teachers: Professor Gelbart, analysis and applications; Professor Exner, statistics. Courses for secondary teachers: Professor Moredock, the new Illinois-Beberman curriculum (demonstration class).

University of Chicago, June 23 to August 29: Professor Schilling, algebra II (vectors and matrices); Professor Chern, algebra IV (group theory), algebraic topology I; Professor Baily, point set topology, several complex variables; Professor Graves, calculus of variations; Visiting Professor Weil, automorphic functions of several variables; Visiting Professor Akizuki, local rings (July only). During July there will be a special program in algebraic geometry and several complex variables, with appropriate seminars.

University of Colorado, June 13 to August 23: Professor Bunt (University of Utrecht), teaching of secondary school mathematics, mathematics workshop in teaching problems; Professor Camp (University of Nebraska), mathematical statistics; Professor Nering (University of Arizona), game theory; Professor Magnus, theory of numbers, college geometry; Professor Briggs, functions of a complex variable.

University of Delaware, June 23 to August 1: Professor Wisner, fundamental concepts in mathematics; Professor Remage, introductory topology.

University of Florida, June 13 to August 9: Professor Ackerson, introduction to mathematical thought; Professor Meyer, mathematical statistics; Professor Sarafyan, higher mathematics for engineers and physicists; Professor Sobczyk, advanced topics in calculus; Professor Gaddum, introduction to topology; Professor Butson, vector analysis; Professor Moore, theory of groups of finite order; Professor Hutcherson, synthetic projective geometry; Professor Morse, foundation of geometry; Professor Kokomoor, history of elementary mathematics.

University of Michigan, June 23 to August 16: Professor Carver, calculus of finite differences; Professor Clarke, mathematical theory of probability, theory of statistics I; Professor Coburn, vector analysis; Professor Conner, advanced mathematics for engineers; Professor Craig, statistical analysis I; Professor Dushnik, introduction to differential equations, Fourier series and applications; Professor Dwyer, statistical analysis II, theory of statistics II; Dr. Galler, methods in high-speed computation; Dr. Griffin, advanced calculus; Professor Harary, introduction to matrices, introduction to the foundations of mathematics; Professor Hay, intermediate course in differential equations; Professor Leisenring, noneuclidean geometry, history of geometry; Dr. Maxwell, introduction to differential equations; Professor McLaughlin, theory of equations and determinants, algebra I; Professor Nesbitt, introduction to differential equations, mathematics of life insurance; Professor Ritt, introduction to functions of complex variable with applications, topics in modern mathematics for teachers; Professor Titus, modern operational mathematics, differential geometry; Professor Ullman, theory of functions of a complex variable; Professor Wilder, general spaces.

University of Minnesota, College of Science, Literature, and the Arts, June 16 to July 19: Professor Gelbaum, critical reasoning in mathematical analysis I, Laplace transforms; Professor Storvick, differential equations. July 21 to August 23: Professor Gelbaum, critical reasoning in mathematical analysis II; Professor Gil de Lamadrid, advanced algebraic theory, intermediate differential equations. It is expected that an Institute for High School Teachers of Mathematics will be conducted from June 16 to August 9.

University of Nebraska: Professor Basoco, differential equations; Professor Abel, topics in geometry; Professor Guy, mathematical analysis; Professor Leavitt, topics in algebra; Professor Jackson, topics in analysis. NSF Institute for High School Teachers: modern geometry, modern algebra.

University of North Carolina, June 3 to July 12: Professor Garner, history of mathematics; Professor Cameron, fundamental concepts; Professor Mackie, theory of equations; Professor Linker, differential equations; Professor Brauer, introduction to the theory of determinants and matrices; Professor Rohrbach, introduction into numerical analysis; Professor Pettis, topics in analysis, functions of a complex variable I. July 12 to August 20: Professor Hill, elementary mathematical statistics; Professor Lasley, analytic geometry from a higher standpoint; Professor MacNerney, functions of a complex variable II, summability; Professor Jones, foundations of geometry; Professor Wall, the theory of matrices.

University of Oklahoma, June 14 to August 12: Professor Bernhart, college geometry, vector analysis; Professor Giever, elementary differential equations; Professor Springer, partial differential equations; Professor Brixey, theory of numbers; Professor Goffman, topics in infinite series.

University of Pennsylvania, June 30 to August 9: introduction to real analysis, introduction to general topology, theory of sets.

University of Pittsburgh, June 9 to July 18, July 21 to August 29: Professor Knipp, differential equations; Professors Blumberg and Christiano, advanced calculus; Professor Bryson, vector analysis; Professor Taylor, functions of a complex variable; Professor

Laush, functions of a real variable; Professor Levine, modern algebraic theories; Professor Elyash, mathematical theory of probability. June 23 to August 15: Professor Elyash, differential equations; Professor Kovacs, mathematical theory of statistics; Professor Myers, recreational mathematics for teachers, theory of equations, teaching of secondary mathematics; Professor Leger, matrix theory. June 23 to August 15 (evenings): Professors Leger and Levine, differential equations; Professor Laird, mathematics of modern engineering; Professor Bryson, Laplace transform theory and applications.

University of Texas, June 10 to August 30: Professor Guy, Fourier and Laplace transforms. June 10 to July 21: Professor Craig, applications of tensor analysis; Professor Moore, theory of sets; Professor Ettlinger, research in differential and integral equations. July 22 to August 30: Professor Wall, theory of functions of real variables, functions of a complex variable; Professor Lane, continued fractions and applications.

University of Washington, June 23 to August 22: Professors Vaught, Pierce, and Blumenthal, linear algebra; June 23 to July 23: Professor Avann, introduction to modern algebra; Professor Livingston, advanced calculus I; Professors Blumenthal and Butler, differential equations; Professor Forrester, topics in applied analysis; Professor McFarlan, advanced analytic geometry; Professor Michael, topology. July 24 to August 22: Professor Beaumont, introduction to modern algebra; Professor Livingston, advanced calculus II; Professors Pierce and Haller, differential equations; Professor Forrester, topics in applied analysis; Professor Brownell, differential geometry; Professor Michael, topology.

University of Wisconsin, June 30 to August 22: Mr. Albright, theory and operation of computing machines; Visiting Lecturer Artzy, applied differential equations, theory of numbers; Professor Bicknell, introduction to mathematical statistics; Visiting Lecturer Dyer, advanced calculus (second half), advanced topics in algebraic topology; Dean Ingraham, determinants and matrices; Visiting Lecturer Dorothy Stone, advanced calculus (first half), theory of integration; staff, modern views of mathematics.

University of Wyoming, June 16 to July 18: Professor Barr, college geometry; Professor S. R. Smith, vector analysis, ordinary differential equations; Professor Steen, theory of equations. July 21 to August 22: Professor Barr, advanced calculus; Professor Schwid, solid analytic geometry, partial differential equations, Fourier series and boundary value problems; Professor S. R. Smith, numerical analysis.

West Virginia University, June 4 to July 16: Professor Cochran, astronomy for teachers; Professor Cunningham, higher plane curves; Mrs. Easton, history of mathematics; Professor Posey, advanced calculus; Professor Vest, complex variables. July 17 to August 27: Professor Cochran, advanced calculus; Mr. Lowenberg, geometry for teachers; Professor Peters, algebraic theories and higher plane curves; Professor Vest, complex variables. An Institute for High School Teachers of Mathematics and Physical Science will be conducted June 4 to July 16.

AEC-ASEE

Nine Institutes on Nuclear Energy for engineering educators will be held this summer under the sponsorship of the Atomic Energy Commission and the American Society for Engineering Education. The purpose of the institutes is to provide special training in the fields of nuclear energy and the nature of nuclear reactor problems so the teachers can incorporate this material in their teaching programs. The 1958 institutes will include four basic courses for teachers with no special background in nuclear energy, four advanced-level courses and one new basic course for teachers in technical institutes.

Participants will receive a minimum of two months' pay in addition to their regular salary for the academic year. For an applicant to be considered, the educational institution will be required to contribute to him a minimum of one month's salary. The AEC grant will match this contribution to a maximum of $750 plus round-trip railroad fare.

Applications for appointment may be obtained from the deans of engineering or from ASEE headquarters. They should be sent to Prof. W. Leighton Collins, Secretary of the ASEE, University of Illinois, Urbana, Illinois.

Each of the basic institutes will be combined programs of a university and a national laboratory, with a quota of from 25 to 30 participants at each location. The dates for all are June 23–August 15 and the locations are: North Carolina State College, Raleigh, with Oak Ridge National Laboratory; Cornell University, Ithaca, with Brookhaven National Laboratory; Purdue University, Lafayette, Ind. with Argonne National Laboratory; and University of California at Berkeley with Radiation Laboratory at Livermore.

The advanced institutes with quotas of from 20 to 25 participants will be as follows: Reactor Theory, University of Michigan, June 23 to August 15; Reactor Instrumentation and Control, Argonne National Laboratory, June 23 to August 15; Chemical Processing, Hanford Laboratories, June 22 to August 15; and Nuclear Metallurgy, Ames Laboratory, June 30 to August 22.

The course for teachers in technical institutes, with a quota of 25 participants, will be held at Pennsylvania State University from June 30 to August 8 and at Argonne National Laboratory from August 11 to August 22.

PERSONAL ITEMS

The establishment of the Richard Courant Lectureship in Mathematical Sciences at New York University was announced on January 8, 1958 at a convocation honoring Dr. Courant, Scientific Director of the University's Institute of Mathematical Sciences.

Hofstra College: Professor Loyal Ollman has been appointed Chairman of the Division of Natural Sciences, Mathematics, and Engineering; Dr. Harold Jacobson, Mr. Thadeus Smith, Mr. Abraham Soble and Miss Lois Chamberlin have been added to the Mathematics staff; Dr. E. R. Stabler will be on leave during the second semester of 1958 to promote a Peace by Education project.

Dr. W. B. Brown, formerly with Northrop Aircraft, Hawthorne, California, has been appointed Professor at Eastern Baptist College, St. Davids, Pennsylvania.

Mr. Horace Homesley, Jr., Graduate Student, University of Florida, has accepted a position on the staff of Atomic Power Division, Newport News Shipbuilding and Dry Dock Company, Virginia.

Dr. M. A. Hyman of Remington-Rand, UNIVAC, Philadelphia, has accepted a position as Senior Mathematician in the Research Center of the International Business Machines Corporation, Ossining, New York.

Dr. Azriel Rosenfeld, Senior Design Engineer, Ford Instrument Company, Long Island City, New York, has been appointed Visiting Assistant Professor at the Institute of Mathematics, Yeshiva University, New York City.

Professor Christopher R. Mitchell, Rhode Island College of Education, died on May 19, 1957.

Professor Emeritus W. A. Hurwitz, Cornell University, died on January 6, 1958. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-FIRST ANNUAL MEETING OF THE ASSOCIATION

The forty-first annual meeting of the Mathematical Association of America was held at the University of Cincinnati and the Hotel Sheraton-Gibson, Cincinnati, Ohio on Thursday and Friday, January 30 and 31, 1958, in conjunction with the annual meeting of the American Mathematical Society. There were registered 750 persons, including 519 members of the Association.

Sessions of the Association were held on Thursday morning in the Ball Room of the Hotel Sheraton-Gibson and on Friday morning and afternoon in the Roof Foyer of the Hotel. President Price presided on Thursday morning and Friday afternoon, and Dean Mina S. Rees on Friday morning. The Program Committee for the meeting consisted of E. A. Cameron, Chairman; E. P. Northrop, and D. E. Richmond.

FIRST SESSION OF THE ASSOCIATION

Retiring Presidential Address: "Mathematics Courses for Mathematics Teachers," by Dean W. L. Duren, Jr., University of Virginia.

"Lectureship Programs of the Association." Panel: Professor B. W. Jones, University of Colorado, Chairman; Professor G. B. Price, University of Kansas; Professor R. D. Schafer, University of Connecticut; Professor J. C. Eaves, University of Kentucky.

SECOND SESSION OF THE ASSOCIATION

"The National Science Foundation and Institutes for Mathematics Teachers," by Dr. Harry Kelly of the National Science Foundation.

"Organization and Operation of Institutes." Panel: Professor L. W. Johnson, Oklahoma State University, Chairman; Professor H. M. Bacon, Stanford University; Professor E. A. Cameron, University of North Carolina; Professor H. F. Fehr, Columbia University; Professor Joseph Landin, University of Illinois; Professor D. E. Richmond, Williams College; Professor R. M. Thrall, University of Michigan; Professor Henry Van Engen, University of Wisconsin.

THIRD SESSION OF THE ASSOCIATION

Annual Business Meeting of the Association.

"Relation of Institutes to the work of the Commission on Mathematics," by Dean A. E. Meder, Jr., Rutgers University, Executive Director of the Commission on Mathematics.

"Appropriate Courses for Institutes." Panel: Professor R. E. Johnson, Smith College; Professor R. A. Rosenbaum, Wesleyan University; Professor George Pólya, Stanford University (paper read by Professor Bacon); Professor E. J. McShane, University of Virginia; Professor J. L. Snell, Dartmouth College.

"Follow-Up Programs to Institutes," by Mrs. Marie S. Wilcox, Thomas Carr Howe High School, Indianapolis, Indiana.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Thursday afternoon in Parlor H of the Hotel Sheraton-Gibson with twenty-three members present. Among the more important items of business transacted were the following:

The Board approved the appointment by President Price of the following Nominating Committee for 1958: Wallace Givens, Chairman, H. M. Bacon and Rothwell Stephens.

The Board elected Lloyd J. Montzingo, Jr., of the University of Buffalo as Associate Secretary for 1958–1962, and E. A. Cameron of the University of North Carolina as member of the Finance Committee for 1958–1961.

The Board approved the following schedule of future meetings: Massachusetts Institute of Technology, August 25–26, 1958; University of Pennsylvania, January 22–23, 1959; University of Utah, August 31–September 1, 1959; and Oklahoma State University, August 28–29, 1961 (sic).

The Board ratified the constitution and by-laws of the Conference Organization of the Mathematical Sciences, which is replacing the Policy Committee for Mathematics.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday, January 31, 1958 in the Roof Foyer of the Hotel Sheraton-Gibson, Cincinnati, Ohio. President G. B. Price presided.

The Secretary announced the results of the balloting for officers in which 2370 votes were cast: Professor G. B. Thomas, Jr., Massachusetts Institute of Technology was elected First Vice-President for the two-year term 1958–1959, and Professors Howard Eves, University of Maine, and J. S. Frame, Michigan State University, were elected Governors for the three-year term 1958–1960.

The membership of the Association was 7094 at the end of 1957, a gain of 603 during the year.

Reports were made by Professor B. W. Jones for the Committee on Visiting Lecturers and by Professor J. S. Frame for the Committee on Employment Opportunities.

MEETING OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday, January 28 through Thursday. The Gibbs lecturer was Professor H. J. Muller. Invited addresses were delivered by Professor Nelson Dunford and Dr. C. D. Papakyriakopoulos. The retiring presidential address of Professor R. L. Wilder was delivered on Wednesday afternoon on the campus of the University of Cincinnati.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: I. A. Barnett, Chairman; R. W. Allen, J. L. Baker, Jr., H. M. Gehman, Arno Jaeger, H. D. Lipsich, C. I. Lubin, G. M. Merriman, B. A. Raymond, W. E. Restemeyer, J. W. T. Youngs.

Registration headquarters was on the mezzanine of the Hotel. Sleeping accommodations were available in the Hotel Sheraton-Gibson and other nearby hotels. Meals were available in the Hotel and at numerous nearby restaurants and cafeterias. The employment register and a text book exhibit were located near the registration desk.

A tea for the attending mathematicians and their guests was given by the University of Cincinnati on Wednesday afternoon in the Union Building on the Campus. The University provided special bus service on Wednesday from the Hotel to the Campus and return.

A complimentary banquet, through the generosity of the companies listed below, was held on Thursday evening in the Roof Garden of the Hotel. The sponsors of the banquet were AVCO Corporation, Cincinnati Gas and Electric Co., Cincinnati Milling Machine Co., General Electric Co. (Aircraft Gas Turbine Division), International Business Machines Corporation, Procter and Gamble Co., Union Central Life Insurance Co., Western and Southern Life Insurance Co. Professor T. H. Hildebrand served as toastmaster. Speakers were Dean H. S. Greene and Professor Emeritus C. N. Moore of

the University of Cincinnati, President Richard Brauer of the Society, and President G. B. Price of the Association. Professor G. A. Hedlund presented a resolution expressing the warm appreciation of those present to those who by their efforts and contributions have aided in making the meetings effective and enjoyable.

HARRY M. GEHMAN, *Secretary-Treasurer*

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The annual fall meeting of the Minnesota Section of the Mathematical Association of America was held on October 5, 1957 at Mankato State College, Mankato, Minn. Professor W. J. Thomsen of Mankato presided at the morning session. The Section Secretary, Professor F. L. Wolf of Carleton College, presided over the afternoon session in place of the Section Chairman, Professor O. E. Stanaitis of St. Olaf College, who was ill. There were 44 persons registered, including 28 members of the Association.

At the business meeting, Professor G. K. Kalisch of the University of Minnesota, Chairman of the Sectional Committee for the High School Contest, reported on the activities of his committee. Professor F. L. Wolf reported on the meeting of section officers held at the summer meeting of the Association in Pennsylvania. It was moved, seconded, and passed that the Section chairman be directed to appoint a Committee on High School-College Relations to investigate and implement ways and means by which the Section may have influence in bettering the mathematical education of students at the high school level in the Minnesota region.

The invitational address was given by Professor L. Hurwicz of the University of Minnesota. The address was entitled *Some Recent Developments in Mathematical Economics and Econometrics.*

The following short papers were presented:

1. *Some remarks on the new high school mathematics curriculum developed at the University of Illinois*, by Professor R. W. Sloan, Carleton College.

The author discussed some of the points wherein the U.I.C.S.M. Program differs from traditional courses at the high school level.

2. *The Apollonius Circle-Contact Problem*, by Dr. C. N. Mills, Sioux Falls College.

Historically the Apollonius Circle-Contact Problem is concerned with the construction of circles tangent to three circles mutually tangent externally. This paper gave the ten different cases from three points, or null circles, to three circles mutually external, but not tangent. The different cases have been constructed making use of many theorems of modern geometry. For this last case, the radius of each of the eight possible circles has been determined recently by Beckman Martin of Toledo, Ohio, using a trigonometric method. As far as the writer knows, no eighth degree algebraic equation has been published which gives the radius of each of the eight circles.

3. *Automatic coding for digital computers*, by Zane Motteler, University of Minnesota.

A good example of automatic coding in use today in large digital computers is the FORTRAN system for the IBM 704. In this system, mathematical formulas punched on cards in a definite format are translated directly into binary code by the machine, with the aid of a long binary tape. Basic problems of machine logic such as setting aside of storage space, loop calculations, subroutines and conditional transfers are very ingeniously handled. Automatic coding is especially advantageous in terms of the time saved in learning the code and in the initial coding of a problem. Input and output is extremely flexible. However, debugging time and printing time are often quite long, and make up for the other savings. Storage is inflexible and often uneconomical; spaces are wasted unless the needs of the problem are constant and can be set exactly in advance. Generally, however, automatic coding is preferred by coders and machine operators to standard symbolic coding because of its simplicity and the excellent output available.

F. L. WOLF, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University, Oklahoma, on October 25, 1957. Professor W. A. Rutledge, Vice-Chairman of the Section, presided. There were 192 persons in attendance, including 63 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor W. A. Rutledge, The University of Tulsa; Vice-Chairman, Professor Eunice Lewis, The University of Oklahoma; Secretary-Treasurer, Professor R. V. Andree, The University of Oklahoma. Professor R. B. Deal, Oklahoma State University of Agriculture and Applied Science, was appointed representative from the section on The Oklahoma Junior Academy of Science Advisory Committee. The committee for the 1958 High School Mathematics Contest was appointed as follows: Professor H. N. Carter, The University of Tulsa, Chairman; Miss Sarah Burkhart, The University of Tulsa; Mr. Richard G. Laatsch, The University of Tulsa.

After a discussion of the recommendations of the Commission on Mathematics of the C.E.E.B., and of the various topics considered at the August meeting of Sectional officers of M.A.A., it was decided to investigate the possibility of providing a limited "visiting lecture service" for high schools in this area. A committee consisting of: Professor Kathrine Mires, Northwestern State College (Chairwoman); Mr. Thomas Hill, Classen High School, Oklahoma City; and Professor R. B. Deal, Oklahoma State University of Agriculture and Applied Science was appointed.

The fall meeting of the Oklahoma Section is held in conjunction with the Oklahoma Education Association and is devoted to papers of particular interest to high school teachers. Research papers are presented in the spring meeting.

The morning meeting consisted of two symposia devoted to *Coming Practices in High School and College Mathematics*, and an hour talk by Professor Gertrude Hendrix, University of Illinois Committee on School Mathematics, on the topic, *The Mathematics Project of the University of Illinois Committee on School Mathematics: Content, Method, and Modus Operandi.*

Discussion in the symposia was spirited and varied with considerable audience participation. The invited speakers were: Eunice Lewis, University High School, Norman; Roy B. Deal, Oklahoma State University of Agriculture and Applied Science, Stillwater; Thomas Hill, Classen High School, Oklahoma City; William N. Huff, The University of Oklahoma, Norman; E. Truman Wester, Central State College, Edmond; and W. R. Orton, The University of Arkansas, Fayetteville; Richard Johnson, Smith College, Northampton, Mass.; Kathrine C. Mires, Northwestern State College, Alva; W. A. Rutledge, The University of Tulsa, Tulsa.

Some of the points made by various speakers were:

Several high schools in the state are currently giving good courses in freshman college mathematics and the universities are responding by permitting well-prepared incoming freshmen to skip the first semester or even the first year of mathematics. After 1959–60, remedial mathematics (Elementary Algebra and Geometry), will not be given at The University of Oklahoma, excepting in summer school. A sharp increase in enrollments in college mathematics was noted in most institutions. One school reported almost 300% increase this term. It was pointed out that, as far as general academic preparation is concerned, the state of Oklahoma is, at least on paper, better situated than most states since 95% of all teachers (grade school, junior high school and high school) have received bachelor's degrees from recognized institutions, and 40% have master's degrees. Relatively few of these degrees are in mathematics (as opposed to mathematics education). However, the belief was expressed that five years of training (rather than the present four) may soon be required for the standard (five year) teaching certificate in mathematics and science. Oklahoma no longer issues "life certificates." Discussion was lively

concerning the need for competent mathematicians to help develop courses for high school teachers which would cut across the boundaries of the usual graduate courses and present in an integrated course, the aspects of modern mathematics most needed by high school teachers. Several speakers urged the requirement of more modern abstract algebra.

The general consensus seemed to be that the mathematical outlook in the state of Oklahoma was encouraging, and that there was promise of further improvement.

R. V. ANDREE, *Secretary*

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twentieth annual meeting of the Northern California Section of the Mathematical Association of America was held at San Francisco State College, January 18, 1958. Professor B. J. Lockhart, Acting Chairman of the Section, presided at both the morning and afternoon sessions. There were 146 persons in attendance, including 80 members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor B. J. Lockhart, U. S. Naval Postgraduate School; Vice-Chairman, Professor Gerald Preston, San Jose State College; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College. The section also voted to continue the high school lectureship program in 1958.

By invitation of the section, Professor George Forsythe, Stanford University, delivered an address at the morning session entitled *Computers and Numerical Analysis in Mathematics Education.* Abstract of this address follows:

A prime need of students of mathematics is to learn that mathematics is a live and interesting field, to which the student himself can make an active contribution. Digital computers, from the hand-cranked model to the most advanced electronic machine, can provide better students with that feeling of life and interest which so fascinates them. The hand-cranked model vastly increases the scope of the arithmetic which a junior high-school student can perform, while the high school student can easily learn to code a machine like the IBM 650. In college a generous admixture of the numerical point of view livens up any course in mathematics. The necessity of actually getting a numerical answer out of a machine will increase anyone's respect for careful analysis. Numerical analysis is defined and discussed. As an example suited to high school or college students, the Seidel iterative solution of a system of linear algebraic equations is illustrated, together with a procedure for accelerating the convergence.

The following papers were presented:

1. *An objectionable quadrangle theorem.* by Professor C. M. Fulton, University of California, Davis.

The following theorem is frequently found in analytic geometry texts: The midpoints of two opposite sides of any quadrilateral and the midpoints of the diagonals are the vertices of a parallelogram. This theorem is not true for a trapezoid or parallelogram. Some suggestions are made for the improvement and proper interpretation of theorems of similar type.

2. *On the Brocard points of a triangle,* by Professor P. Yff, Fresno State College.

Given any triangle and any point in the Euclidean plane and a fixed angle θ, they determine a path which converges to the perimeter of a unique triangle and independent of the given point. Convergence occurs only for a certain set of values of θ. Each side of the second triangle forms the angle θ with the corresponding side of the given triangle. Each Brocard point of the given triangle is the limit point of a sequence of such inscribed triangles, the result being independent of θ.

3. *Reports on contests and lectureships,* by Professor E. M. Beesley, University of Nevada; Professor D. W. Blakeslee, San Francisco State College; and Professor Roy Dubisch, Fresno State College.

Professor Beesley reported that the Mathematics Department of the University of Nevada, with the assistance of the Nevada Chapter of Pi Mu Epsilon, held its first annual high school prize examination in mathematics on April 7, 1957. The examination was prepared and administered by the university's mathematics staff. Two hundred ninety-one students took the examination. Cash prizes, medals, and certificates were awarded.

Professor Blakeslee reported that 112 high schools and 3300 students participated in the 1957 Mathematical Association of American contest as compared to 73 schools and 2300 students in 1956.

Professor Dubisch reported on the first visiting lectureship program for high schools sponsored by the section. Not as many high schools participated as had been anticipated but the reaction of those who did was most favorable.

4. *Some properties of a generalized Euler ϕ-function*, by Professor H. L. Alder, University of California, Davis.

It is shown that many of the well-known properties of the Euler ϕ-function also hold with some modification for the generalized Euler ϕ-function $\phi(n, m)$, which was recently introduced by the author (see "A generalization of the Euler ϕ-function," to appear soon in this MONTHLY).

5. *On the digital control of machine tools*, by Professor H. D. Huskey, University of California, Berkeley.

The high performance of present day aircraft requires the precise machining of airframe structures merely to control weight. The specification of the motion of the parts of a milling machine to produce a given cutting path is an interesting problem in three dimensional analytic geometry. Methods used to cut required contours and surfaces are discussed in detail and ramifications in design techniques and engineering drawing are mentioned.

6. *On the curriculum for prospective high school teachers*, by Professor G. Pólya, Stanford University.

The author reports on the aims and contents of lectures he gave, or is giving, in three different institutes for high school teachers. This report will appear in the MONTHLY.

7. *Geometry as a way of thinking*, by Dr. E. Greer, Lockheed Aircraft Corporation, Sunnyvale.

The declining enrollment in geometry in the secondary schools is, in part, due to the way instructors are presenting the subject matter. The prime importance of this course is to train the student to make conclusions through scientific reasoning, developed so easily through the demonstrative proofs used in its exercises. The facts learned are secondary in importance. To increase, only slightly, the percentage of correct decisions made by people by this method of thinking would be valuable to society. The purpose of this paper is to point out this important function of geometry and to encourage improvement in its presentation.

8. *On looking backward at solutions—some remarks and examples*, by Professor C. M. Larsen, San Jose State College.

A student of mathematics can learn a great deal by examining his work after reaching a solution to a problem, instead of abandoning the problem as "done." Examining the solution, however, means more than merely gazing at it. Specifically, checking the solution and variation of the problem are among the useful activities that should enter into "Looking Back." These activities were illustrated in connection with two examples from elementary calculus, and the belief was expressed that such efforts can add considerable interest to otherwise routine problems.

ROY DUBISCH, *Secretary*

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CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, Ball State Teachers College, Muncie, May 3, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS, Kansas State Teachers College, Emporia, April 12, 1958.

KENTUCKY, University of Kentucky, Lexington, April 26, 1958.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Randolph-Macon Woman's College, Lynchburg, Virginia, April 26, 1958.

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April 19, 1958.

MICHIGAN

MINNESOTA, St. John's University, Collegeville, Minnesota, May 17, 1958.

MISSOURI, University of Missouri, Columbia, May 3, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, June 17, 1958 (joint meeting with ASEE, mathematics division).

OHIO, Denison University, Granville, April 26, 1958.

OKLAHOMA, Central State College, Edmond, April 18–19, 1958.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA, Lehigh University, Bethlehem, November 29, 1958.

ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April 18–19, 1958.

UPPER NEW YORK STATE, University of Montreal, Montreal, Quebec, Canada, May 10, 1958.

WISCONSIN, Carroll College, Waukesha, May 3, 1958.

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VARIATION, MULTIPLICITY, AND SEMICONTINUITY

LAMBERTO CESARI, Purdue University

The concept of total variation of a real function of one real variable is generally discussed in courses and books on real functions, or on integration. Nevertheless, the recent research on continuous varieties in Euclidean or metric spaces (curves, surfaces, *etc.*) have focused the attention on certain less-known aspects of the concept of total variation, whose extensions have an essential part in that research. In retrospect, these less-known facets of the familiar concept of total variation have become interesting. In dealing with these more elusive aspects of the concept of total variation we will come across a number of well-known properties, and we will list them without proof (See, *e.g.*, E. W. Hobson, *Theory of Functions of a Real Variable*, 1927; S. Saks, *Theory of the Integral*, 1937).

1. Jordan total variation. Given any single-valued, real, everywhere-finite function $f(u)$, $a \leqq u \leqq b$, and any subinterval $I = [\alpha, \beta]$ of $[a, b]$, by *variation* of f in I is meant, as usual, the number var $(f; I) = |f(\beta) - f(\alpha)|$ and by *oscillation* of f in I, the number osc $(f; I) = M - m$, where M, m denote the supremum and the infimum of f in I. Thus $0 \leqq \text{var}\ (f; I) \leqq \text{osc}\ (f; I) \leqq +\infty$. If $f(I)$ denotes the set of all real numbers $x = f(u)$ for some $u \in I$, *i.e.* $f(I) = [f(u),\ u \in I]$, and by diameter of a set we denote the supremum of the distances between pairs of points of the set, then we have again $M - m = \text{diam}\ f(I)$ and, therefore, osc $(f; I) = \text{diam}\ f(I)$.

If f is continuous in I, then osc $(f; I)$ is the difference $M - m$ between the maximum and the minimum of f in I. If f is monotone in I, *i.e.*, always nondecreasing or always nonincreasing, then var $(f; I) = \text{osc}\ (f; I)$.

Let $D = [a = u_0 < u_1 < \cdots < u_n = b]$ denote any finite subdivision of $[a, b]$ into subintervals $I_i = [u_{i-1}, u_i]$, $i = 1, \cdots, n$. Then the following two definitions of *total variation* appear natural:

$$V_1 = \sup_D \sum_{i=1}^{n} \text{var}\ (f; I_i), \qquad V_2 = \sup_D \sum_{i=1}^{n} \text{osc}\ (f; I_i),$$

where the supremum is taken in both with respect to all possible subdivisions D of $[a, b]$ as above. As a matter of fact, the following theorem holds:

(1.i) *For every function $f(u)$, $a \leqq u \leqq b$, we have $V_1 = V_2$.*

Proof. Obviously $V_1 \leqq V_2$. On the other hand, if $\epsilon > 0$ is any number, there is a subdivision D of $[a, b]$ in N parts such that, if $\Delta_i = \text{osc}\ (f; u_{i-1}, u_i)$, we have $\sigma = \sum_{i=1}^{N} \Delta_i > V_2 - \epsilon$ or $> \epsilon^{-1}$ according as $V_2 < +\infty$ or $V_2 = +\infty$. Now in each interval $[u_{i-1}, u_i]$ there are two points ξ_i', ξ_i'' with $|f(\xi_i'') - f(\xi_i')| > \Delta_i - \epsilon/N$ if $\Delta_i < +\infty$, $> \epsilon^{-1}$ if $\Delta_i = +\infty$. Now the points ξ_i', ξ_i'' divide each $[u_{i-1}, u_i]$ into one, or two, or three subintervals. If $D' = [a = v_0 < v_1 < \cdots < v_n = b]$, $N \leqq n \leqq 3N$, is the new subdivision of $[a, b]$, then the sum $\sum |f(v_j) - f(v_{j-1})|$ relative to the intervals $[v_{j-1}, v_j]$ contained in $[u_{i-1}, u_i]$ is $\geqq \Delta_i - \epsilon/N$ if $\Delta_i < +\infty$, $> \epsilon^{-1}$ if

$\Delta_i=+\infty$, and finally $\sum_{j=1}^{N} |f(v_j)-f(v_{j-1})| > \sigma-\epsilon$ or $>\epsilon^{-1}$. Thus $V_1 \geqq V_2$ and finally $V_1=V_2$. Thereby (1.i) is proved.

As usual we shall define as *total variation* of $f(u)$ in $[a, b]$, the common value of V_1 and V_2, *i.e.*, $V=V(f; a, b)=V_1=V_2$. Then $f(u)$ is said to be of bounded variation, or BV, in $[a, b]$ if $V<+\infty$. Obviously, if f is BV in $[a, b]$, then

$$\sum_{i=1}^{n} \text{var } (f; I_i) \leqq V < +\infty, \qquad \sum_{i=1}^{n} \text{osc } (f; I_i) \leqq V < +\infty,$$

for every subdivision D. Also f is bounded in $[a, b]$, since

$$|f(u)| = |f(u)-f(a)+f(a)| \leqq |f(a)| + V < +\infty.$$

If f is monotone in $[a, b]$ then $V=|f(b)-f(a)|$ and f is BV. If f has finitely-many relative maxima and minima, say at the points $a<\xi_1< \cdots <\xi_k<b$, then $V=|f(a)-f(\xi_1)|+|f(\xi_1)-f(\xi_2)|+ \cdots +|f(\xi_n)-f(b)|$ and f is BV. Thus for $f(u)=\sin u$, $0\leqq u \leqq 2\pi$, we have $V=4$. The two functions $f(u)$, $g(u)$, $0\leqq u\leqq 1$, defined by $f(u)=u^2 \sin u^{-1}$, $g(u)=u \sin u^{-1}$, $0<u\leqq 1$, $f(0)=g(0)=0$, need a less elementary analysis. Both f, g are continuous, f is BV, and g is not BV in $[0, 1]$.

Remark 1. In the considerations above we have used the two interval functions $\phi_1(I)=\text{var } (f; I)$ and $\phi_2(I)=\text{osc } (f; I)$. Both are nonnegative, subadditive functions, *i.e.*, if we denote either one of these functions by $\phi(I)$, then for every I and every finite subdivision $(I_1, \cdots, I_m)$ of I into subintervals we have $0 \leqq \phi(I) \leqq \phi(I_1)+ \cdots +\phi(I_m)$. Nevertheless $V(I)=V(f; I)$, *i.e.*, the total variation of f restricted to the interval $I=[\alpha, \beta]$, is an additive interval function, *i.e.*, $V(I)=V(I_1)+ \cdots +V(I_m)$ [cf. (5.iii)].

Remark 2. If $f=g\pm h$, then $V(f)\leqq V(g)+V(h)$. If f, g are BV, then $f\pm g$, fg, $|f|$, $|g|$ are BV, and also f/g is BV provided $|g|\geqq m>0$.

2. The Jordan total variation as a limit. Suppose that $f(u)$ is any continuous function of u in $[a, b]$. For every subdivision D let us denote by d the number $d=\max (u_i-u_{i-1})$, $i=1, \cdots, n$; and by δ the number $\delta=\max \text{osc } (f; u_{i-1}, u_i)$. Either $d\geqq 0$ or $\delta\geqq 0$ can be thought of as a *norm* of D. By the uniform continuity of $f(u)$ in $[a, b]$ it follows that $\delta \rightarrow 0$ as $d\rightarrow 0$, but the converse is not true, as it becomes clear when f is constant on some subinterval of $[a, b]$.

(2.i) *For every continuous function f we have*

$$V_1 = V_2 = \lim_{\delta\rightarrow 0} \sum_{i=1}^{n} \text{osc } (f; I_i) = \lim_{\delta\rightarrow 0} \sum_{i=1}^{n} \text{var } (f; I_i) \tag{2.1}$$

and the same holds as $d\rightarrow 0$.

Proof. The last part is a trivial consequence of (2.1) and of the remark above. Also the whole statement is trivial if $f(u)$ is a constant in $[a, b]$. Because of (1.i) we have

$$V_1 = V_2 \geqq \sum_{i=1}^{n} \operatorname{osc}(f; u_{i-1}, u_i) \geqq \sum_{i=1}^{n} \left| f(u_i) - f(u_{i-1}) \right|$$

for every subdivision D. Hence it is enough to prove that the last limit (2.1) exists and equals V_1. We may suppose f not constant in $[a, b]$. Given $\epsilon > 0$ let $D_0 = [a = u_0 < u_1 < \cdots < u_N = b]$ be any subdivision of $[a, b]$ such that

$$\sum_{i=1}^{N} \left| f(u_i) - f(u_{i-1}) \right| > V - \epsilon \text{ or } 1/\epsilon \tag{2.2}$$

according as $V < +\infty$ or $V = +\infty$. The existence of D_0 follows from the definition of V. We may suppose that f is constant in no interval $[u_{i-1}, u_i]$, since in such a case, we could well suppress (or add) points u_i in D_0 without reducing the sum (2.2) until D_0 satisfies such a requirement. Let $\eta_0 > 0$ denote the minimum of the oscillations of f in the intervals $[u_{i-1}, u_i]$, $i = 1, \cdots, n$, and denote by $\eta > 0$ the number $\eta = \min [\eta_0/3, \epsilon/N]$. Now let $D = [a = v_0 < v_1 < \cdots < v_n = b]$ be any subdivision of $[a, b]$ with norm $\delta < \eta$. Since $\eta \leqq \eta_0/3$, no part $[v_{j-1}, v_j]$ may contain completely a part $[u_{i-1}, u_i]$. We may separate the intervals $[v_{j-1}, v_j]$ into two classes C', C'' by putting in C' those parts which contain in their interior points u_i, $i = 1, \cdots, N-1$, and in C'' all others (Fig. 1). There are at

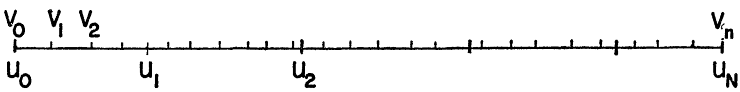

FIG. 1.

most $N-1$ intervals in C'. For each $[u_{i-1}, u_i]$ we have now a subdivision, say $u_{i-1} \leqq v_{j_s} \leqq v_{j_s+1} \leqq \cdots \leqq v_{j_s+t} \leqq u_i$, into parts, of which at most two are parts of intervals $[v_{j-1}, v_j]$ of C', and all others (at least one) are intervals of C''. We have

$$\begin{aligned} \left| f(u_i) - f(u_{i-1}) \right| &\leqq \left| f(u_{i-1}) - f(v_{j_s}) \right| + \left| f(v_{j_s+t}) - f(u_i) \right| + \sum \left| f(v_j) - f(v_{j-1}) \right| \\ &\leqq 2\eta + \sum \left| f(v_j) - f(v_{j-1}) \right|, \end{aligned} \tag{2.3}$$

where Σ ranges over all intervals $[v_{j-1}, v_j]$ of C'' in $[u_{i-1}, u_i]$. Obviously the first two terms in the second member of (2.3) do not exceed the oscillations of f in $[u_{i-1}, v_{j_s}]$, $[v_{j_s+t}, u_i]$ and hence each is less than η. Thus

$$\sum_{i=1}^{N} \left| f(u_i) - f(u_{i-1}) \right| \leqq 2N\eta + \sum{}' \left| f(v_j) - f(v_{j-1}) \right|,$$

where Σ' ranges over all $[v_{j-1}, v_j]$ of C'', and also

$$V_1 \geqq \sum_{i=1}^{n} \left| f(v_j) - f(v_{j-1}) \right| \geqq \sum{}' \left| f(v_j) - f(v_{j-1}) \right|$$

$$\geqq \sum_{i=1}^{N} \left| f(u_i) - f(u_{i-1}) \right| - 2N\eta \geqq V_1 - \epsilon - 2\epsilon \text{ or } > 1/\epsilon$$

according as $V_1 < +\infty$ or $V_1 = +\infty$. This holds for all subdivisions D with $\delta < \eta$ and this proves that the last limit (2.1) is V_1. Thereby (1.i) is proved.

Remark 1. Statement (2.i) does not hold necessarily for functions f which are not continuous.

Remark 2. The reasoning used in the proof of (2.i) is a typical one. An analogous reasoning is used in the proof that the upper and lower Darboux integrals in the theory of Riemann integration are limits (and not just infimum and supremum of the respective sums). The same reasoning proves that the Jordan length of a continuous curve is a limit.

A theorem of C. Scheeffer concerning subadditive interval functions may be given which is general enough to comprehend these different situations. (See C. Scheeffer, Acta Math., vol. 5, 1884–85, p. 49, and extensions in L. Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. I, 1921–23, p. 39). A general theory of subadditive (or overadditive) interval- (and set-) functions has been given by S. Banach (Fund. Math., vol. 6, 1924, pp. 170–188). On the same subject see also the exposition of S. Kempisty, *Fonctions d'intervalle non-additive*, Actualités Sci. Ind. 824, 1939.

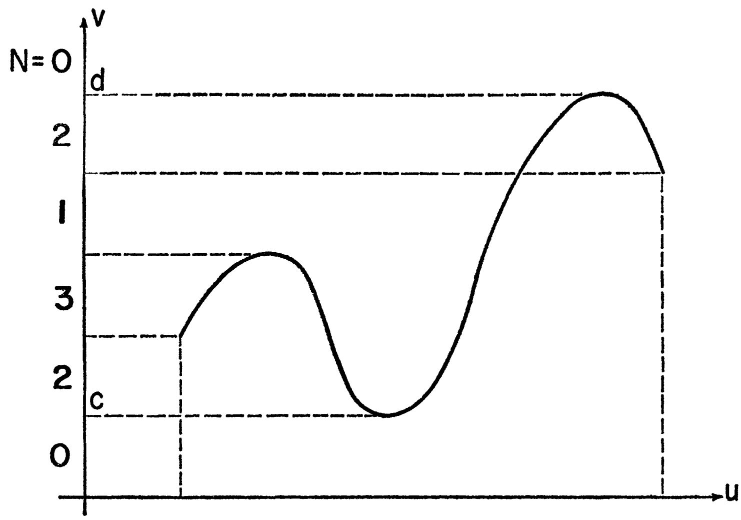

FIG. 2.

3. The multiplicity function. Again let $x=f(u)$, $a \leqq u \leqq b$, be any single-valued real continuous function, and let us denote by c and d its minimum and maximum in $[a, b]$. For every number $\bar{x}$, $-\infty < \bar{x} < +\infty$, let us denote by $N(\bar{x}) = N(\bar{x}; f, a, b)$ the number [zero, any positive integer, $+\infty$] of the solutions

of the equation $f(u)=\bar{x}$. This function $N(x)$, $-\infty<x<+\infty$, is usually denoted as the (*crude*) *multiplicity* function of $f(u)$. Obviously $N(x)$, $-\infty<x<+\infty$, is nonnegative and $N(x)=0$ for all $x>d$ and all $x<c$ (Fig. 2). We will prove below that $N(x)$ is a Borel-measurable function and thus the number

$$(3.1)\qquad V_3=\int_{-\infty}^{+\infty} N(x)dx$$

exists, $0\leqq V_3\leqq+\infty$. This number has been proposed by Banach as an alternative definition of total variation. Indeed the following theorem holds:

(3.i) (S. Banach). *For every continuous function f we have $V_1=V_3$.*

Proof. Let D_n denote the subdivision of $I=[a, b]$ into 2^n equal parts, and more precisely let I_{n1} denote the closed interval $[a, a+(b-a)/2^n]$, and I_{ni} all remaining intervals open at the left and closed at the right

$$I_{ni}=(a+(i-1)(b-a)/2^n,\ a+i(b-a)/2^n],\ i=2, 3, \cdots, 2^n.$$

The function f maps each interval I_{ni} into a segment (closed or not) of the x-axis, namely, the segment, from c_i to d_i, where $c_i=\min f$, $d_i=\max f$ in I_{ni}. The characteristic function $\phi_{ni}(x)$ of the set $f(I_{ni})$ is therefore zero for $x>d_i$ and $x<c_i$, one for $c_i<x<d_i$, while it may be zero or one at the two end points. Thus $\phi_{ni}(x)$, $-\infty<x<+\infty$, is certainly B-measurable, and so is the function $\phi_n(x)=\sum_{i=1}^{2^n}\phi_{ni}(x)$, $-\infty<x<+\infty$. In addition,

$$(3.2)\qquad \int_{-\infty}^{+\infty}\phi_n(x)dx=\sum_{i=1}^{2n}\int_{-\infty}^{+\infty}\phi_{ni}(x)dx=\sum_{i=1}^{2n}(d_i-c_i)=\sum_{i=1}^{2n}\operatorname{osc}(f, I_{ni})$$

and $\phi_n(x)\geqq 0$, $\phi_n(x)\leqq\phi_{n+1}(x)$ for all x and n. Thus the limit

$$\phi(x)=\lim_{n\to\infty}\phi_n(x),\qquad -\infty<x<+\infty,$$

exists everywhere in $(-\infty, +\infty)$ and $\phi(x)$ is also B-measurable. Finally, by a theorem of B. Levi (*cf.* Hobson, *loc. cit.*, Vol. I, p. 582), we have

$$\int_{-\infty}^{+\infty}\phi(x)dx=\lim_{n\to\infty}\int_{-\infty}^{+\infty}\phi_n(x)dx=V_2,$$

where the last equality follows from (3.2) and (2.i). All we have to prove now is that $\phi=N$. Indeed, if for a point $\bar{x}$ and an integer k (finite) we have $N(\bar{x})\geqq k$, there must be some k *distinct* points $\bar{u}$ with $f(\bar{u})=\bar{x}$, and these k points must belong to distinct intervals I_{ni} for all n large enough. Thus $\phi_{ni}(\bar{x})=+1$ for at least k different intervals I_{ni}, and hence $\phi(\bar{x})\geqq\phi_n(\bar{x})\geqq k$. Since k is any number $k\leqq N(\bar{x})$, we conclude that $\phi(\bar{x})\geqq N(\bar{x})$. Suppose now that $\phi(\bar{x})\geqq k$ for some $\bar{x}$ and integer k, then, since $\phi_n(\bar{x})\to\phi(\bar{x})$ and ϕ_n is necessarily an integer, we have $\phi_n(\bar{x})\geqq k$ for all large n, and hence $\bar{x}$ belongs to at least k distinct sets $f(I_{ni})$. Thus $f(\bar{u})=\bar{x}$ for at least k points $\bar{u}$ of k different intervals I_{ni}. Thus $N(\bar{x})\geqq k$,

and also $N(\bar{x}) \geqq \phi(\bar{x})$ since k is any integer $k \leqq \phi(\bar{x})$. This assures that $N(x) = \phi(x)$ for all x. Theorem (3.i) is proved. (See for another proof, S. Saks, *Theory of the Integral*, Warsawa, 1937, p. 280.)

Remark. If $I = [\alpha, \beta]$ and $[I_1, \cdots, I_m]$ is any finite subdivision of I into parts by means of the points $\alpha = u_0 < u_1 < \cdots < u_m = \beta$, then

$$N(x; f, I) = \sum_{i=1}^{m} N(x; f, I_j)$$

for all $x \neq f(u_j)$, $j = 0, 1, \cdots, m$. The proof is trivial.

4. The Jordan total variation as a Jordan length. The Jordan length of a curve is defined as the supremum of the elementary lengths of the inscribed polygonal lines. Now $x = f(u)$, $a \leqq u \leqq b$, can be thought of as a mapping of the interval $[a, b]$ of the u-axis E_1 into the x-axis E_1', hence as a "curve" C contained in the 1-dimensional space E_1'. Thus as u moves from a to b the point $x = f(u)$ moves from $f(a)$ to $f(b)$ but not necessarily monotonically. For instance, in Figure 3, $f(x)$ moves from $f(a)$ backwards to the minimum c of f, then after a number of forward and backward movements reaches the maximum d of f,

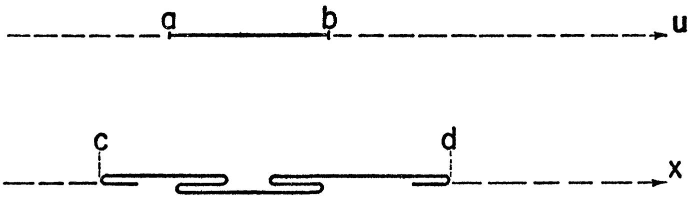

FIG. 3.

and ends in $f(b)$. We shall denote by V_4 the Jordan length of the wormlike curve C. Let us prove that $V_1 = V_4$. For every subdivision $D = [a = u_0 < u_1 < \cdots < u_n = b]$ of $[a, b]$ let $p_i = f(u_i)$ be the image of u_i on C, $i = 0, 1, \cdots, n$. Then the polygonal line $P = p_0 p_1 p_2 \cdots p_n$ inscribed in C, has sides $p_{i-1}p_i$ each of length equal to the distance of p_{i-1} and p_i, *i.e.*, $|f(u_i) - f(u_{i-1})|$. Thus the elementary length $l(P)$ of P is the sum of these numbers, and

$$V_4 = \sup l(P) = \sup \sum_{i=1}^{n} |f(u_i) - f(u_{i-1})| = V_1.$$

5. Positive and negative total variations. If m denotes any real number we shall denote by m^+, m^- the numbers $m^+ = m$ if $m \geqq 0$, $m^+ = 0$ if $m \leqq 0$; $m^- = 0$ if $m \geqq 0$, $m^- = |m|$ if $m \leqq 0$. In other words $m^+ = 2^{-1}(|m| + m) \geqq 0$, $m^- = 2^{-1}(|m| - m) \geqq 0$, and hence $m^+ + m^- = |m|$, $m^+ - m^- = m$.

We shall now consider a single-valued real function $x = f(u)$, $a \leqq u \leqq b$. As usual we will denote by $D = [a = u_0 < u_1 < \cdots < u_n = b]$ any finite subdivision of $[a, b]$ and we will then denote by *positive* and *negative total variations*, V_+

$= V_+(f; a, b)$, $V_- = V_-(f; a, b)$ of f in $[a, b]$ the numbers $0 \leqq V_+, V_- \leqq +\infty$,

$$(5.1) \qquad V_+ = \sup_D \sum_{i=1}^n [f(u_i) - f(u_{i-1})]^+, \qquad V_- = \sup_D \sum_{i=1}^n [f(u_i) - f(u_{i-1})]^-.$$

Obviously, $0 \leqq V_+ \leqq V \leqq +\infty$, $0 \leqq V_- \leqq V \leqq +\infty$. Hence V_+, V_-, V are all finite if f is BV.

If we denote, for the sake of brevity, by $\sum$, $\sum_+$, $\sum_-$ the sums

$$\sum = \sum_{i=1}^n |f(u_i) - f(u_{i-1})|, \qquad \sum_+ = \sum_{i=1}^n [f(u_i) - f(u_{i-1})]^+,$$
$$\sum_- = \sum_{i=1}^n [f(u_i) - f(u_{i-1})]^-,$$

we have

$$(5.2) \qquad \sum_+ + \sum_- = \sum, \qquad \sum_+ - \sum_- = f(b) - f(a),$$

and hence

$$(5.3) \qquad 2\sum_+ = \sum + f(b) - f(a), \qquad 2\sum_- = \sum - f(b) + f(a).$$

These relations show that when $\sum$ approaches its supremum V, then also $\sum_+$ and $\sum_-$ approach their suprema V_+, V_- respectively, and the converse is also true. As a corollary of (2.i) we have then

(5.i) *For every continuous function f we have*

$$V_+ = \lim_{\delta \to 0} \sum_{i=1}^n [f(u_i) - f(u_{i-1})]^+, \qquad V_- = \lim_{\delta \to 0} \sum_{i=1}^n [f(u_i) - f(u_{i-1})]^-$$

and the same holds if $d \to 0$.

(5.ii) *For every function $f(u)$, $a \leqq u \leqq b$, we have $V_+ + V_- = V$, and for every BV function, we have also $V_+ - V_- = f(b) - f(a)$.*

Proof. This is a consequence of the formulas (5.2) and (5.3) and of the remark above.

(5.iii) *For every function $f(u)$, $u \in I = [a, b]$, and any finite subdivision $[I_1, \cdots, I_k]$ of I into subintervals we have*

$$V(I) = \sum_{j=1}^k V(I_j), \qquad V_+(I) = \sum_{j=1}^k V_+(I_j), \qquad V_-(I) = \sum_{j=1}^k V_-(I_j).$$

We omit the simple proof.

6. The functions $v(u)$, $v_+(u)$, $v_-(u)$. Given any function $f(u)$, $a \leqq u \leqq b$, we shall denote by $v(u)$, $v_+(u)$, $v_-(u)$, respectively, the total variation, the positive, and the negative total variations of f in $[a\ \ u]$:

$$v(u) = V(f; a, u), \qquad v_+(u) = V_+(f; a, u), \qquad v_-(u) = V_-(f; a, u).$$

Obviously, $v(x)$, $v_+(x)$, $v_-(x)$ are nonnegative, monotone nondecreasing functions, $0 \leqq v, v_+, v_- \leqq +\infty$, and $v(b) = V$, $v_+(b) = V_+$, $v_-(b) = V_-$. Thus v, v_+, v_- are everywhere finite if and only if f is BV. Also, $v(a) = v_+(a) = v_-(a) = 0$. As a corollary of (5.ii) we have now

(6.i) *For every BV function f we have $v_+(u) + v_-(u) = v(u)$, $v_+(u) - v_-(u) = f(u) - f(a)$ for all $a \leqq u \leqq b$.*

(6.ii) *A function f is BV if and only if $f = g - h$ is the difference of two monotone (everywhere-finite) functions.*

Statement (6.ii) is a corollary of (6.i) and of Remark 2 at the end of Section 1.

Remark 1. The decomposition mentioned in (6.i) is unique. Indeed, $v_+(u) = 2^{-1}[f(u) - f(a) + v(u)]$, $v_-(u) = 2^{-1}[-f(u) + f(a) + v(u)]$. A function $f(u)$, $a \leqq u \leqq b$, discontinuous at the point $\bar{u}$ is said to have a discontinuity of the first kind at $u = \bar{u}$ if both limits $f(\bar{u} - 0)$, $f(\bar{u} + 0)$ exist and are finite (and are not both equal to $f(\bar{u})$). Monotone (everywhere-finite) functions have only discontinuities of the first kind. Consequently, BV-functions also have only discontinuities of the first kind.

(6.iii) *If $f(u)$, $a \leqq u \leqq b$, is BV in $[a, b]$ and is continuous at $u = u_0$, then also $v(u)$, $v_+(u)$, $v_-(u)$ are continuous at $u = u_0$.*

This property is usually given in all expositions.

(6.iv) *A continuous function f is BV if and only if it is the difference of two monotone nondecreasing continuous functions.*

A consequence of (6.i) and (6.iii).

As is well known, a function f is said to be AC in $[a, b]$ if, given $\epsilon > 0$, there is a $\delta > 0$, such that, for every finite set of nonoverlapping intervals $[\alpha_i, \beta_i]$, $i = 1, \cdots, n$, with $\sum(\beta_i - \alpha_i) < \delta$, we have $\sum |f(\beta_i) - f(\alpha_i)| < \epsilon$. The following statements too are well known, and easily proved: (α) An AC function $f(u)$, $a \leqq u \leqq b$, is continuous and BV. (β) A function $f(u)$, $a \leqq u \leqq b$, is AC if and only if the function $v(u)$ is AC.

Remark. Examples of continuous BV functions which are not AC are usually given in all expositions. One of them is the nondecreasing continuous function $\phi(u)$, $0 \leqq u \leqq 1$, $\phi(0) = 0$, $\phi(1) = 1$, which is constant on each of the complementary intervals of the ternary Cantor set (Hobson, *loc. cit.*, Vol. I, p. 123, p. 368).

7. Lower semicontinuity of the total variation. We come now to one of the most important properties of the total variation: its lower semicontinuity. By this we mean that, if $f(u)$, $a \leqq u \leqq b$, is any function and $f_n(u)$, $a \leqq u \leqq b$, $n = 1, 2, \cdots$, a sequence of functions approaching $f(u)$, *i.e.*, $f(u) = \lim f_n(u)$ as

$n \to \infty$ for all u, if V, V_n are the total variations of f, f_n in $[a, b]$, then (7.i, 7.ii)

$$V \leqq \liminf_{n\to\infty} V_n. \tag{7.1}$$

To understand this property, let us suppose first that f is continuous and has finitely-many maxima and minima at points $a < \xi_1 < \cdots < \xi_k < b$ (Fig. 4). Let S be the sum $|f(a) - f(\xi_1)| + |f(\xi_1) - f(\xi_2)| + \cdots + |f(\xi_n) - f(b)|$ and S_n, the analogous sum for f_n and the *same points* $\xi_1, \cdots, \xi_k$. Since $f(a) = \lim f_n(a)$, $f(\xi_i) = \lim f_n(\xi_i)$, $f(b) = \lim f_n(b)$ as $n \to \infty$, $i = 1, \cdots, k$, we have $S = \lim S_n$ as $n \to \infty$. By $V = S$, $S = \lim S_n$, $S_n \leqq V_n$, we conclude that (7.1) holds. Also, we understand that because of small oscillations of each of the functions f_n, in the $k+1$ intervals $[a, \xi_1], \cdots, [\xi_k, b]$ (oscillations not affecting the convergence of the sequence $f_n(u)$), it may occur that V_n is very large in comparison with V. That this situation actually may occur is shown by the examples below. On the other hand, the property (7.1) is very general as the statements (7.i), (7.ii) below show.

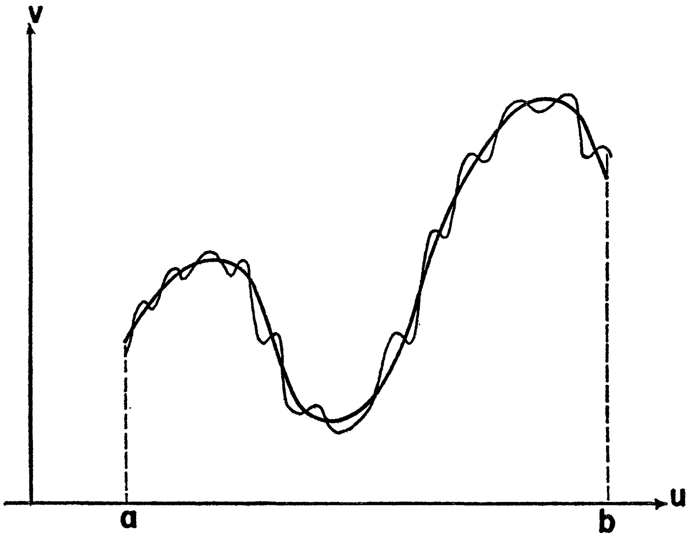

FIG. 4.

Example 1. Suppose $x = f(u) = 0$, $0 \leqq u \leqq 1$, and $x = f_n(u)$, $0 \leqq u \leqq 1$, is represented by the polygonal lines OBA, OC_1MC_2A, $OD_1N_1D_2MD_3N_2D_4A$, $\cdots$ of Figure 5, *i.e.*, $f_n(u) = u - 2^{-n}i$ if $w^{-n}i \leqq u \leqq 2^{-n}i + 2^{-n-1}$, $f_n(u) = 2^{-n}(i+1) - u$ if $2^{-n}i + 2^{-n-1} \leqq u \leqq 2^{-n}(i+1)$, $i = 0, 1, \cdots, 2^n - 1$, $n = 1, 2, \cdots$. Then $f = \lim f_n$ uniformly in $[0, 1]$, $V = 0$, $V_n = 1$, $n = 1, 2, \cdots$, and $0 = V < \lim V_n = 1$.

Example 2. Let $f(u) = 0$, $f_n(u) = n^{-1} \sin n^s u$, $0 \leqq u \leqq 2\pi$, for some real s and those $n \geq 0$ for which n^s is an integer. Then $f = \lim f_n$ uniformly in $[0, 2\pi]$, $V = 0$, $V_n = 4n^{s-1}$. Thus for $s = 1$, we have $0 = V < \lim V_n = 4$; for $s = 2$, we have

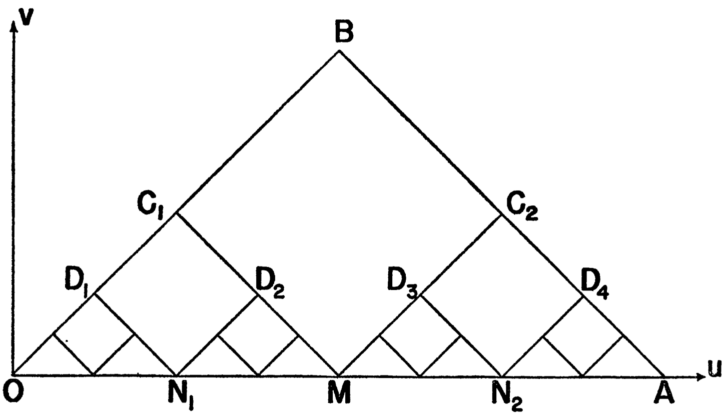

FIG. 5.

$0=V<\lim V_n=+\infty$; for $s=2^{-1}$ if $n=1, 4, 9, \cdots$, $s=1$ otherwise, we have $0=V=\liminf V_n<\limsup V_n=1$.

(7.i) *If $f(u)$, $f_n(u)$, $a\leqq u\leqq b$, are functions in $[a, b]$ and $f_n(u)\to f(u)$ as $n\to\infty$ for all $u\in[a, b]$, then*

$$(7.2)\quad V(f) \leqq \liminf_{n\to\infty} V(f_n), \qquad V_+(f) \leqq \liminf_{n\to\infty} V_+(f_n), \qquad V_-(f) \leqq \liminf_{n\to\infty} V_-(f_n).$$

Proof. Given $\epsilon>0$ there is a subdivision D such that

$$\sum_{i=1}^{N} |f(u_i) - f(u_{i-1})| > V - \epsilon \text{ or } 1/\epsilon$$

according as $V<+\infty$ or $V=+\infty$. Since $f_n(u_i)\to f(u_i)$ as $n\to\infty$, $i=0, 1, \cdots, N$, there is an n_i such that $|f_n(u_i)-f(u_i)|<\epsilon/2N$ for $n\geqq n_i$, $i=0, 1, \cdots, N$, and hence also for $n\geqq n_0=\max n_i$. For $n\geqq n_0$ we have also $|f_n(u_i)-f_n(u_{i-1})| > |f(u_i)-f(u_{i-1})| -\epsilon/N$ and finally

$$\left.\begin{matrix} V-\epsilon \\ 1/\epsilon \end{matrix}\right\} < \sum_{i=1}^{N} [|f_n(u_i) - f_n(u_{i-1})| + \epsilon/N] \leqq V(f_n) + \epsilon,$$

for all $n\geqq n_0$. This implies (7.2). Analogous proof holds for V_+ and V_-.

A variant of (7.i) is the following, for which we omit the proof.

If $f(u)$, $a\leqq u\leqq b$, is a continuous function, if $f_n(u)$, $a\leqq u\leqq b$, $n=1, 2, \cdots$, are functions with $f_n(u)\to f(u)$ as $n\to\infty$ at least for all u of a set which is everywhere dense in $[a, b]$, then relations (7.2) *hold.*

Remark. In neither of the theorems above uniform convergence is required, and in the first one the continuity of the functions is not required.

8. On maxima and minima of continuous functions. Before we discuss further questions concerning the multiplicity function N, we mention some known results on maxima and minima of continuous functions, which are of interest in themselves.

A point u_0 is said to be a point of weak [strict] relative maximum for a function $f(u)$, $a \leqq u \leqq b$, if (i) $a < u_0 < b$; and (ii) there is a neighborhood U of u_0 where $f(u) \leqq f(u_0)$ for all $u \in U[f(u) < f(u_0)$ for all $u \in U$, $u \neq u_0]$. Analogous definitions hold for weak and strict relative minima. Let E_w, e_w, E_s, e_s be the sets of all points $u \in (a, b)$ of weak maximum, weak minimum, strict maximum, strict minimum, respectively. Let M_w, m_w, M_s, m_s be the sets of all values x taken by $f(u)$ at the points u of E_w, e_w, E_s, e_s respectively; that is,

$$M_w \equiv [f(u),\, u \in E_w] = f(E_w), \qquad m_w \equiv [f(u),\, u \in e_w] = f(e_w),$$

$$M_w \equiv [f(u),\, u \in E_s] = f(E_s), \qquad m_s \equiv [f(u),\, u \in e_s] = f(e_s),$$

$$E_s \subset E_w, \qquad M_s \subset M_w, \qquad e_s \subset e_w, \qquad m_s \subset m_w.$$

(8.1) *The sets E_s, e_s, M_s, m_s, M_w, m_w are countable.*

Proof. Consider the set A_n of all points $u_0 \in E_s$ such that $f(u) < f(u_0)$ for all $u_0 - 1/n < u < u_0 + 1/n$, $u \neq u_0$. Obviously $E_s = A_1 + A_2 + \cdots$. Let us prove that, if two points $u_1, u_2 \in A_n$, then $|u_1 - u_2| \geqq 1/n$. Indeed, in the contrary case, we have $f(u_1) < f(u_2)$, $f(u_2) < f(u_1)$, a contradiction. Thus each set A_n is finite and E_s is countable. Analogous proof holds for e_s. The sets M_s, m_s are also countable since they are the sets of values taken by $f(u)$ on E_s, e_s respectively.

Consider the set B_n of all points $u_0 \in E_w$ such that $f(u) \leqq f(u_0)$ for all $u_0 - 1/n < u < u_0 + 1/n$ and $C_n = f(B_n)$. Obviously $M_w = C_1 + C_2 + \cdots$. Let us prove that if two numbers $x_1, x_2 \in C_n$, $x_1 \neq x_2$, and $x_1 = f(u_1)$, $x_2 = f(u_2)$, $u_1, u_2 \in B_n$, then $|u_1 - u_2| \geqq 1/n$. Indeed, in the contrary case, we would have $x_1 \geqq x_2$, $x_1 \leqq x_2$, and hence $x_1 = x_2$, a contradiction. Thus each set C_n is finite and M_w is countable.

Remark. The sets E_w, e_w may well be uncountable since, if f is constant in an interval $[\alpha, \beta]$, then $[\alpha, \beta] \subset E_w$, $[\alpha, \beta] \subset e_w$. For some of the statements above cf. S. Saks, loc. cit., p. 261.

9. The multiplicity functions N_c, N_+, N_- and their properties of lower semicontinuity. The number $N(\bar{x})$ answers the question of how many times the continuous function $f(u)$ takes the value $\bar{x}$ in $[a, b]$, or, geometrically, how many points the curve $x = f(u)$, $a < u < b$, has in common with the straight line $x = \bar{x}$. Analogously we may ask how many times the same curve "crosses" the straight line $x = \bar{x}$, or how many times it does so from below to above, or from above to below. The answer is given by the functions $N_c(x)$, $N_+(x)$, $N_-(x)$ defined below, the first of which coincides with $N(x)$ for all but countably many points, and all related to the corresponding total variations V, V_+, V_- of $f(u)$ in $[a, b]$ by the Banach-type relations

$$V = \int_{-\infty}^{+\infty} N(x)dx = \int_{-\infty}^{+\infty} N_c(x)dx, \quad V_+ = \int_{-\infty}^{+\infty} N_+(x)dx, \quad V_- = \int_{-\infty}^{+\infty} N_-(x)dx.$$

The functions $N_c(x)$, $N_+(x)$, $N_-(x)$ are called the *corrected multiplicity*, the *positive* and *negative multiplicity* of $f(x)$ in $[a, b]$. The functions $N_c(x)$, $N_+(x)$, $N_-(x)$ have properties of lower semicontinuity, both with respect to x and to f, which will be stated below. These properties are analogous to the ones proved above for V, V_+, V_- (8.i). The crude multiplicity function $N(x)$ is not lower semicontinuous as examples immediately show.

For any interval $I = [\alpha, \beta] \subset [a, b]$ with $\alpha < \beta$, let $\phi(x; I, f)$ be the function defined by

$$\phi(x) = \phi(x; I, f) = \begin{cases} +1 & \text{if } f(\alpha) < x < f(\beta), \\ -1 & \text{if } f(\alpha) > x > f(\beta), \\ 0 & \text{otherwise.} \end{cases} \qquad -\infty < x < +\infty,$$

The function $\phi(x; I, f)$ has the following property of addition:

If $\alpha = u_0 < u_1 < \cdots < u_m = \beta$ is any finite subdivision of $I = [\alpha, \beta]$ into the m intervals $I_i = [u_{i-1}, u_i]$, $i = 1, \cdots, m$, and $f(u_i) \neq x$, $i = 0, 1, \cdots, m$. then

$$\phi(x; I, f) = \sum_{i=1}^{m} \phi(x; I_i, f). \tag{9.1}$$

Indeed, if $d_0 = f(\alpha) - \bar{x}$, $d_m = f(\beta) - \bar{x}$ have opposite signs, say $d_0 < 0 < d_1$, and $d_i = f(u_i) - \bar{x}$, $i = 0, 1, \cdots, m$, then the sequence $d_0, d_1, \cdots, d_m$ is made up of numbers all $\neq 0$, and it must have at least one variation, more precisely, one variation from $-$ to $+$ more than variations from $+$ to $-$. The other cases can be dealt with analogously.

Let us now denote, as usual, by $\phi^+(x)$, $\phi^-(x)$ the functions

$$\phi^+(x) = \phi^+(x; I, f) = 2^{-1}[\,|\phi(x)| + \phi(x)] = \begin{cases} +1 & \text{if } f(\alpha) < x < f(\beta), \\ 0 & \text{otherwise;} \end{cases}$$

$$\phi^-(x) = \phi^-(x; I, f) = 2^{-1}[\,|\phi(x)| - \phi(x)] = \begin{cases} +1 & \text{if } f(\alpha) > x > f(\beta), \\ 0 & \text{otherwise.} \end{cases}$$

Obviously we have $0 \leqq \phi^+, \phi^- \leqq |\phi|$, $\phi^+ + \phi^- = |\phi|$, $\phi^+ - \phi^- = \phi$ for all $-\infty < x < +\infty$. Let us observe that (9.1) implies

$$|\phi(x; I, f)| \leqq \sum_{i=1}^{m} |\phi(x; I_i, f).|$$

and, by using the definitions above, we have also

$$\phi^+(x; I, f) \leqq \sum_{i=1}^{m} \phi^+(x; I_i, f), \qquad \phi^-(x; I, f) \leqq \sum_{i=1}^{m} \phi^-(x; I_i, f). \tag{9.2}$$

Let $S=[I]$ be any finite system of nonoverlapping intervals $I=[\alpha, \beta]$ in $[a, b]$ and denote by $N_c(x)=N_c(x; f)$, $N_+(x)=N_+(x; f)$, $N_-(x)=N_-(x; f)$ the functions defined by

$$N_c(x) = \sup_S \sum_{I\in S} \mid \phi(x;f,I) \mid, \qquad N_+(x) = \sup_S \sum_{I\in S} \phi^+(x; f, I),$$
$$N_-(x) = \sup_S \sum_{I\in S} \phi^-(x; f, I).$$

Obviously, $0\leqq N_c$, N_+, $N_-\leqq +\infty$, $N_+\leqq N$, $N_-\leqq N$ for all $-\infty<x<+\infty$.

(9.i) *For every continuous function $x=f(u)$, $a\leqq u\leqq b$, we have $0\leqq N_c(x) \leqq N(x)$ for all $-\infty<x<+\infty$, and also $N_c(x)=N(x)$ for all but countably many x.*

(9.ii) *For every continuous function $x=f(u)$, $a\leqq u\leqq b$, we have $N_c(x)=N_+(x) +N_-(x)$ for all $-\infty<x<+\infty$.*

Proof of (9.i) *and* (9.ii). If $k\leqq N_c(\bar{x})$ for some $\bar{x}$ and integer k, then $f(u)-\bar{x}$ changes sign in at least k nonoverlapping intervals I, and thus $\bar{x}$ is taken in interior points of at least k nonoverlapping intervals, hence in at least k distinct points. Thus $k\leqq N(\bar{x})$ and finally $N_c(\bar{x})\leqq N(\bar{x})$. Suppose that $\bar{x}$ is any real number with $\bar{x}\neq f(a)$, $\bar{x}\neq f(b)$, $\bar{x}\notin M_w+m_w$, and suppose $k\leqq N(\bar{x})$ for some integer k. There must be at least k distinct points $\bar{u}$ in $[a, b]$ with $f(\bar{u})=\bar{x}$. Consider k disjoint open intervals U each containing one point $\bar{x}$. Since $\bar{x}\neq f(a)$, $\bar{x}\neq f(b)$, $\bar{x}\in M_w+m_w$, the difference $f(u)-\bar{x}<f(u)-f(\bar{u})$ changes sign in each of these intervals, otherwise $\bar{x}$ would belong to M_w+m_w. Therefore, in each U there is a closed subinterval $I=[\alpha, \beta]$ such that $f(u)-\bar{x}$ has values of opposite signs at α and β. Thus $|\phi(\bar{x};f, I)|=1$ and, since the k intervals I are not overlapping, we have also $N_c(\bar{x})\leqq k$. Since k is any integer $k\leqq N(\bar{x})$ we conclude that $N(\bar{x})\leqq N_c(\bar{x})$ and finally, $N(\bar{x})=N_c(\bar{x})$ for all but countably many $\bar{x}$. Thus (9.i) is proved.

For any $\bar{x}$ let k be any integer with $k\leqq N_c(\bar{x})$. Then there are at least k nonoverlapping intervals $I=[\alpha, \beta]$ such that $f(\alpha)<\bar{x}<f(\beta)$ or $f(\alpha)>\bar{x}>f(\beta)$, that is, either $\phi(\bar{x}; I, f)=+1$ or $\phi(\bar{x}; I, f)=-1$. If k_1, k_2 are the numbers of intervals I of the first and of the second type we have $k=k_1+k_2\leqq N_+(\bar{x})+N_-(\bar{x})$. Since this holds for every $k\leqq N_c(\bar{x})$ we conclude that $N_c(\bar{x})\leqq N_+(\bar{x})+N_-(\bar{x})$. Suppose now that $k_1\leqq N_+(\bar{x})$, $k_2\leqq N_-(\bar{x})$ for some two integers k_1, k_2. There must be k_1 nonoverlapping intervals, say I_+, with $\phi^+(\bar{x}; I_+, f)=+1$; and k_2 nonoverlapping intervals, say I_-, with $\phi^-(\bar{x}; I_-, f)=+1$. On the other hand, it may well occur that intervals I_- are partially overlapping with intervals I_+ and vice versa. Let D be the finite subdivision of $[a, b]$ into parts which is obtained by using all end points of the intervals I_+, I_- as points of subdivision. By (9.2) we deduce that in each interval I_+ there must be at least one part, say J_+, with $\phi^+(\bar{x}; J_+, f) =+1$; and in each interval I_- there must be at least one part, say J_-, with $\phi^-(\bar{x}; J_-, f)=+1$. Thus there are in D at least k_1 intervals J_+ and at least k_2 intervals J_-, and these intervals are all distinct and nonoverlapping. Obviously, $|\phi(\bar{x}; J, f)|=+1$ for all intervals J_+ and J_-; hence $k_1+k_2\leqq N_c(\bar{x})$. We conclude that $N_+(\bar{x})+N_-(\bar{x})\leqq N_c(\bar{x})$ and, finally, $N_+(\bar{x})+N_-(\bar{x})=N_c(\bar{x})$.

(9.iii) *For every continuous function* $f(u)$, $a \leqq u \leqq b$, *the functions* $N_c(x)$, $N_+(x)$, $N_-(x)$ *are lower semicontinuous in* $(-\infty, +\infty)$; *that is, for every* x_0, *we have*

$$(9.3)\quad N_c(x_0) \leqq \liminf_{x\to x_0} N_c(x), \quad N_+(x_0) \leqq \liminf_{x\to x_0} N_+(x), \quad N_-(x_0) \leqq \liminf_{x\to x_0} N_-(x).$$

Proof. If $k \leqq N_c(x_0)$ for some integer k, then there are k disjoint intervals $I=[\alpha, \beta]$ with $f(\alpha)<x_0<f(\beta)$ or $f(\alpha)>x_0>f(\beta)$. Thus if $m>0$ is the minimum of the $2k$ differences $|f(\alpha)-x_0|$, $|f(\beta)-x_0|$, we have $f(\alpha)<x<f(\beta)$ or $f(\alpha)>x>f(\beta)$ for every x with $|x-x_0|<m$ and the same k disjoint intervals I. This implies that $N_c(x) \geqq k$ for all $x_0-m<x<x_0+m$. Thus if $N_c(x_0)<+\infty$ and we take $k=N_c(x_0)$, then we have $N_c(x) \geqq N_c(x_0)$ for all x of a neighborhood of x_0; if $N_c(x_0)=+\infty$, then for every integer k we have $N_c(x) \geqq k$ in some neighborhood of x_0. This certainly implies the first of (9.3). Analogous reasoning holds for N_+ and N_-.

(9.iv) *If* $f(u)$, $f_n(u)$, $a \leqq u \leqq b$, $n=1,2,\cdots$, *are continuous functions and* $f_n(u)\to f(u)$ *everywhere in* $[a, b]$, *or at least for all* u *of a set which is everywhere dense in* $[a, b]$, *then*

$$N_c(x, f) \leqq \liminf_{n\to\infty} N_c(x; f_n), \qquad N_+(x, f) \leqq \liminf_{n\to\infty} N_+(x; f_n),$$
$$N_-(x; f) \leqq \liminf_{n\to\infty} N_-(x; f_n).$$

The proof is analogous to the one for (9.iii).

Remark 1. Statements (9.iii) and (9.iv) assure that the functions N_c, N_+, N_- are lower semicontinuous both with respect to x and with respect to f separately. Actually the functions N_c, N_+, N_- are lower semicontinuous with respect to the pair (x, f). That is, if $x\to x_0$, $f_n\to f$, then $N_c(x_0, f) \leqq \liminf N_c(x, f_n)$ as $x\to x_0$, $f_n\to f$. We omit the proof.

Remark 2. If for $-\infty<\bar{x}<+\infty$, one of $N_c(\bar{x})$, $N_+(\bar{x})$, $N_-(\bar{x})$ is finite, then all three are finite, and, if $\bar{x}\neq f(a)$, $\bar{x}\neq f(b)$, then $N_+(\bar{x})-N_-(\bar{x})=\phi(\bar{x}; f, [a, b])$. Also, $N_c(\bar{x})$ is attained for a subdivision of $[a, b]$ if and only if $N_+(\bar{x})$ and $N_-(\bar{x})$ are attained for the same subdivision. We omit the proofs.

10. A necessary and sufficient condition for absolute continuity.

(10.i) *A continuous BV function* $x=f(u)$, $a \leqq u \leqq b$, *is AC if and only if every set* $E\subset[a, b]$ *of measure zero is mapped by* f *into a set* $f(E)$ *of measure zero.*

This statement is usually given in all expositions.

11. The theorem of Lebesgue and its corollaries.

(11.i) (H. Lebesgue) *Every real monotone nondecreasing function* $F(u)$, $a \leqq u \leqq b$, *has a finite derivative* $0 \leqq F'(u)<+\infty$ *almost everywhere,* $F'(u)$ *is L-integrable in* $[a, b]$, *and* $F(b)-F(a) \geqq \int_a^b F'(u)du$. *The equality sign holds if and only if* F *is AC in* $[a, b]$.

This statement is usually given in the expositions together with the theorem (11.ii) below. The proof of (11.i) is based on Vitali's covering theorem.

(11.ii) *If* $f(u)$, $a \leqq u \leqq b$, *is* BV *and* $v(u)$ *is the corresponding total variation in* $[a, u]$, *then the derivatives* f', v' *exist almost everywhere*, $|f'(u)| = v'(u)$ *almost everywhere* (a.e.) *in* $[a, b]$.

(11.iii) *If* $f(u)$, $a \leqq u \leqq b$, *is* BV, *then* $f'(u)$ *exists and is finite* a.e. *in* $[a, b]$ *and* $\int_a^b |f'(u)| du \leqq V(f; a, b)$. *The equality sign holds if and only if* $f(u)$ *is* AC.

(11.iv) *If* $f(u)$, $a \leqq u \leqq b$, *is* BV, *then the derivatives* $v'(u)$, $v'_+(u)$, $v'_-(u)$ *exist and are finite* a.e. *in* $[a, b]$, *and* (a) $v' = |f'|$, $v' = v'_+ + v'_-$, $f' = v'_+ - v'_-$ a.e.; (b) *either* $v'_+ = f'$, $v'_- = 0$, *or* $v'_+ = 0$, $v'_- = |f'|$ a.e.; (c) $\int_a^b v' du \leqq V$, $\int_a^b v'_+ du \leqq V_+$, $\int_a^b v'_- du \leqq V_-$.

Proof of (11.iii) *and* (11.iv). The derivatives v', v'_+, v'_- exist a.e. by virtue of (11.i) and then f' exists a.e. and $f' = v'_+ - v'_-$ by (6.i). Also $|f'| = v' = v'_+ + v'_-$ by (6.i) and (11.ii). Thus (a) is proved. As a consequence we have $v'_+ = 2^{-1}[|f'| + f']$, $v'_- = 2^{-1}[|f'| - f']$, and thus $v'_+ = f'$, $v'_- = 0$ if $f' > 0$; $v'_+ = 0$, $v'_- = |f'|$ if $f' < 0$; $v'_+ = v'_- = 0$ if $f' = 0$. Thus (b) is proved. Relations (c) follow from (11.i). Finally $\int_a^b |f'| du = \int_a^b v' du \leqq v(b) - v(a) = V(f; a, b)$. Now the equality sign holds if and only if $v(u)$ is AC and, by statement (β) of Section 6, we know that this occurs if and only if $f(u)$ is AC. Thus (11.iii) and (11.iv) are proved.

Remark. Part (b) of (11.iv) is a first instance of a property of separation of v_+, v_- of which also curves and surfaces show very important extensions. Less elementary interpretations of (11.iv.b) are given in the following lines.

Let $f(u)$ be continuous and BV in $[a, b]$, and $0 < V = v(b) < +\infty$. Then $s = v(u)$ is constant on a subinterval of $[a, b]$ if and only if $f(u)$ is constant there and thus the function $F(s)$, $s \in J = [0, V]$, defined by $F(s) = f(u)$ whenever $s = v(u)$, is continuous in J, and $F(0) = f(a)$, $F(V) = f(b)$. If $w(s)$, $w_+(s)$, $w_-(s)$, $s \in J$, denote the total variations of F in $[0, s]$ (Sec. 6), we have $F(s) - F(0) = w_+(s) - w_-(s)$, $w_+(s) + w_-(s) = w(s) \equiv s$ for all $s \in J$. Also, a.e. in $[0, V]$, we have $F' = w'_+ - w'_-$, $w'_+ + w'_- = w' \equiv 1$ and, by (11.iv.b), either $w'_+ = 1$, $w'_- = 0$, or $w'_+ = 0$, $w'_- = 1$ a.e. in J. Since $0 \leqq w_+(s') - w_+(s)$, $w_-(s') - w_-(s) \leqq s' - s$ for all $0 \leqq s < s' \leqq V$, we conclude that $w_+(s)$, $w_-(s)$ are Lipschitzian with constant 1 in J, hence AC. Thus $w_+(s) = \int_0^s w'_+ ds$, $w_-(s) = \int_0^s w'_- ds$. If $\tilde{\mathfrak{B}}$ denotes the ring of all B-measurable sets $\tilde{M} \subset J$, then for every set $\tilde{M} \in \tilde{\mathfrak{B}}$ we shall denote by $w(\tilde{M}) = |M|$, $w_+(\tilde{M})$, $w_-(\tilde{M})$, $\omega(\tilde{M})$ the integrals taken on $\tilde{M}$ of the following functions $w' \equiv 1$, w'_+, w'_-, F'. Then $w(\tilde{M})$, $w_+(\tilde{M})$, $w_-(\tilde{M})$, $\omega(\tilde{M})$ are now set functions defined in $\tilde{\mathfrak{B}}$, and w, w_+, w_- are measures, ω is a signed measure, and $w(\tilde{M}) = w_+(\tilde{M}) + w_-(\tilde{M}) = |\tilde{M}|$, $w_+(\tilde{M}) - w_-(\tilde{M}) = \omega(\tilde{M})$ for all $\tilde{M} \in \tilde{\mathfrak{B}}$. Finally, if J_+ is the set where $w'_+ = 1$, and $J_- = J - J_+$, then, for every set $\tilde{M} \in \tilde{\mathfrak{B}}$, we have $\omega(\tilde{M}J_+) = w(\tilde{M}J_+) = w_+(\tilde{M}J_+) \geqq 0$, $-\omega(\tilde{M}J_-) = w(\tilde{M}J_-) = w_-(\tilde{M}J_-) \geqq 0$, that is, J_+, J_- is a Hahn decomposition of J with respect to the signed measure ω [see, for these general concepts, P. R. Halmos, *Measure Theory*, 1950, p. 121].

Let $\Gamma = \{g\}$ denote the decomposition of $[a, b]$ into all maximal closed intervals g of constancy for $f(u)$ in $[a, b]$ (at most countably many, if any), and

all remaining single points g of $[a, b]$. The transformation $M_0 = s^{-1}(\tilde{M})$ transform the ring $\tilde{\mathfrak{B}}$ into the ring $\mathfrak{B}_0$ of all B-measurable sets $M_0 \subset [a, b]$ which are sums of elements $g \in \Gamma$. The same transformation transforms $w(\tilde{M})$, $w_+(\tilde{M})$, $w_-(\tilde{M})$, $\omega(\tilde{M})$ into corresponding measures $v(M_0)$, $v_+(M_0)$, $v_-(M_0)$, $\mathfrak{V}(M_0)$ in the ring $\mathfrak{B}_0$, and the decomposition J_+, J_- of J into a Hahn decomposition H_+, H_- of $[a, b]$ with respect to the signed measure $\mathfrak{V}$. Finally, if $\mathfrak{B}$ is the ring of all B-measurable subsets $M \subset [a, b]$, and, for every $M \in \mathfrak{B}$, we denote by M_0 the minimum set $M_0 \supset M$, $M_0 \in \mathfrak{B}_0$, we may put $v(M) = v(M_0)$, and analogously for v_+, v_-, $\mathfrak{V}$. Then the set functions $v(M)$, $v_+(M)$, $v_-(M)$, $\mathfrak{V}(M)$ are defined for all sets $M \in \mathfrak{B}$, and v, v_+, v_- are measures, $\mathfrak{V}(M)$ is a signed measure in the ring $\mathfrak{B}$. Also, H_+, H_- is still a Hahn decomposition of $[a, b]$ for $\mathfrak{V}$, and we have $v(M) = v_+(M) + v_-(M)$, $\mathfrak{V}(M) = v_+(M) - v_-(M)$, $\mathfrak{V}(MH_+) = v(MH_+) = v_+(MH_+) \geqq 0$, $-\mathfrak{V}(MH_-) = v(MH_-) = v_-(MH_-) \geqq 0$ for all sets $M \in \mathfrak{B}$. (For modern developments on continuous varities in Euclidean spaces, see T. Rado, *Length and Area*, Amer. Math. Soc., 1948; L. Cesari, *Surface Area*, Princeton, 1956.)

MATHEMATICIANS IN THE MARKET PLACE

MINA REES, Hunter College

My theme is the extent and nature of the need for mathematicians in industry and government;* my concern is with some of the inadequacies in our college and university environment if we are to meet this need. In a world in which technology has assumed an importance which is drawing in its wake public realization of the need for sound scientific education and for the support of basic research, mathematicians of every level of originality and training are needed: industrial mathematicians to support advances in our technology and in our business operations, as well as research men to push forward the frontiers of mathematical research and lend imaginative insights to the corpus of mathematics, and teachers for our colleges and schools.

That adequate support and adequate recognition must be given to research in pure mathematics is accepted as obvious. That gifted young people must be given the fullest opportunity to taste its pleasures and be seduced by its charms is a first requirement of any program of mathematical education. But the times demand that additional groups of young mathematical scholars be trained who will seek their careers in the market place to provide business, industry, and government with the mathematical support for which they are finding increasing need. Fortunately the presence of increasingly large numbers of young people in our colleges makes the time propitious for us to take stock. Fortunately, too,

* I shall make little attempt to distinguish between government and industry. The existence of a number of industrial companies in which substantially all the business is directly or indirectly with the government makes such a distinction virtually impossible if we are discussing the nature of the work—though salary scales and the nature of tenure are usually quite distinct.

there are many mathematicians now engaged in industrial research who find it a fascinating and intellectually rewarding pursuit.

We need additional mathematicians not only in industry, but in the universities and in the secondary schools. Our present dilemma in staffing the schools and universities is caused partly by the drift of well-trained mathematicians to industry because of the combined lure of higher salaries and interesting work. The need is clear to train more able mathematicians and to raise academic salaries to a level competitive with those offered by industry. The second point I need not emphasize; but on the first point I should like to expand.

Since 1950 we have been producing an average of a little over 200 Ph.D.'s in mathematics a year in the United States (and the average before 1950 was considerably smaller). This is a staggeringly small number to fill the needs that exist. It is imperative that we attract more young people to careers in mathematics. But many able boys and girls for whom teaching is not attractive still have the idea that teaching is the only career to which training in mathematics leads; and the nature of industrial mathematics often is not understood by the teachers who provide the only contact young people have with mathematics. We need to acquaint our young people with the great variety of interesting careers open to mathematicians. Inevitably some who embark on the study of mathematics as a road to a career in industry will find themselves diverted by their enthusiasm for research in pure mathematics—a quid pro quo for those lost to business from the university. It may be worth noting that over one-third of living Americans with Ph.D.'s in mathematics came to mathematics from some other field—the most frequent fields being engineering, chemistry, and physics.

One additional point needs mention. The problem is not to steal all the ablest youth of the land from other pursuits but merely to maintain the position of mathematics. Mathematicians themselves have not recognized the strength of their position.

The occasion of these observations is not the latest Sputnik in the sky. On the contrary, in considering mathematical training, as in the larger educational universe, the family of Sputniks merely verifies what we knew already.

As a part of the recently completed *Survey of Research Potential and Training in the Mathematical Sciences*,* a subcommittee on nonteaching opportunities† gathered data and assessed attitudes that give some information on the following questions:

What is the size of the population employed as mathematicians in industry and government?

* A Survey of Research Potential and Training in the Mathematical Sciences, University of Chicago, 1957.

† The members were: Hendrik Bode, Bell Telephone Laboratories, Inc.; A. H. Flax, Cornell Aeronautical Laboratory, Inc.; Alston Householder, Oak Ridge National Laboratory; Harlan Mills, General Electric Co.; Mina Rees, Hunter College, Chairman.

What is their quality as research men?

What is the nature of their mathematical activity in industry?

What training and personal qualifications are needed for success in industry?

Are there qualities or attitudes which distinguish industrial mathematicians as a group from university mathematicians?

I want now to report on some of these findings, and to put forth some recommendations based on the Subcommittee's report, as well as on interviews with persons engaged in mathematical research in industry and on discussions held by the Subcommittee.

The number of persons employed as mathematicians in nonacademic work is startling. Between 7000 and 8000 men and women are thought of by their employers as mathematicians, and nearly 7000 of these are employed without the Ph.D. Among Ph.D.'s over 900 are functioning as mathematicians—this in spite of the evidence of the Survey that there are at most 700, and probably between 600 and 700, American Ph.D.'s with original training in mathematics now engaged in nonteaching activities. The larger number reported by employers probably reflects several discrepancies of which the most important are (1) the fact that many Ph.D.'s originally trained as physicists or engineers are now operating as mathematicians, and (2) the fact that persons engaged in work making extensive use of statistics are thought of by their employers as mathematical statisticians, though their basic concern is with psychology, or economics, or some other field.

Also to be noted in this connection is the reply given by university departments of mathematics to the Survey question, "In what field or fields of mathematics do you most need personnel in order to fill gaps or to broaden your mathematical coverage?" About 100 vacancies were listed, with well over half lying in applied mathematical fields. Clearly the demand for Ph.D.'s with competence in applied mathematics or interest in the applications of mathematics is very great, and the need to re-examine our university environment is compelling.

With respect to non-Ph.D's employed as mathematicians we have a phenomenon with which all of us in academic work are familiar. The new A.B. or B.S. who goes out to a job paying a higher salary than his professor's is all too familiar; and the moral that professorial salaries must be raised is all too clear. It is, however, also clear that the colleges and universities are facing unprecedented demands from industry and government for graduates of mathematics curricula.

It may be worth a pause to note that the results of the Survey, indicating the size of the demand, are not old and familiar information, though many of us may be inclined to say we knew it all along. In the calendar year 1957, a thoroughly competent book by Blank and Stigler,* dealing with the demand and supply of scientific personnel and relying on the 1950 Census and the

* Demand and Supply of Scientific Personnel, David M. Blank and George J. Stigler, National Bureau of Economic Research, 1957.

latest available statistics (1951) concerning employment of mathematicians, reported that less than 13% of Ph.D. mathematicians were employed outside the colleges and universities. (The authors did suspect a bias in their data because of the emphasis, in the collection of data, on membership in professional societies.) The Survey data show nearly 23% of Ph.D. mathematicians in nonacademic positions, with the trend among recent Ph.D.'s approaching 30%.

What is the quality of these Ph.D.'s as research men? In spite of the prejudices on both sides (and these are very real) there seems to be very little evidence that there is much difference in quality between teachers and industrial mathematicians. The Survey Committee, like all other groups, had great difficulty in finding an acceptable measure of quality. The results do make clear that, since World War II, an increasingly large percentage of those with very high undergraduate grades are turning to nonacademic pursuits; and that the publication record of industrial mathematicians with postwar Ph.D.'s is somewhat better than the average for the whole group of Ph.D.'s. This is in spite of the fact that much original work done in industry and government does not find its way into publication because of the restrictions imposed by security or competition. A further note concerning quality is the election to the National Academy of Sciences, within the last two years, of two men who made their reputations in industrial mathematics.

With no evidence to indicate that the quality of the work done in industry is substantially different from that done at the university, it is none the less clear that the motivation, and often the nature of the research, is quite different in the two settings. Unfortunately there are few opportunities for academic mathematicians to understand the nature of the industrial mathematician's work and this lack of understanding is reflected in the impressions given to our students.

I was interested in receiving, a short time ago, the "Abbreviated Proceedings"* of a conference held at Oxford University, April 8–18, 1957, for school teachers and industrialists. This was supported by 18 groups, mostly industrial, a few government, and one American (The Office of Naval Research) to provide liaison between preparatory school teachers and industry. The school teachers were struck by the scale, complexity, and variety of mathematical work in industry. As a setting for some of the things I have to say, I want to quote from the introductory remarks by John Hammersley of Trinity College, Oxford, who organized the conference:

"Only a handful of school teachers have ever used mathematics in practice. Mathematical examination problems are usually considered unfair if insoluble or improperly described; whereas the mathematical problems of real life are almost invariably insoluble and badly stated, at least in the first instance. In real life, the mathematician's main task is to formulate problems by building an abstract mathematical model consisting of equations, which shall be simple

* Oxford Mathematical Conference (Abbreviated Proceedings), London, 1957.

enough to solve without being so crude that they fail to mirror reality. Solving equations is a minor technical matter compared with this fascinating and sophisticated craft of model-building, which calls for both clear, keen commonsense and the highest qualities of artistic and creative imagination."

A further quotation from the introductory remarks of W. E. Scott of the English Electric Company may also be enlightening:

"The growing complexity of some modern industries has . . . increased the opportunity for mathematicians to fit themselves in and become most valuable to industry. A main reason for this trend has been growing development costs. It is nowadays most expensive to carry out a new major project, and prohibitively so if the right approach is not made from the beginning. Thus the maximum amount of thought must be given to a design before it is built or to an experiment before it is started, and this often involves considerable theoretical study."

What is expected of the mathematician in industry? I speak here primarily of the Ph.D. mathematician, though some of my remarks will apply to the young college graduate.

The first consideration is the question of personality. Unless a man enjoys working with others, unless he is interested in considering other people's problems, unless he finds it interesting to evolve the appropriate mathematical model for handling situations that are often not correctly or clearly described, and to bring his mathematical maturity to bear on situations he has not himself selected he probably does not belong in industry. One relevant result of the Survey was not a surprise, but the statistics do serve to corroborate a strong prior impression. Group research is viewed more favorably by industrial mathematicians, and there is more of it among them than among university people.

An aspect of this collaborative feature of industrial research is the critical importance of the mathematician's ability to communicate with the nonmathematician. In industry the audience may be highly sophisticated scientists and engineers who do not, however, have the specific language and habits of thought of the mathematician. In government, the audience may well be a business man or a military officer. In both cases the effectiveness of the mathematician is largely conditioned by his ability to understand the other man's language, and to present to him, simply and comprehensibly, the power and the limitations of his craft. Thus the mathematician in a nonacademic environment requires to a marked degree some of the "teaching" interest and skill needed also by his academic colleagues. In order to understand the problems that are brought to him, he needs to acquire some of the basic ideas of the sciences and of engineering; in order to present his own ideas with cogency he needs experience in presenting mathematical ideas to nonmathematicians. To quote from the Survey Report: " . . . the applied mathematician ought to have significant subject matter knowledge in other scientific and technical areas. More important

than detailed subject matter knowledge as such, however, is a fairly extensive acquaintance with the "bare bones" logical structure of a number of areas. This is evidently a sort of background on the basis of which abstract mathematical model building and axiomatizing generally can be most readily and reliably applied in particular new situations. . . . The graduate student in mathematics who contemplates a nonacademic career would be well advised to acquire experience at the university in presenting his ideas to students and colleagues in other departments. At the minimum this experience should include engineering and science departments, but it is desirable that the social scientists also be included in the audiences."

The report of the Oxford Conference includes this statement by D. G. Owen of the British Iron and Steel Federation:

"We also need, especially in our mathematicians, a practical outlook and good personality: a problem may take three months to formulate and solve, but unless the results are 'sold' to management and made to work *in practice* the job cannot be regarded as complete. High quality investigators . . . are wanted, and they must be able and willing to talk with laymen in acceptable language."

What of the specific mathematical training needed for work in industry? The striking thing about modern applications of mathematics is that fields that were thought of as highly abstract even a decade ago have proved to be highly applicable in the modern setting. There is the obvious relevance of many parts of analysis, including differential equations, particularly of the nonlinear variety, to the new problems of propulsion and aerodynamics of high speed aircraft on the one hand, and, on the other, to problems arising from nuclear weapons research and other explosion phenomena, from radar and other electronics problems. If I may quote still another time from the Oxford Conference report, P. L. Taylor of Metropolitan-Vickers Electrical Co. Ltd., in his paper on "The Mathematics of Linear Electric Circuit Theory" has this to say:

"Since the properties of a circuit are essentially independent of the geometrical arrangement of the connecting wires—they may be arranged at will provided only that no connections are broken or new ones made—it is evident that it is the topological properties of the linear graph that are of interest."

And further:

"In manipulating the system of equations that describe a circuit the ideal tool is matrix algebra."

The application of Boolean algebra to computer design, of group theory to communications theory, of number theory to numerical analysis, and of noncommutative algebra and group theory to nuclear physics leaves one with the quite correct impression that almost any mathematics is applicable, and that broadly educated mathematicians are needed.

I have chosen to quote from a British report to emphasize the similarity of the situation in England and the United States. In our country, too, a sampling

of problems confronted in industry shows a broad spectrum of mathematical disciplines in use in the attempt to find solutions. On a recent visit to the Bell Telephone Laboratories, I was told of a few of the problems in which some members of the mathematical research group are interested. One of these originated in the Financial Department of the American Telephone and Telegraph Company, and concerned the charges for private wire services. The mathematical result that was derived at Bell Laboratories gave a new theorem in the theory of trees. Interestingly enough the same result seems to have been found at about the same time at the Mathematisch Centrum in Amsterdam when, in the design of a new computer, the old computer was asked to determine what wiring would keep the total length of wire as small as possible. In the Telephone Company's problem, the Federal Communications Commission requires that the charges be based on the shortest network that can be constructed connecting plants at several locations; in the computer problem, the effort is to minimize the network of wires connecting a collection of pins. The theorem previously existing in the literature required that the minimum distance between *any* two points be found—a requirement poorly adapted to computation. The new result gives a systematic procedure that can easily be translated into a machine program. We have here a situation that illustrates the new requirement often found in the industrial applications of mathematics—to find a new theorem which will enable us to exploit the speed of electronic computers. For industrial applications, the computability of a method is even more important than elegance is for a proof in pure mathematics.

Another Bell Laboratories problem concerns communication in the presence of noise, and involves quantizing the sending of bits of information. The problem is to translate into usable form an existence proof and asymptotic theorems due to Shannon. Though the problem is unsolved, some progress has been made using the theory of groups.

A new kind of demand for mathematicians has been produced by the current expansion of the use of operations research in industry. This is a type of nonacademic employment that relies less on specific techniques than do the traditional fields of applied mathematics. A large expansion in this type of activity is in process, with a significant increase in the opportunities for mathematicians. When I was asked last year by the publisher of *Career* magazine to suggest mathematicians who might write accounts of mathematics in industry for high school students, I gave him the names of a group of men, who, I thought, would provide a broad spectrum of experience, and not emphasize computing exclusively. What was my dismay to find when the magazine appeared that there was substantially nothing in it but operations research. Even the communications industry and the aircraft industry—old standbys for classical applied mathematics—are now emphasizing operations research.

I have not mentioned the familiar fields of statistics and computing. The uses of statistics are expanding in academic research as well as in industrial work, and the need for statisticians who can carry on independent work con-

tinues to grow. In England, the emphasis on more training in statistics in the schools parallels some of the suggestions of our own Commission on Mathematics. One company, Imperial Industries, Ltd., reports that they hope to meet their shortage of statisticians by "recruiting mathematicians straight from graduation at universities and then sending them back to the most appropriate university departments for postgraduate training in mathematical statistics"* after they have spent a little time at work getting an idea of the sort of problems faced by the industry.

In computing, the availability of large fast machines has changed the face of industrial research, and we shall be hard pressed to provide trained users of these machines. As the telephone company's leased lines problem suggests, one of the boundary conditions imposed on many industrial problems is the ease with which the proposed formulation or solution yields to evaluation by automatic computation. Moreover the availability of large-scale fast computers has multiplied the number of problems for which industry seeks mathematical solution, and has given industry a powerful tool for attacking some problems for which no analytical procedures were formerly sought. One of the problems currently being reduced to numerical solution at Westinghouse may serve to illustrate the point.†

At an appliance plant in Ohio, a problem was posed very simply: "Given a product that is to be manufactured on an assembly line, what is the best way to arrange the assembly line in order to minimize the amount of human labor." There are boundary conditions, like the existence of zones where certain operations must be performed, and the permutability of some types of operation and nonpermutability of others. Traditionally, such assembly lines have been set up by industrial engineers, with no certainty that the solution which has been designed is optimal. The mathematician who was consulted on this problem used point set theory to find an optimal solution (not unique); and proceeded to show, by hand computation, that the mathematical solution for an existing assembly line would reduce the number of people needed for the job from 25 to 22—a result of substantial interest for management. The programming of the solution for a digital computer makes the solution available to the company's various management divisions which can get the answers to their own specific problems with digital computer speed.

I have referred several times to the situation in England. Though this is in many ways similar to our own, it is not the same. The British, as well as the Russians, have been conspicuous in the high quality of their research in applied mathematics. It probably is not an accident that this excellence is found in a country where the traditional training of a mathematician includes substantial formal work in applied fields. The delightful essay of Littlewood on "A Mathe-

* *Loc. cit.*, p. 80.

† Based on a description provided by M. Ostrofsky, Manager, Department of Mathematics, Westinghouse Electric Corp.

matical Education"* discloses without comment, and in a matter-of-fact way, that his education at Cambridge, beginning in 1903, included the study of dynamics, hydrodynamics, spherical harmonics, electricity, and the foundations of mechanics. This tradition continues. The same subjects will be found in most university schedules today. For example, at one of the colleges at the University of London, there are four hours of lecture in applied mathematics during a student's first year at the University, and six hours during his second year. During his third year he may elect to specialize in pure mathematics, or in applied mathematics, or to continue in both.

Whether American universities will wish to encourage formal course work in applied mathematics is problematical, and whether they should is controversial. But students should be given the opportunity to learn something about the nature of industrial mathematics. The training needed for effective work as an industrial mathematician has been described to me by Hendrik Bode, on the basis of his experience at Bell Telephone Laboratories.† In his judgment, an applied mathematician needs to serve one, or perhaps several, interneships. For example, in these days of automatically sequenced computing machines a thorough acquaintance with numerical methods, standard approximating procedures, and the various ways of attacking problems numerically are obviously of special importance. This is the sort of ability which is best learned by actual practice in a computing center. The computing facility need not be a large one—in fact there may be a positive advantage in having to use one's ingenuity to overcome mechanical inadequacies; but a first-hand acquaintance with numbers, of the sort that is developed only by a certain amount of laboratory work, is almost essential.

Another significant interneship could be spent in a statistical center. This would give the aspiring applied mathematician a better appreciation than he could probably have otherwise of the reliability of the factual statements to which he will be subjected later, and would also give him a feeling for the probable significance, or lack of significance, of a given element of a logical framework in a particular situation.

Another profitable, if brief, interneship might be served as a participant in an actual functioning applied mathematics group in an industrial or similar environment. This would be particularly useful if it occurred before the end of the formal training period. It might, of course, be possible to find ways of meeting most, if not all, of these interneship requirements in a single place.

The young applied mathematician should also find time, if possible while he is still at the university, to participate to some extent in seminars or research activities in other disciplines.

Although there are few university departments of mathematics which are ideal for this type of training, a sympathetic point of view on the part of the department as a whole goes far toward making adequate training possible. In a

* J. E. Littlewood, A Mathematician's Miscellany, London, 1953.

† The following five paragraphs are paraphrased from a letter from Hendrik Bode.

large university especially, where there is always ample and diversified research activity, the young applied mathematician should be able to find most of what he needs if he is given reasonable support and encouragement.

The basic need is for a cordial atmosphere at the university. For decades chemistry departments have accepted as part of their established function the training of industrial chemists. More recently, since World War II, the physicists have come to accept a similar role. At present three-fourths of our chemistry Ph.D.'s are seeking their careers in industry, and three-fifths of our physicists. Mathematics is now emerging into a situation like that of chemistry and physics. While it is true that there are university professors who believe that good mathematics is useless mathematics, it is interesting that the statistics of the Survey show that Harvard, Princeton and Yale perform in an average manner in the number of Ph.D.'s they send into nonacademic mathematical pursuits. But it is also interesting that, although the number of Ph.D.'s going into industrial work is increasing, few of them were aware of the opportunities in these fields when they started graduate work. Better information is needed in the schools, the colleges and the universities about the nature of the work mathematicians do in industry.

On this front some progress is being made. On the secondary level, the Committee on Applications of Mathematics of the National Research Council, in collaboration with the National Council of Teachers of Mathematics, is preparing for publication a booklet of profiles of successful young mathematicians, most of them in industrial positions. This should be completed within a year; it will be very widely distributed in the secondary schools of the country. At the college and university level, also, some progress is indicated by the replies to one of the Survey questions. Of the university and college professors who responded, a substantial number indicated that consultation with industry or government was among their four most important contributions to mathematics. Thus there is some flow of information about industrial mathematics into university faculties. But more needs to be done. The flow of information from university to industry is accomplished in significant part by the summer visits of many professors to industrial laboratories. These visits often have an instructional flavor, with university men bringing up-to-date or sophisticated methods to some of the young mathematicians who have been recruited into industry without extensive training. Often, however, the professor is participating with senior industrial colleagues in an enriching experience. More extended work experience by academic people, and extended loans to universities by industry would benefit both. More general use of visits by students to industrial establishments and by industrial mathematicians to college and university classes should be explored. There are already extensive fellowship programs in some industries which provide selected employees with an opportunity to do advanced graduate work at neighboring universities at company expense; and there are companies which provide lecturers in applied topics to near-by universities. There are also extensive in-service graduate training programs in some companies.

The Survey Subcommittee recommended that other ways be sought to establish better cooperation between university and industry. The possibility of summer institutes in areas of applied interest should be explored, with participation by academic and nonacademic mathematicians. This is a problem which I hope will be one of the first items on the agenda of the newly established Conference Organization of the Mathematical Sciences. Among activities that the Society for Industrial and Applied Mathematics (SIAM) might well undertake, I have two suggestions. The first draws its inspiration from the economists. The Foundation for Economic Education, now in its eleventh year, provides summer fellowships, intended merely to pay expenses, which enable professors to visit one of the participating industries for a period of about six weeks, and become acquainted with management aspects of the company. Last year there were 80 colleges and universities that participated, and 56 business firms. There is now at least one aircraft company which makes similar arrangements for visits by university mathematicians. An organized effort like this might well prove mutually beneficial. Similar activities for secondary school mathematics teachers have been carried on in the Los Angeles area.

Probably easier to achieve would be a visiting lecturer program similar to that administered by the Mathematical Association. If SIAM would provide for distinguished lecturers from industry to visit the colleges and universities, on a schedule comparable but suitably different from that used by the Association, it is probable that an important new element might be introduced into the atmosphere on many campuses.

One of the few significant differences between nonacademic and academic mathematicians which the Survey disclosed, was the dissatisfaction of the industrial mathematicians with the publication policies of the mathematical world. This is particularly significant since between 25 and 30% of the members of the American Mathematical Society are employed outside the university. Various suggestions have been made to try to remedy this situation; and it is pleasant to note that the Society has recently added the Journal of SIAM to the group of its subsidized publications, in the hope that this Journal may provide another desirable publication outlet for some of the research results of nonacademic mathematicians.

It is important that all members of the mathematical community exert their maximum effort to eliminate the friction that exists between pure and applied mathematicians. As F. J. Weyl says in his excellent report on Applied Mathematics in the United States:*

"Training and research in mathematics has now become of the same immediate importance to the public welfare as training and research in the experimental

* Survey of Training and Research in Applied Mathematics in the United States conducted by the National Research Council, F. J. Weyl, Principal Investigator, November, 1954. Published by the Society for Industrial and Applied Mathematics, Philadelphia, 1956.

sciences. Rather than indirectly, as in the past, through services rendered to the other sciences along the way of their development, it is making progressively more of its contributions to the concerns of the day directly and in concert with the other sciences .... This not only gives a broadened scientific scope and golden new opportunities to mathematics, but it also throws an added burden of social responsibility on its practitioners, making it of importance that the applied aspects of mathematics and those who have concern therewith are recognized and heeded in the representative organizations of American mathematics."

Mathematics is an entity. The welfare of each part is essential to the other. Salaries at the university advance with the increased demand for mathematicians in industry; and financial support for research in pure mathematics accompanies support for applied research. Wisdom and far-sightedness demand that all mathematicians join forces in attracting more students to the study of mathematics, and in offering them a broad view of the role of mathematics in our civilization—in research, in teaching, and in industrial applications. We should find a way to put an end to the attitude among university mathematicians that industrial mathematicians are second-class citizens; and we should recognize them for what they are: a source of power for mathematics, a reason for the respect in which pure mathematics is held by many who do not understand it, and sometimes a leavening influence in determining the directions of interesting new mathematical research.

COMPLETENESS AND PARSEVAL'S EQUATION

JOHN M. H. OLMSTED, University of Minnesota

For a fixed finite closed interval $[a, b]$, let L^2 denote the space of all real-valued functions that are defined and measurable on $[a, b]$ and whose squares are Lebesgue integrable there, with identification of functions equal almost everywhere [1]. If f and g are any two members of L^2, then the *inner product* (f, g), the *norm* $\|f\|$, and the *distance* $d(f, g)$ are defined by the equations $(f, g) = \int_a^b f(x)g(x)dx$, $\|f\| = (f, f)^{1/2}$, $d(f, g) = \|f-g\|$. A sequence $\{\phi_n\}$ of members of L^2 is *orthonormal* if and only if $(\phi_m, \phi_n) = 0$ or 1 according as $m \neq n$ or $m = n$. The *Fourier series* of a member f of L^2 with respect to an orthonormal sequence $\{\phi_n\}$ is the series $\sum \alpha_n \phi_n$, where the numbers α_n are the *Fourier coefficients* $\alpha_n = (f, \phi_n)$, $n = 1, 2, \cdots$.

Certain facts hold for a Fourier series without further assumption [2]. One is Bessel's inequality

$$(1) \qquad \sum \alpha_n^2 \leqq \|f\|^2.$$

Another is the "least squares" property of the Fourier coefficients, which states

that the distance between a given f and any linear combination of a fixed finite set of $\{\phi_n\}$ is minimized if the coefficients are the Fourier coefficients.

For questions of convergence, point-wise or in the mean, and Cesàro summability [3, 4] it is important to include a further hypothesis guaranteeing that in some sense or other there are "enough" members in the orthonormal family. This added hypothesis can be expressed in many equivalent forms including, in particular, the following six:

1. For any $f \in L^2$ the Fourier series $\sum \alpha_n f_n$ *converges in the mean* to f:
$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} \alpha_n f_n - f \right\| = 0.$$
2. The finite linear combinations of members of $\{\phi_n\}$ are *dense* in L^2.
3. *Parseval's equation* holds for any $f \in L^2$:
$$\sum \alpha_n^2 = \|f\|^2.$$
4. The sequence $\{\phi_n\}$ is *complete* or *maximal*; that is, there is no strictly larger orthonormal sequence containing $\{\phi_n\}$.
5. The only member of L^2 orthogonal to every member of $\{\phi_n\}$ is 0.
6. A member of L^2 is uniquely determined by its Fourier coefficients.

The proof that all of these six conditions are equivalent rests in part on the completeness of L^2 as a metric space [2]. The question now is the exact extent to which the completeness of L^2 is needed. To make the question more precise, let us replace the space L^2 by the (incomplete) subspace R^2, the Lebesgue integral being replaced by the standard proper Riemann integral. Which of the preceding six conditions imply which others? At least a partial answer is based on standard and rather elementary techniques [2]. This is that the first three are equivalent, the last three are equivalent, and the first three imply the last three:

$$1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4 \Leftrightarrow 5 \Leftrightarrow 6. \tag{2}$$

The one remaining unanswered question is whether the central implication in (2) can be reversed. It will be the purpose of the remaining portion of this paper to show that *the reverse implication does not hold*:

$$3 \not\Leftarrow 4, \tag{3}$$

and that a simple demonstration can be provided by extending the space R^2 to its completion L^2.

For convenience of notation we shall denote members of the two spaces R^2 and L^2 by lower-case and upper-case letters, respectively: $f, g, h \in R^2$, $F, G, H \in L^2$. Let F be an arbitrary member of L^2 not equivalent to a member of R^2 (for example, F might be an unbounded function, or the characteristic function of a Cantor set of positive measure), normalize F with an appropriate factor so

that $\|F\|=1$, and let Π be the orthogonal complement of F in L^2 [2].

LEMMA. *The set* $R^2\cap\Pi$ *is dense in* Π.

Proof. Let G be an arbitrary member of Π, and let $\{f_n\}$ and $\{h_n\}$ be sequences in R^2 such that $f_n\to F$, $h_n\to F+G$, and assume that $\|f_n-F\|<1$ for all n. Define $g_n=h_n-\lambda_nf_n$, where λ_n is determined so that $g_n\in\Pi:\lambda_n=(h_n, F)/(f_n, F)$. As $n\to\infty$, by the continuity of the inner product, $(f_n, F)\to(F, F)=1$, $(h_n, F)\to(F+G, F)=(F, F)=1$, and $\lambda_n\to 1$. Therefore $g_n\to(F+G)-1(F)=G$.

Since the space L^2 is separable [1], so is Π, and hence there exists a sequence $\{f_n\}$ in $R^2\cap\Pi$ dense in Π. Assuming without loss of generality that no f_n is a linear combination of members of $\{f_n\}$ with subscripts less than n, we can apply the Gram-Schmidt process [2] and obtain an orthonormal sequence $\{\phi_n\}$ of members of $R^2\cap\Pi$. Since this sequence lies in the closed linear subspace Π, which is not dense in L^2, it follows that the finite linear combinations of members of $\{\phi_n\}$ cannot be dense in L^2. Therefore these finite linear combinations are not dense in the subspace R^2 (which *is* dense in L^2), and the sequence $\{\phi_n\}$ does not satisfy any of the first three of the six listed conditions, applied to the space R^2. On the other hand, if a member H of L^2 is orthogonal to every ϕ_n it is orthogonal to every member of Π and hence must be a multiple of F. Therefore if H belongs to R^2 it must equal 0, and the sequence $\{\phi_n\}$ is complete in R^2 and satisfies the last three of the six listed conditions. This completes the proof.

The preceding argument applies to any incomplete linear subspace of L^2, and we conclude that *Parseval's equation is a consequence of completeness for orthonormal sequences only in the presence of completeness for the linear space concerned.*

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California

A WRONSKIAN

DAVID ZEITLIN, Remington Rand Univac, St. Paul

Let u and v be differentiable functions of x and consider the Wronskian of the set of functions $u, uv, uv^2, \cdots, uv^n$. We denote the Wronskian by $W(u, uv, uv^2, \cdots, uv^n)=|a_{sj}|$, $s, j=1, 2, \cdots, n+1$, where $a_{sj}=D^{s-1}(uv^{j-1})$

and $D \equiv d/dx$. We will show that

$$W(u, uv, uv^2, \cdots, uv^n) = u^{n+1} \cdot [Dv]^{(n(n+1))/2} \cdot \prod_{t=1}^{n} t!. \tag{1}$$

To establish (1), we will show that the Wronskian can be transformed into a triangular form, *i.e.*, every element above the principal diagonal is zero.

Proceeding from right to left, we add to the elements of the $(j+1)$th column $(-v)$ times the elements in the jth column, starting with $j=n$ and continuing on, respectively, with $j=n-1, n-2, \cdots, 1$. The above steps will be called a *process*. We repeat the *process* again, where now $j=n, n-1, \cdots, 2$, respectively. All told, we repeat the *process* n times, where in the pth process, $p \leqq n$, $j=n, n-1, \cdots, p$. We note that the elements of the jth column assume their final values at the completion of the $(j-1)$th *process*.

Let $a_{sj}^{(p)}$ denote the element in the sth row and jth column after p applications of the *process*, where $j \leqq n+1$, $p \leqq j-1$. Then, we have

$$\begin{aligned} a_{sj}^{(1)} &= a_{sj} - va_{sj-1} \text{ (note that } a_{1j}^{(1)} = 0 \text{ for } j > 1) \\ a_{sj}^{(2)} &= a_{sj}^{(1)} - va_{sj-1}^{(1)} = a_{sj} - 2va_{sj-1} + v^2 a_{sj-2} \\ a_{sj}^{(3)} &= a_{sj}^{(2)} - va_{sj-1}^{(2)} = a_{sj} - 3va_{sj-1} + 3v^2 a_{sj-2} - v^3 a_{sj-3}. \end{aligned}$$

Denoting, as usual, $\binom{p}{k} = p!/k!(p-k)!$, we claim that $a_{sj}^{(p)} = a_{sj}^{(p-1)} - va_{sj-1}^{(p-1)}$ satisfies the relation

$$a_{sj}^{(p)} = \sum_{k=0}^{p} (-1)^k \binom{p}{k} v^k a_{sj-k}. \tag{2}$$

We may prove (2) by mathematical induction on p. (2) is true for $p=1$. Suppose now that (2) is true for $p=P$. Then

$$a_{sj}^{(P+1)} = a_{sj}^{(P)} - va_{sj-1}^{(P)} = \sum_{k=0}^{P} (-1)^k \binom{P}{k} v^k a_{sj-k} - \sum_{k=0}^{P} (-1)^k \binom{P}{k} v^{k+1} a_{sj-(k+1)}.$$

In the second sum, let $t=k+1$. Since $\binom{P}{k}=0$ for $k>P$, $k<0$, we have

$$\begin{aligned} a_{sj}^{(P+1)} &= \sum_{k=0}^{P+1} (-1)^k \binom{P}{k} v^k a_{sj-k} + \sum_{t=0}^{P+1} (-1)^t \binom{P}{t-1} v^t a_{sj-t} \\ &= \sum_{k=0}^{P+1} (-1)^k \left[\binom{P}{k-1} + \binom{P}{k}\right] v^k a_{sj-k} = \sum_{k=0}^{P+1} (-1)^k \binom{P+1}{k} v^k a_{sj-k}. \end{aligned}$$

This completes the proof of (2). Let $b_{sj} = a_{sj}^{(j-1)}$; then from (2), for $p=j-1$, we have

$$b_{sj} = \sum_{k=0}^{j-1} (-1)^k \binom{j-1}{k} v^k D^{s-1}(uv^{j-1-k}), \quad s, j = 1, \cdots, n+1. \tag{3}$$

We claim now that

(4) $$b_{sj} = 0 \text{ if } j > s,$$

(5) $$b_{ss} = (s-1)!\cdot u(Dv)^{s-1}, \qquad s = 1, 2, 3, \cdots, n+1.$$

From Leibniz's rule, we have

$$D^{s-1}(uv^{j-1-k}) = \sum_{m=0}^{s-1} \binom{s-1}{m} D^{s-1-m}(uv^{j-1}) D^m(v^{-k});$$

and so

(6) $$b_{sj} = \sum_{m=0}^{s-1} \binom{s-1}{m} D^{s-1-m}(uv^{j-1}) \sum_{k=0}^{j-1} (-1)^k \binom{j-1}{k} v^k D^m(v^{-k}).$$

Now, $v^k D^m(v^{-k})$ is a polynomial in k of degree m, *i.e.*,

(7) $$v^k D^m(v^{-k}) = (-v)^{-m}(Dv)^m k^m + \sum_{i=0}^{m-2} g_i(v) k^{m-1-i},$$

where the explicit expressions for $g_i(v)$ are of no concern. Recalling the well-known identity,

(8) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m = \begin{bmatrix} 0, & \text{if } m < n \\ n!(-1)^n, & \text{if } m = n \end{bmatrix},$$

we find from (6), using (7) and (8), that $b_{sj}=0$ if $j>s$; for $j=s$, there is only one nonzero term, when $m=s-1$. Thus, $b_{ss}=(-v)^{-(s-1)}\cdot(Dv)^{s-1}\cdot(-1)^{s-1}\cdot[(s-1)!]$ $uv^{s-1}=(s-1)!\cdot u(Dv)^{s-1}$. Since the determinant $|b_{sj}|$ is triangular, we have

$$W(u, uv, uv^2, \cdots, uv^n) = \prod_{s=1}^{n+1} b_{ss},$$

thus yielding (1) as claimed.

For $v=u$, $n=k-1$, we obtain from (1)

(9) $$W(u, u^2, u^3, \cdots, u^k) = u^k\cdot[Du]^{(k(k-1))/2}\cdot \prod_{t=1}^{k-1} t!.$$

From (1), with $v=1$, $W(u, u, \cdots, u)\equiv 0$. Some applications of (1) and (9) are now given.

(i) For linear differential equations with constant coefficients, the following Wronskians may be of interest:

If $u=e^{ax}$, $v=x$, then

(10) $$W(e^{ax}, xe^{ax}, x^2e^{ax}, \cdots, x^ne^{ax}) = e^{a(n+1)x}\cdot \prod_{t=1}^{n} t!.$$

If $a=0$ in (10), then

$$W(1, x, x^2, \cdots, x^n) = \prod_{t=1}^{n} t!. \tag{11}$$

If $u=e^{ax}\cos bx$, $v=x$, then

$$W(e^{ax}\cos bx, xe^{ax}\cos bx, \cdots, x^n e^{ax}\cos bx) = (e^{ax}\cos bx)^{n+1}\cdot \prod_{t=1}^{n} t!. \tag{12}$$

Since the above Wronskians are defined for $x=0$, they satisfy Abel's identity for $x=0$.

(ii) Using (9), we find:

If $P(x)$ is any polynomial in x, then

$$W(P(x), P^2(x), \cdots, P^k(x)) = P^k(x)\cdot [DP(x)]^{(k(k-1))/2}\cdot \prod_{t=1}^{k-1} t!. \tag{13}$$

If $f(x)$ is a differentiable function of x, then

$$\begin{aligned} &W\left(\int_a^x f(t)dt, \left[\int_a^x f(t)dt\right]^2, \cdots, \left[\int_a^x f(t)dt\right]^k\right) \\ &\qquad = \left[\int_a^x f(t)dt\right]^k \cdot [f(x)]^{(k(k-1))/2}\cdot \prod_{m=1}^{k-1} m!. \end{aligned} \tag{14}$$

Since $\int_a^x f(t)dt = (x-a)f(u)$, where $a<u<x$ or $x<u<a$, we have

$$\lim_{x\to a}\left[\frac{W\left(\int_a^x f(t)dt, \cdots, \left[\int_a^x f(t)dt\right]^k\right)}{(x-a)^k}\right] = [f(a)]^{(k(k+1))/2}\cdot \prod_{m=1}^{k-1} m!. \tag{15}$$

If $u=f(x)e^{mx}$, where m is a constant, $m\neq 0$, then

$$\begin{aligned} &W(f(x)e^{mx}, f^2(x)e^{2mx}, \cdots, f^k(x)e^{kmx}) \\ &\qquad = [f(x)]^k\cdot e^{(k(k+1)mx)/2}\cdot [mf(x)+Df(x)]^{k(k-1))/2}\cdot \prod_{t=1}^{k-1} t!. \end{aligned} \tag{16}$$

Factoring out e^{jmx} from the jth column of the Wronskian, we have $e^{\sum_{j=1}^{k} jmx} = e^{k(k+1)mx/2}$. Thus, we find the value of the resultant determinant of order k:

$$\begin{aligned} |c_{sj}| &= \left| \sum_{i=0}^{s-1} (jm)^i \binom{s-1}{i} D^{s-1-i}[f(x)]^j \right| \\ &= [f(x)]^k\cdot [mf(x)+Df(x)]^{(k(k-1))/2}\cdot \prod_{t=1}^{k-1} t!, \end{aligned} \tag{17}$$

where $s, j=1, 2, 3, \cdots, k$.

(iii) From (1), if $n=k$, $u=x^n$, $v=\ln x$, then

$$W(x^n, x^n \ln x, \cdots, x^n \ln^k x) = x^{((2n-k)(k+1))/2} \cdot \prod_{t=1}^{k} t!. \tag{18}$$

(18) solves the elementary problem E1213 [1956, 253] of this MONTHLY, vol. 63.

(iv) Let $f=f(x)$ and $g=g(x)$ be differentiable functions of x. Then we find that

$$\begin{aligned} W(f^n, gf^{n-1}, g^2f^{n-2}, \cdots, g^{n-1}f, g^n) &= (f^n)^{n+1}\left[D\left(\frac{g}{f}\right)\right]^{(n(n+1))/2} \prod_{t=1}^{n} t! \\ &= (g^n)^{n+1}\left[-D\left(\frac{f}{g}\right)\right]^{(n(n+1))/2} \prod_{t=1}^{n} t!. \end{aligned} \tag{19}$$

If we define $u=f^n$, $v=g/f$, then the first form of (19) is obtained from (1). Noting that $D(g/f)=-(g/f)^2D(f/g)$, we obtain the second form of (19). We observe that the first row of the Wronskian in (19) is obtained as the successive terms of the expansion $(f+g)^n$, with omission of the binomial coefficients.

SUMMING OF TRIGONOMETRIC SERIES WITH COEFFICIENTS WHICH HAVE A PERIODIC FACTOR

HARVEY J. FLETCHER, Brigham Young University

Introduction. There are certain Fourier series which converge to elementary functions. The question arises as to whether these series still converge to elementary functions if certain terms are removed. For example, the following series

$$F(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

converges to

$$\frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \quad \text{when} \quad 0 \leqq x \leqq 2\pi. \tag{1}$$

Can we find an elementary expression for the series

$$\frac{\cos x}{1^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \frac{\cos 9x}{9^2} + \frac{\cos 11x}{11^2} + \cdots ? \tag{2}$$

To answer this, we first define a function $f(x)$ to be in a class S if it can be expressed in open intervals as a finite sum of terms of the type Ax^le^{bx} where A and b are complex constants and l is a nonnegative integer.

The following theorem will answer the above problem:

THEOREM. *If $\sum_{n=1}^{\infty} r(n)$ sin nx is in S and $f(n)$ is an even function of period p, then $\sum_{n=1}^{\infty} r(n)f(n)$ sin nx is in S. If $\sum_{n=1}^{\infty} r(n)$ cos nx is in S and $f(n)$ is an odd function of period p, then $\sum_{n=1}^{\infty} r(n)$ cos nx is in S.*

Proof. If $f(n)$ is an even function of period p, then it can be expanded as (see [1])

$$(3)\qquad \begin{aligned} f(n) &= \sum_{k=1}^{p} a_k \cos\frac{2n\pi k}{p}, \quad \text{where} \\ a_k &= \frac{1}{p}\sum_{m=1}^{p} f(m)\cos\frac{2m\pi k}{p}. \end{aligned}$$

The series $\sum_{n=1}^{\infty} r(n)f(n)\sin nx$ can be written as

$$\begin{aligned} &\sum_{k=1}^{p} a_k \sum_{n=1}^{\infty} r(n)\cos\frac{2n\pi k}{p}\sin nx \\ &\qquad = \frac{1}{2}\sum_{k=1}^{p} a_k \sum_{n=1}^{\infty} r(n)\left[\sin n\left(x+\frac{2\pi k}{p}\right)+\sin n\left(x-\frac{2\pi k}{p}\right)\right]. \end{aligned}$$

Since the series $F(x)=\sum r(n)\sin nx$ is in S, the above series reduces to

$$\frac{1}{2}\sum_{k=1}^{p} a_k\left[F\left(x+\frac{2\pi k}{p}\right)+F\left(x-\frac{2\pi k}{p}\right)\right],$$

which, from the definition, is still in S. A similar argument holds if $f(n)$ is an odd function. The theorem is valid, in particular, if, in the sine series, $r(n)$ is an odd function or in the cosine series $r(n)$ is an even function. (See [2] or [3]).

Example. Consider the series

$$(4)\quad G(x) = \frac{\cos x}{1^2}+\frac{\cos 4x}{4^2}+\frac{\cos 6x}{6^2}+\frac{\cos 9x}{9^2}+\cdots+\frac{f(n)\cos nx}{n^2}+\cdots,$$

where
$$f(n) = \begin{cases} 1 & \text{if } n\equiv 1 \text{ or } 4 \bmod 5, \\ 0 & \text{if } n\equiv 0,\ 2, \text{ or } 3 \bmod 5. \end{cases}$$

Since $f(n)$ is an even function of period 5, it has the expansion

$$f(n) = \frac{2}{5}+\frac{4}{5}\left[\cos\frac{2\pi}{5}\cos\frac{2\pi n}{5}+\cos\frac{4\pi}{5}\cos\frac{4\pi n}{5}\right].$$

Using the above theorem, this sums to

$$\begin{aligned} G(x) = \frac{2}{5}F(x) &+ \frac{2}{5}\cos\frac{2\pi}{5}\left[F\left(x+\frac{2\pi}{5}\right)+F\left(x-\frac{2\pi}{5}\right)\right] \\ &+ \frac{2}{5}\cos\frac{4\pi}{5}\left[F\left(x+\frac{4\pi}{5}\right)+F\left(x-\frac{4\pi}{5}\right)\right], \end{aligned}$$

where

$$F(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}$$

if $0 \leqq x \leqq 2\pi$, and $F(x+2\pi) = F(x)$ if $-\infty < x < \infty$.

Sometimes the original series is not in S, whereas the series obtained by multiplying by a periodic factor is in S. For example,

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

is not in S, whereas

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \cdot f(n),$$

where

$$f(n) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 3, \\ 1 & \text{if } n \equiv 1 \bmod 3, \\ -1 & \text{if } n \equiv 2 \bmod 3, \end{cases}$$

has the sum

$$\frac{2}{3} \sin \frac{2\pi}{3} \left[F\left(x - \frac{2\pi}{3}\right) - F\left(x + \frac{2\pi}{3}\right) \right]$$

which is in S.

References

1. H. Weyl, Theory of Groups and Quantum Mechanics, New York, 1928, p. 34.

2. C. C. Chao, The closed-form summation of some common Fourier series, Quart. J. Mech. Appl. Math., vol. 9, 1956, p. 508.

3. H. J. Fletcher and C. J. Thorne, Sine and Cosine Transforms, Tech. Rep. #5, OOR Contract No. DA-04-495-ORD. 160, Univ. of Utah, 1952.

A NOTE ON LINEAR DIFFERENCE EQUATIONS

R. W. WAGNER, University of Massachusetts

This note contains a modification of the method given by Friedman* for solving homogeneous equations which yields a formal solution of the nonhomogeneous linear difference equation. Strangely enough, I have not been able to find this solution in standard reference books. For simplicity, the present discussion will be confined to the second-order equation, but the modification for higher order equations is obvious.

Assume that $f(0)$ and $f(1)$ are given as initial conditions, and consider the

* B. Friedman, Principles and Techniques of Applied Mathematics, New York, 1956, pp. 122–123. Milne-Thompson gives essentially the same method.

linear difference equation

$$f(x+2) = p(x)f(x+1) + q(x)f(x) + r(x), \tag{1}$$

and the vectors $(f(x+2), f(x+1), 1)$ and $(f(x+1), f(x), 1)$. By using equation (1) we find that

$$\begin{aligned}(f(x+2), f(x+1), 1) &= (p(x)f(x+1) + q(x)f(x) + r(x), f(x+1), 1) \\ &= (f(x+1), f(x), 1)\begin{pmatrix} p(x) & 1 & 0 \\ q(x) & 0 & 0 \\ r(x) & 0 & 1 \end{pmatrix}.\end{aligned}$$

This constitutes a homogeneous recurrence relation, and

$$(f(n), f(n-1), 1) = (f(1), f(0), 1)\left[\prod_{k=0}^{n-2}\begin{pmatrix} p(k) & 1 & 0 \\ q(k) & 0 & 0 \\ r(k) & 0 & 1 \end{pmatrix}\right] \tag{2}$$

with the matrices in the product arranged so that the factor for $k=j$ is immediately to the right of the factor for $k=j-1$. The first components of these vectors can be isolated by forming the scalar product with the column vector $(1, 0, 0)$ to get

$$f(n) = (f(1), f(0), 1)\left[\prod_{k=0}^{n-2}\begin{pmatrix} p(k) & 1 & 0 \\ q(k) & 0 & 0 \\ r(k) & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

as the solution of (1).

Note that each matrix in the product can be written in terms of blocks by distinguishing the last row and column in the form

$$\begin{pmatrix} p(k) & 1 & 0 \\ q(k) & 0 & 0 \\ r(k) & 0 & 1 \end{pmatrix} = \begin{pmatrix} P_k & 0 \\ R_k & 1 \end{pmatrix} \quad \text{with} \quad \begin{aligned} P_k &= \begin{pmatrix} p(k) & 1 \\ q(k) & 0 \end{pmatrix}, \\ R_k &= (r(k) \quad 0). \end{aligned}$$

Hence the matrix product will have the form

$$\begin{pmatrix} H_{n-2} & 0 \\ S_{n-2} & 1 \end{pmatrix} = \prod_{k=0}^{n-2}\begin{pmatrix} P_k & 0 \\ R_k & 1 \end{pmatrix}. \tag{4}$$

If U_n denotes the row vector $(f(n+1), f(n))$, equation (2) can be written as

$$(U_n, 1) = (U_0, 1)\begin{pmatrix} H_{n-1} & 0 \\ S_{n-2} & 1 \end{pmatrix}. \tag{5}$$

In case $r(x)=0$ for all x, the equation (1) is homogeneous, $R_k=0$ and $S_k=0$ for all k, and the solution is

$$U_n = U_0 H_{n-2}. \tag{6}$$

Moreover, each element of H_n is a solution of the homogeneous form of equation (1) *for special initial values.*

If $h(n)$, $k(n)$, and $s(n)$ denote the elements of the first column of the matrix product in equation (3), we get the standard form of the solution

$$f(x) = h(x)f(1) + k(x)f(0) + s(x) \tag{7}$$

in which $s(x)$ is called the particular solution.

The form of the particular solution* is of some interest and can be easily deduced from equation (4). Observe that

$$\begin{pmatrix} H_{n-1} & 0 \\ S_{n-1} & 1 \end{pmatrix} = \begin{pmatrix} H_{n-2} & 0 \\ S_{n-2} & 1 \end{pmatrix}\begin{pmatrix} P_{n-1} & 0 \\ R_{n-1} & 1 \end{pmatrix}, \tag{8}$$

so that

$$H_{n-1} = \prod_{k=0}^{n-1} P_k, \tag{9}$$

and

$$S_{n-1} = S_{n-2}P_{n-1} + R_{n-1}. \tag{10}$$

A consequence of (8) or (9) is that, if H_{n-2} is nonsingular,†

$$P_{n-1} = H_{n-2}^{-1} H_{n-1},$$

so that

$$S_{n-1} = S_{n-2}H_{n-2}^{-1}H_{n-1} + R_{n-1},$$

or

$$S_{n-1}H_{n-1}^{-1} - S_{n-2}H_{n-2}^{-1} = R_{n-1}H_{n-1}^{-1}.$$

This can be written in terms of the difference operator as

$$\Delta(S_{n-2}H_{n-2}^{-1}) = R_{n-1}H_{n-1}^{-1}.$$

Thus the particular solution can be derived by a summation process applied to an expression involving the nonhomogeneous term of equation (1) and the solutions of the homogeneous form of the equation. This same result can be derived by the method called variation of parameters.

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* This part of the note was added at the suggestion of the referee.

† H_n will be singular if, and only if, $q(k) = 0$ for $k \leqq n$.

REMARK ON A NOTE OF P. TURÁN

T. S. NANJUNDIAH, Mysore University

Using some properties of the Legendre polynomials, P. Turán [1] has given a proof of the combinatorial identity

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k} = \binom{n+k}{k}^2, \tag{1}$$

which occurred without proof in a book of the Chinese mathematician Le-Jen Shoo from 1867 (from the author's summary as reported in the Mathematical Reviews, vol. 16, 1955, p. 13). Rewriting the sum in (1) in the reverse order and changing the notation slightly, (1) takes the form

$$\sum_{r=0}^{n} \binom{n}{r}^2 \binom{\mu+r}{2n} = \binom{\mu}{n}^2, \qquad n = 0, 1, \cdots. \tag{2}$$

Now (2) is obviously a special case of the identity

$$\sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r}\binom{\mu+r}{m+n} = \binom{\mu}{m}\binom{\mu}{n}, \qquad m, n = 0, 1, \cdots. \tag{3}$$

Here we give a very simple and brief proof of (3) which involves just three applications of Vandermonde's identity.

Plainly we may take the summation in (3) over the range $0 \leqq r \leqq n$. We have

$$\begin{aligned}
\sum_{r=0}^{n} \binom{m}{r}\binom{n}{r}\binom{\mu+r}{m+n} &= \sum_{r=0}^{n} \binom{m}{r}\binom{n}{r} \sum_{s=0}^{r} \binom{r}{s}\binom{\mu}{m+n-s} \\
&= \sum_{s=0}^{n} \binom{\mu}{m+n-s}\binom{m}{s} \sum_{r=s}^{n} \binom{m-s}{r-s}\binom{n}{n-r} \\
&= \sum_{s=0}^{n} \binom{\mu}{m+n-s}\binom{m}{s}\binom{m+n-s}{n-s} \\
&= \binom{\mu}{m} \sum_{s=0}^{n} \binom{m}{s}\binom{\mu-m}{n-s} = \binom{\mu}{m}\binom{\mu}{n},
\end{aligned}$$

and the proof of (3) is complete. The same method of proof actually yields the more general identity

$$\sum_{r=0}^{n} \binom{m-\mu+\nu}{r}\binom{n+\mu-\nu}{n-r}\binom{\mu+r}{m+n} = \binom{\mu}{m}\binom{\nu}{n}, \qquad m, n = 0, 1, \cdots, \tag{4}$$

holding for arbitrary complex μ, ν.

Reference

1. P. Turán, On a problem in the history of Chinese mathematics, Mat. Lapok, vol. 5, 1954, pp. 1–6 (Hungarian, Russian and English summaries).

CONDITIONS FOR A POSITIVE-DEFINITE QUADRATIC FORM ESTABLISHED BY INDUCTION

C. J. SEELYE, Victoria University College, Wellington, New Zealand

We give an inductive proof that the necessary and sufficient conditions for a quadratic form to be positive-definite are that the leading principal (abbreviated $l-p$) minors of the matrix of the form should be positive. Although A. Mirsky in *An Introduction to Linear Algebra* proves sufficiency by induction with effectively a condensation process, the present account may be of interest because it treats both sufficiency and necessity together and avoids much of the mechanical manipulation appearing in most textbook proofs. This is accomplished by the use of some simple properties of determinants and matrices and for these reference may be made to Chapters I and II of A. C. Aitken's *Determinants and Matrices*.

(i) *One variable.* Obviously $a_{11}x_1^2>0$ for all nonzero x_1 if and only if $a_{11}>0$. Here the $l-p$ minor is a_{11}, the sole element of the matrix, so that the theorem is true for this case.

(ii) *n variables.* Assume the theorem true for n variables x_i which can be considered as the elements of a column vector $\mathbf{x}$. Transposing, we write $\mathbf{x}'=[x_1 \cdots x_n]$ and take the matrix of the quadratic form as $\mathbf{A}=[a_{ij}]$ where, without loss of generality, $a_{ij}=a_{ji}$. i, j run 1 to n. The assumed necessary and sufficient conditions for positive-definiteness are that the n $l-p$ minors $|a_{rs}|$, obtained by letting r, s run 1 to 1, 2, $\cdots$ or n respectively, are all positive.

(iii) *$n+1$ variables.* Consider now the case of $n+1$ variables, denoting the extra variable as x_0 and the additional matrix elements as

$$a_{00} \text{ and } \mathbf{a}_0' = [a_{0i}] = [a_{i0}], \qquad i = 1 \text{ to } n,$$

so that the corresponding quadratic form is

$$T = [x_0\ \mathbf{x}']\begin{bmatrix} a_{00} & \mathbf{a}_0' \\ \mathbf{a}_0 & \mathbf{A} \end{bmatrix}\begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix}.$$

The new matrix is

$$\mathbf{A}^+ = [a_{ij}], \qquad i, j = 0 \text{ to } n.$$

The matrices are partitioned conformably, so by multiplying out and noting symmetry we have

$$T = a_{00}x_0^2 + 2\mathbf{x}'\mathbf{a}_0x_0 + \mathbf{x}'\mathbf{A}\mathbf{x}.$$

Regarding this as a quadratic in x_0, we have $T>0$ for all values of the variables, exclusive of the case when all $n+1$ are zero, if and only if

$$(C_1) \qquad a_{00} > 0$$

and

$$(C_2) \qquad (\mathbf{x}'\mathbf{a}_0)^2 < a_{00}(\mathbf{x}'\mathbf{A}\mathbf{x}).$$

This latter condition is

$$\mathbf{x}'(a_{00}\mathbf{A} - \mathbf{a}_0\mathbf{a}_0')\mathbf{x} > 0,$$

involving a new quadratic form with matrix

$$\mathbf{B} = a_{00}\mathbf{A} - \mathbf{a}_0\mathbf{a}_0'.$$

Its elements $b_{ij} = a_{00}a_{ij} - a_{i0}a_{0j}$ are the 2×2 minors of $\mathbf{A}^+$ with a_{00} as leading pivot. A relation between the $l-p$ minors of $\mathbf{A}^+$ and $\mathbf{B}$ can be obtained as follows:*

If $\boldsymbol{\alpha} = -a^{-1}\mathbf{a}_0$ and $\mathbf{I}$ is the $n\times n$ unit matrix,

$$\begin{bmatrix} 1 & 0 \\ \boldsymbol{\alpha} & \mathbf{I} \end{bmatrix}\begin{bmatrix} a_{00} & \mathbf{a}_0' \\ \mathbf{a}_0 & \mathbf{A} \end{bmatrix}\begin{bmatrix} 1 & \boldsymbol{\alpha}' \\ 0 & \mathbf{I} \end{bmatrix} = \begin{bmatrix} a_{00} & 0 \\ 0 & a_{00}^{-1}\mathbf{B} \end{bmatrix}.$$

Since the prefactor merely adds multiples of the first row of $\mathbf{A}^+$ to subsequent rows and the postfactor operates on columns similarly, any leading determinant of $\mathbf{A}^+$ and of this product have the same value. Thus for the $(r+1)$th $l-p$ minor

$$(1) \qquad \left| a_{00}a_{11} \cdots a_{rr} \right| = a_{00}^{1-r} \left| b_{11}b_{22} \cdots b_{rr} \right|, \qquad r = 1 \text{ to } n,$$

a relationship often used in evaluating determinants by condensation.

Because we are assuming the results expressed in (ii) apply to $n\times n$ matrices, condition C_2 is effectively that all the $l-p$ minors of $\mathbf{B}$ are positive. If also $a_{00}>0(C_1)$, it follows from (1) that the $l-p$ minors of $\mathbf{A}^+$ of orders 2 to $n+1$ are positive and a_{00} is its first minor.

Therefore C_1 and C_2 together are equivalent to the condition that the $n+1$ $l-p$ minors of $\mathbf{A}^+$ are all positive. Conversely if the last statement is true, it is immediately apparent that C_1 and C_2 hold.

The validity of the first case has been noted, the necessity and sufficiency of the conditions given in the theorem have been shown to apply to the $n+1$th case if they apply to the nth case and the inductive proof is thus complete.

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* I wish to thank the referee for suggesting this form of demonstration of (1) for inclusion here.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

FOR THE HOMOGENEOUS DIFFERENTIAL EQUATION $M(x, y)dx+N(x, y)dy=0$, $y=vx$ or $x=vy$?

MORTON J. HELLMAN, Rutgers University

In most elementary differential equations texts when the homogeneous equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{1}$$

is considered, it is shown that either substitution $y=vx$ or $x=vy$ will always transform the equation to one where the variables are separable. Usually the remark is made that either substitution will work. Most often, however, it is not clear which substitution will lead to the simpler integration on v, and it is necessary to study both resulting integrands to decide this. Thus, the equation is almost solved twice; the following device indicates how this process can be substantially shortened.

Write (1) in the form

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}. \tag{2}$$

Since M and N are homogeneous functions of the same degree, (2) can be put into the form

$$\frac{dy}{dx} = -f\left(\frac{y}{x}\right) = -g\left(\frac{x}{y}\right). \tag{3}$$

Setting $y/x=v$, and $x/y=u$ for convenience, (3) yields

$$\frac{dx}{x} + \frac{dv}{v + f(v)} = 0, \tag{4}$$

$$\frac{dy}{y} + \frac{du}{u + \dfrac{1}{g(u)}} = 0. \tag{5}$$

The question is now which is simpler to integrate, (4) or (5). To decide this, use the facts that $f(v)=g(u)$ and $v=1/u$. Then (5) becomes

$$\frac{dy}{y} + \frac{du}{u + \dfrac{1}{f(1/u)}} = 0. \tag{6}$$

The problem has been reduced to finding which is easier to evaluate,

$$\int \frac{dv}{v + f(v)}, \tag{7}$$

$$\int \frac{dv}{v + \dfrac{1}{f(1/v)}}. \tag{8}$$

This analysis leads to a convenient rule for solving such a differential equation. Except for those special cases in which it is evident that the substitution $x = vy$ will prove simpler, set $y = vx$ and consider equation (4) which results. In particular, from (7) which occurs in the integration of equation (4), form (8). If (7) is simpler to evaluate than (8), the solution of the equation will be

$$\ln |x| + \int \frac{dv}{v + f(v)} = c \quad \text{where} \quad v = \frac{y}{x}. \tag{9}$$

If (8) is simpler to find than (7), the solution will be

$$\ln |y| + \int \frac{dv}{v + \dfrac{1}{f(1/v)}} = c \quad \text{where} \quad v = \frac{x}{y}. \tag{10}$$

An example will illustrate the technique.
Solve:

$$(2xy + 3y^2)dx + (5y^2 - x^2)dy = 0. \tag{11}$$

Since it is not clear which substitution will result in the simpler integration on v, set $y = vx$, and (11) becomes

$$\int \frac{dx}{x} + \int \frac{5v^2 - 1}{5v^3 + 3v^2 + v}\, dv = c. \tag{12}$$

Although this second integral can be evaluated by the method of partial fractions, a look at the integral that would have resulted from the substitution $x = vy$ is justified. To obtain this write this integral in the form (7). Thus

$$\int \frac{5v^2 - 1}{5v^3 + 3v^2 + v}\, dv = \int \frac{dv}{v + \dfrac{3v^2 + 2v}{5v^2 - 1}}. \tag{13}$$

Here

$$f(v) = \frac{3v^2 + 2v}{5v^2 - 1}$$

and

$$\frac{1}{f(1/v)} = \frac{1}{\dfrac{\dfrac{3}{v^2} + \dfrac{2}{v}}{\dfrac{5}{v^2} - 1}} = \frac{5 - v^2}{3 + 2v}.$$

Now (8) becomes

$$\int \frac{(2v + 3)dv}{v^2 + 3v + 5} \tag{14}$$

which is immediately integrable; now using (14) in (10), the solution to equation (11) reduces to

$$x^2 + 3xy + 5y^2 = cy. \tag{15}$$

The same device could be used to replace (8) by (7) when starting with the substitution $x = vy$ when it is desired to see the integral on v that would have resulted from $y = vx$. That is, from (8) occurring in the solution (10) find $[f(1/v)]^{-1}$, hence $f(1/v)$ and therefore $f(v)$, and thus form (7). If this is simpler to find than (8), (9) gives the complete solution.

A PROJECTIVE DEFINITION OF DOUBLE RATIO

FRANK C. GENTRY, University of New Mexico

Most textbooks in projective geometry introduce the idea of double ratio of four points on a line by means of a metric definition in terms of distances between the points and then proceed to prove that the double ratio is a projective invariant. The following definition, although not new, does not seem to be very well known.

With a projective system of coordinates in n-space, let $(\alpha x) \equiv \alpha_0 x_0 + \cdots + \alpha_n x_n = 0$ and $(\beta x) = 0$ be the equations of two hyperplanes or primes and let $a \equiv (a_0, \cdots, a_n)$ and $b \equiv (b_0, \cdots, b_n)$ be two points not incident with either of the two primes.

DEFINITION. *The double ratio D of the two points a and b with respect to the two primes $(\alpha x) = 0$ and $(\beta x) = 0$ is*

$$D = \frac{(\alpha a)(\beta b)}{(\alpha b)(\beta a)} \tag{1}$$

where $(\alpha a) \equiv (\alpha_1 a_0 + \cdots + \alpha_n a_n)$, etc. The expression D is obviously an invariant under linear transformation since each of its factors is. Moreover it is independent of any constant of proportionality in the coordinates of the points or the coefficients of the primes. It is therefore a geometric invariant of the points and primes.

In (1) let D, α_i, β_i, b_i $(i=0, 1, \cdots, n)$ be constants while a_i is replaced by a variable x_i. Then (1) becomes a linear equation, obviously satisfied by the point a and the base of the pencil of primes determined by $(\alpha x)=0$ and $(\beta x)=0$. Call this prime $(\gamma x)=0$ and let $(\delta x)=0$ be the equation of the similar prime on b and the base of the same pencil of primes. Then D is the double ratio of the four primes α, β, γ, δ and a and b are any two points of γ and δ respectively.

Dually, let D, a_i, b_i, β_i be constants and let α_i be replaced by a variable μ_i. Then (1) becomes the prime equation of the point c in which the prime α meets the line ab. Similarly let the point d be the meet of the prime β and the line ab. Hence D is the double ratio of the four points a, b, c, d on a line.

Now if α, β, γ, and δ are primes of a parallel pencil, equivalent projectively to any pencil of primes, and a and b are points of γ and δ respectively, then the quantities (αa), *etc.*, are proportional to the actual distances from the primes to the points and D satisfies the usual metric definition of double ratio.

In order to compute the double ratio of four primes of a pencil we may choose any one of them for α and any other one for β. Then a is chosen as any point of either of the remaining primes and b is any point of the other. Hence we obtain the 24 possible double ratios. If D is represented by the symbol $(\alpha\beta, ab)$ then, obviously $(\alpha\beta, ab)=(\alpha\beta, ba)=(ab, \alpha\beta)=(ba, \alpha\beta)$ since $(\alpha a)\equiv(a\alpha)$, *etc.*

In particular, in a linear space S_1 let two points be given by their coordinates: $a(a_0, a_1)$, $b(b_0, b_1)$ and two by their equations: $c(\gamma x)=0$, $d(\delta x)=0$. Then the double ratio

$$(ab, cd) = \frac{(\gamma a)(\delta b)}{(\gamma b)(\delta a)}\,.$$

But if $\gamma_0 x_0+\gamma_1 x_1=0$ then $x_0:x_1=-\gamma_1:\gamma_0$ and if c has the coordinates (c_0, c_1), then $c_0:c_1=-\gamma_1:\gamma_0$. Similarly if d has the coordinates (d_0, d_1), then $d_0:d_1 = -\delta_1:\delta_0$ and D may be written

$$(ab, cd) = \frac{(c_1a_0 - c_0a_1)(d_1b_0 - d_0b_1)}{(c_1b_0 - c_0b_1)(d_1a_0 - d_0a_1)} = \frac{|ac|\;|bd|}{|ad|\;|bc|}\,.$$

Similarly the double ratio of four points of a line given by their equations $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ is

$$(\alpha\beta, \gamma\delta) = \frac{|\alpha\gamma|\;|\beta\delta|}{|\alpha\delta|\;|\beta\gamma|}\,.$$

FINDING THE RATE IN A CONTINUOUS ANNUITY

HUGH E. STELSON, Michigan State University

In this note some formulas are presented for very accurate determination of the rate of interest in a continuous annuity. The present value of a continuous

annuity is given by the formula,

$$B = Ra = \frac{R(1 - e^{-n\delta})}{\delta} \tag{1}$$

where R = periodic payment, n = number of periodic payments, B = amount of loan, δ = rate of interest (to be determined).

Now formula (1) can be written in the form

$$B = \frac{I}{\dfrac{x}{1 - e^{-x}} - 1} \tag{2}$$

where $I = Rn - B$, the cost of the loan, and $x = n\delta$.

The right member of (2) may be expanded in the following series:

$$B = \frac{2I}{x}\left(1 - \frac{x}{6} + \frac{x^2}{36} - \frac{x^3}{540} - \frac{x^4}{6480} \cdots\right). \tag{3}$$

We now invert (3) and obtain

$$x = 2M\left(1 - \frac{M}{3} + \frac{2M^2}{9} - \frac{22M^3}{135} - \frac{7M^4}{135} \cdots\right) \tag{4}$$

where $M = I/B$.

Now (4) is approximately equal to

$$x' = \frac{2M}{\sqrt[3]{1 + M}} = \frac{2I}{\sqrt[3]{B^2Rn}} \tag{5}$$

with an error less than $+8M^4/405$, the result being a little too small (*i.e.*, $x - x' \doteq +8M^4/405$). Formula (5) is an analog of Schurig's formula.

Also (4) may be expressed as a continued fraction:

$$x = 2M\left(\cfrac{1}{1 + \cfrac{\frac{M}{3}}{1 + \cfrac{\frac{M}{3}}{1 + \cfrac{2M}{15}}}} \cdots\right). \tag{6}$$

By considering convergents, we obtain some very excellent approximations to the rate [1]

$$x_1 = \frac{2M}{1+\frac{M}{3}}\left(\text{error} < +\frac{2M^3}{9}\right), \tag{7}$$

$$x_2 = 2M\left[\frac{3+M}{3+2M}\right]\left(\text{error} < -\frac{4M^4}{135}\right), \tag{8}$$

$$x_3 = \frac{6M(15+7M)}{45+36M+2M^2}\left(\text{error} < -\frac{698}{2025}M^5\right). \tag{9}$$

A very good approximation is the average of (5) and (8),

$$x \doteq \frac{x'+x_2}{2}\left(\text{error} < -\frac{2M^4}{405}\right). \tag{10}$$

δ can be found by recalling that $\delta = x/n$. It should be noted that (7) gives too small a result, (8) gives a better result, but too large, while (9) gives a still better result, but slightly large.

Example. Let $B = 1{,}906{,}083.04$ and $R = 100{,}000$ then $I = 493{,}916.94$ and $M = .2591266747$. By computation with a desk calculator, we find

$$\begin{aligned}
x_1 &= .477048046, & \delta_1 &= .019877 \text{ (error } + .000123),\\
x_2 &= .480082912, & \delta_2 &= .02000345 \text{ (error } - .0000034),\\
x_3 &= .479988792, & \delta_3 &= .019999533 \text{ (error } + .000000467),\\
x' &= .4799397, & \delta' &= .0199975 \text{ (error } + .0000025).
\end{aligned}$$

References

1. Hugh E. Stelson, Note on finding the interest rate, this MONTHLY, 1953, vol. 60, pp. 703–705.

2. Erich Michalup, Some approximation formulas of the effective rate and the force of interest, Skandinavisk Aktuarietidskrift, 1955, Haft 3–4, p. 163.

3. J. F. Steffensen, A table of the function $G(x) = x/(1-e^{-x})$ and its application to problems in compound interest, Skandinavisk Aktuarietidskrift, 1938, pp. 47–71.

4. Hugh E. Stelson, Some continuous annuity approximations, Giorn. Mat. Finanz., 1955. vol. 1.

PROOF OF THE MEAN VALUE THEOREM

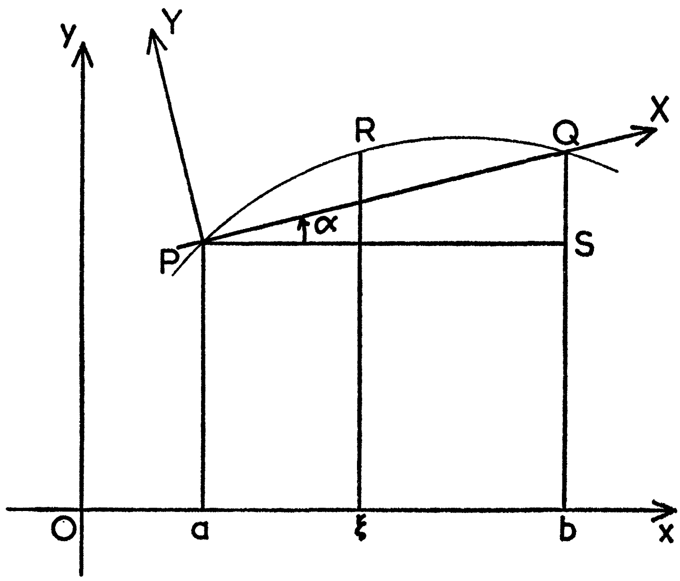
CHUNG LIE WANG, University of South Carolina

A proof is given here of the mean value theorem, based on a translation and rotation of axes, which is believed to be new.

The theorem in its usual form is as follows: Let $y = f(x)$ be continuous for $a \leqq x \leqq b$ with a derivative, $f'(x)$, such that $a < x < b$. Then, there is at least one number ξ such that $f(b) - f(a) = (b-a)f'(\xi)$, where $a < \xi < b$.

In the accompanying figure let PRQ be a portion of the graph of $y = f(x)$ and let chord PQ make angle α with PS, where PS is parallel to the x-axis. Then,

$$\tan \alpha = \frac{SQ}{PS} = \frac{f(b) - f(a)}{b - a}. \tag{1}$$



Refer $y=f(x)$ to a new system of axes (X, Y) with origin at $P[a, f(a)]$ and with α as the angle through which the (x, y) system is rotated to bring it into coincidence with the (X, Y) system. We then have

$$X \cos \alpha - Y \sin \alpha = x - a, \qquad X \sin \alpha + Y \cos \alpha = y - f(a),$$

from which we find

$$(2) \quad \begin{aligned} X &= \frac{\begin{vmatrix} x - a & -\sin \alpha \\ y - f(a) & \cos \alpha \end{vmatrix}}{\begin{vmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{vmatrix}} = [y - f(a)] \sin \alpha + (x - a) \cos \alpha, \\ Y &= \frac{\begin{vmatrix} \cos \alpha & x - a \\ \sin \alpha & y - f(a) \end{vmatrix}}{\begin{vmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & -\cos \alpha \end{vmatrix}} = [y - f(a)] \cos \alpha - (x - a) \sin \alpha. \end{aligned}$$

Then, if $Y=F(X)$ is the equation of the curve with respect to the new system of axes, we have

$$F(X) = [f(x) - f(a)] \cos\alpha - (x - a) \sin\alpha, \tag{3}$$
$$X = [f(x) - f(a)] \sin\alpha + (x - a) \cos\alpha,$$

from which it follows that

$$\frac{dF(X)}{dx} = f'(x) \cos\alpha - \sin\alpha, \qquad \frac{dX}{dx} = f'(x) \sin\alpha + \cos\alpha.$$

Hence,

$$F'(X) = \frac{dF(X)}{dX} = \frac{f'(x)\cos\alpha - \sin\alpha}{f'(x)\sin\alpha + \cos\alpha}.$$

From (3), when $x = a$, $F(X) = 0$, and since from (1),

$$\sin\alpha = \frac{f(b) - f(a)}{\sqrt{[(f(b) - f(a)]^2 + (b - a)^2}}, \qquad \cos\alpha = \frac{b - a}{\sqrt{[f(b) - f(a)]^2 + (b - a)^2}},$$

we also have $F(X) = 0$ when $x = b$. Hence, by Rolle's theorem, $F'(X)$ must vanish for some value of x between a and b.* Let this value be $x = \xi$. We then have

$$\frac{f'(\xi)\cos\alpha - \sin\alpha}{f'(\xi)\sin\alpha + \cos\alpha} = 0 \quad \text{or} \quad f'(\xi)\cos\alpha - \sin\alpha = 0.$$

It follows from this last equation that $\tan\alpha = f'(\xi)$.

The result of substituting this in (1) is $f(b) - f(a) = (b - a)f'(\xi)$.

* In exceptional cases where $F(X)$ does not satisfy all the conditions of Rolle's theorem, the proof given is not valid.

ON THE DERIVATIVE OF x^c

S. Leader, Rutgers University

For real constants a and c with $a > 0$, the differentiation formula

$$Dx^c\big]_{x=a} = ca^{c-1} \tag{1}$$

can be obtained directly from the special case in which $a = 1$ and $c > 0$:

$$\lim_{x\to 1} \frac{x^c - 1}{x - 1} = c. \tag{2}$$

To derive (1) from (2) we have only to use the identities

$$\frac{x^c - a^c}{x - a} = \left(\frac{w^c - 1}{w - 1}\right) a^{c-1} = -\, w^c \left(\frac{w^{-c} - 1}{w - 1}\right) a^{c-1} \tag{3}$$

where $w = x/a$, noting that w approaches 1 as x approaches a. Thus, we need only prove (2).

For the case $c=n$, a positive integer, (2) follows from the identity

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + x^3 + \cdots + x^{n-1}. \tag{4}$$

For $c=m/n$, a positive rational, we have, setting $y=x^{1/n}$,

$$\frac{x^{m/n} - 1}{x - 1} = \frac{\dfrac{y^m - 1}{y - 1}}{\dfrac{y^n - 1}{y - 1}} \tag{5}$$

which gives (2) *via* the preceding case.

For arbitrary $c>0$ consider any rationals p and q such that $0<p<c<q$. Then for $0<x<1$ we have $x^q<x^c<x^p$ so $x^q-1<x^c-1<x^p-1$. Since $x-1<0$,

$$\frac{x^p - 1}{x - 1} < \frac{x^c - 1}{x - 1} < \frac{x^q - 1}{x - 1} \cdot \tag{6}$$

Now (6) also holds for $x>1$, for then $x^p<x^c<x^q$ and $x-1>0$. Letting x approach 1 in (6) we obtain

$$p \leqq \lim_{x\to 1} \frac{x^c - 1}{x - 1} \leqq q. \tag{7}$$

Since p and q may be arbitrarily close to c, (7) gives (2).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1316. *Proposed by R. C. Buck, University of Wisconsin*

Let S be a set of elements with an associative multiplication. Suppose that S has a special element u with the property that u is a left and a right divisor of every element in S. Does S have to possess a unit?

E 1317. *Proposed by N. A. Court, University of Oklahoma*

Let $(T)=DABC$ be a tetrahedron and let (M) be a sphere having its center M on the axis of the circle ABC. If one and only one point is taken in each

pair of points determined by (M) on the edges of (T) issued from the vertex D, show that the $2^3=8$ planes thus determined are cut by the plane ABC along four pairs of isotomic transversals of the triangle ABC.

E 1318. *Proposed by Gordon Raisbeck, Bell Telephone Laboratories, Inc.*

A. A. Mullin has proved (this MONTHLY [1957, p. 669]) that

$$\Phi(n) = \sum_{r=0}^{n} {}_nP_r = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \sim (n!)e$$

for large n, where $\sim$ denotes asymptotic equality. Prove that

$$\Phi(n) = [(n!)e], \qquad n \geqq 1,$$

where $[x]$ denotes the largest integer not greater than x.

E 1319. *Proposed by S. W. Golomb, California Institute of Technology*

A quiz contestant selects a category containing n questions, k of which are too difficult for him. The questions are selected from the category at random, and the contestant continues answering until he misses a question. What is the probability that he will miss on the ath question?

E 1320. *Proposed by G. S. Stoller, Polytechnic Institute of Brooklyn*

(a) Let $P(x)$ be any polynomial in x with integral coefficients. Prove that there exists an infinite number of integers t such that $P(t)$, $P(t+1)$, $\cdots$, $P(t+m)$ are all composite for any given positive integer m.

(b) Let $P(x)=x^2+1$. Find a value of t satisfying part (a) for $m=5$.

SOLUTIONS

Two Equal Sums of n Distinct Squares

E 1286 [1957, 670]. *Proposed by F. S. Stancliff, Springfield, Ohio*

Establish the identity

$$\sum_{r=0}^{n-2} (2^r - 1)^2 + [3(2^{n-1}) - 1]^2 = \sum_{r=2}^{n} (2^r - 1)^2 + (2^n - 4)^2,$$

thus obtaining a general method for finding two equal sums of n distinct squares.

Solution by C. F. Pinzka, University of Cincinnati. The identity is established by noting that

$$\begin{aligned}\sum_{r=2}^{n} (2^r + 1)^2 - \sum_{r=0}^{n-2} (2^r - 1)^2 &= \sum_{r=0}^{n-2} [(4\cdot 2^r + 1)^2 - (2^r - 1)^2] \\ &= 5\sum_{r=0}^{n-2} (3\cdot 2^{2r} + 2^{r+1}) = 5(2^{2n-2} - 1 + 2^n - 2) \\ &= (5\cdot 2^{n-1} - 5)(2^{n-1} + 3) = [3\cdot 2^{n-1} - 1]^2 - (2^n - 4)^2.\end{aligned}$$

Also solved by D. S. Adorno, W. A. Al-Salam, Winifred Asprey, D. A. Breault, D. R. Brillinger, P. L. Chessin, W. J. Cody, A. E. Danese, Underwood Dudley, E. S. Eby, W. V. Gamzon, Lawrence Glasser, Cornelius Groenewoud, Emil Grosswald, Leonard Hauer, Vern Hoggatt, Edgar Karst, M. A. Kirchberg, J. D. E. Konhauser, Joe Lipman, D. C. B. Marsh, J. B. Muskat, Tim Nolan, F. D. Parker, M. J. Pascual, Susan Pyeatt, L. A. Ringenberg, D. A. Robinson, R. E. Shafer, Ya'akov Shima, Paul Slepian, Peter Treuenfels, C. W. Trigg, Dale Woods, and David Zeitlin. Late solutions by E. F. Allen, D. W. Bailey, Julian Braun, Eugene Famolari, Jr., R. H. Hou, A. R. Hyde, D. L. Muench, and Dmitri Thoro.

Equilateral Triangles on the Sides of a Given Triangle

E 1287 [1957, 671]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the radical center P of the circles inscribed in the equilateral triangles BCA', CAB', ABC' constructed exteriorly (or interiorly) on the sides of a triangle ABC having centroid G divides the distance from G to the point of concurrence Q of the lines AA', BB', CC' in the ratio $GP/GQ=1/4$.

Solution by H. E. Fettis, Dayton, Ohio. The proposition may be established with the aid of the following lemma:

The segments of the line joining the midpoints of any two sides of the triangle ABC intercepted by the circles inscribed in the corresponding equilateral triangles are equal. Let the midpoints of the sides be designated by L, M, N and the centers of the equilateral triangles by D, E, F. Then, if MN intersects circle (E) again in R and circle (F) in S,

$$EM/RM = AC/2h, \qquad FN/NS = AB/2h,$$

where h is the altitude drawn from A. Hence

$$(AC)(RM)/EM = (AB)(NS)/FN.$$

But, since the triangles are equilateral, $AC/EM=AB/FN$, so that $RM=NS$.

Proceeding to the main problem, it is known that DEF is equilateral, with sides mutually perpendicular to AA', BB', CC'. Consider, now, the parallelogram $MNVW$, where V and W are the remaining points of contact of circle (D). If the common midpoint T of VW and LA' is joined to the midpoint K of LA, intersecting GQ at P, this join will be parallel to AA'. Also, since VN and WM are parallel to AA', K must also be the midpoint of MN. Further, this line which, by construction, divides the median segment GA at K in the ratio $1/4$ must also divide GQ at P in the ratio $1/4$. Now, by virtue of the lemma, K has equal powers with respect to the circles (E) and (F), and since KT is perpendicular to EF, it is the radical axis of these circles. Thus the radical axis of any pair of circles divides GQ at P in the ratio $1/4$, and therefore P is the radical center of the circles (D), (E), (F).

An analogous proof may be given for the case where the triangles are described internally.

Also solved by W. B. Carver, J. W. Clawson, Roscoe Woods, and the proposer.

Woods gave a synthetic proof based on the lemma: *Let O be the circumcenter of a triangle ABC and let P be any point in the plane of the triangle. The radical center M of the circles whose centers are A, B, C and whose radii are k(AP), k(BP), k(CP), where k is a parameter, lies on the line OP and such that* $OM/OP=k^2$. Applying the lemma to triangle DEF, letting Q play the role of P in the lemma, and taking $k=1/2$, the theorem readily follows.

Carver employed the method of conjugate coordinates. Late solutions by E. F. Allen and Josef Langr.

Odd Binomial Coefficients

E 1288 [1957, 671]. *Proposed by S. H. Kimball, University of Maine*

The number of odd binomial coefficients in any finite binomial expansion is a power of 2 (Putnam Mathematical Competition, this MONTHLY [1957, p. 24]). Prove that the power of 2 is the number of 1's in the binary scale expression for n in $(x+y)^n$.

I. *Solution by T. R. Hatcher and J. A. Riley, Parke Mathematical Laboratories, Carlisle, Mass.*

Let h and n be positive integers with $h<n$. We define the *binary length* of n, $L(n)$, to be the number of ones in the binary representation of n, and the *binary capacity* of n, $C(n)$, to be the exponent of the highest power of two which divides n. We say "h is contained in n," written $h\subset n$, if when h has a one in a certain binary place, n also has a one in the corresponding binary place; that is, the binary representation of h can be obtained from that of n by changing ones to zeros.

The following properties are easily proved:

(1) $C(n)$ is the number of terminating zeros in the binary representation of n.
(2) $C(n)=0$ if and only if n is odd.
(3) $C(ab)=C(a)+C(b)$, $C(a/b)=C(a)-C(b)$.
(4) $L(n)=L(h)+L(n-h)$ if and only if $h\subset n$.
(5) $C(n)=1+L(n-1)-L(n)$.
(6) $C(n!)=n-L(n)$.
(7) $C\binom{n}{h}=L(h)+L(n-h)-L(n)$.

The corollary of the following theorem gives the solution.

THEOREM. $\binom{n}{h}$ *is odd if and only if* $h\subset n$.

Proof. If $\binom{n}{h}$ is odd, $C\binom{n}{h}=0$ and by (7) $L(n)=L(h)+L(n-h)$. Thus, by (4), $h\subset n$. Conversely, if $h\subset n$, then $L(n)=L(h)+L(n-h)$ and $C\binom{n}{h}=0$.

COROLLARY. *The number of integers h such that* $\binom{n}{h}$ *is odd is* $2^{L(n)}$.

Proof. The number of integers h with $h\subset n$ and $L(h)=j$ is $\binom{L(n)}{j}$. Thus the number of integers h for which $\binom{n}{h}$ is odd is simply

$$\sum_{j=0}^{L(n)}\binom{L(n)}{j}=2^{L(n)}.$$

II. *Remarks by Leo Moser, University of Alberta.* Problem E 1288 is a special case of 4723 [1957, 116]. The solution of that problem is the following:

If $n=a_0+a_1p+a_2p^2+\cdots+a_kp^k$, $0\leqq a_i<p$, $i=0, 1, 2, \cdots, k$, then the number of solutions of $(\binom{n}{r}, p)=1$, $r=0, 1, \cdots, n$, is $\prod_{i=0}^{k}(a_i+1)$.

The result in E 1288 is contained in J. W. L. Glaisher, "On the residue of a binomial coefficient with respect to a prime modulus," *Quarterly Journal of Mathematics*, vol. 30, 1899, pp. 150–156. More recently a proof was given by J. B. Roberts, "On binomial coefficient residues," *Canadian Journal of Mathematics*, vol. 9, 1957, pp. 363–370.

Also solved by D. R. Brillinger, Leonard Carlitz, Joe Lipman, D. C. B. Marsh, Paul Schillo, and the proposer.

The Fermat Equation

E 1289 [1957, 671]. *Proposed by Marvin Shinbrot, National Advisory Committee for Aeronautics, Moffett Field, California*

Show that the Fermat equation $x^n+y^n=z^n$ has no nontrivial solution in integers for $n>2$ if $z<2^{1/n}/(2^{1/n}-1)$.

Solution by G. S. Stoller, Polytechnic Institute of Brooklyn. Suppose there is a solution in integers of $x^n+y^n=z^n$ for $n>2$ and integral. Then, since $0<x<z$ and $0<y<z$, $z^n\leqq 2(z-1)^n$, or $z\geqq 2^{1/n}/(2^{1/n}-1)$.

Also solved by Leonard Carlitz, M. A. Kirchberg, D. C. B. Marsh, L. F. Meyers, J. B. Muskat, R. E. Shafer, and the proposer. Late solution by Walter Leighton and M. C. Sholander (jointly).

Inverse Tangents

E 1290 [1957, 671]. *Proposed by F. E. Shafer, University of California Radiation Laboratory, Livermore, California*

If $|n \tan^{-1} x| \leqq \pi/2$, show that

$$n \tan^{-1} x = \tan^{-1} \frac{\operatorname{Im}(1+ix)^n}{\operatorname{Re}(1+ix)^n},$$

where $\operatorname{Im} f(z)$ denotes the imaginary part of $f(z)$ and $\operatorname{Re} f(z)$ denotes the real part of $f(z)$.

Solution by W. A. Al-Salam, Duke University. Consider the complex number $z=1+ix$. We have $\arg z=\tan^{-1} x$. Hence it follows that $\arg z^n=n\tan^{-1} x$. Now, since $|n \tan^{-1} x| \leqq \pi/2$, we have the desired result.

Also solved by D. S. Adorno, R. J. Arguello, D. A. Breault, Tien Chi Chen and Bryant Tuckerman (jointly), P. L. Chessin, Underwood Dudley, E. S. Eby, Lawrence Glasser, Cornelius Groenewoud, Juris Hartmanis, Vern Hoggatt, J. R. Holdsworth, M. A. Kirchberg, M. S. Klamkin, J. D. E. Konhauser, Joe Lipman, D. C. B. Marsh, Morris Morduchow, C. S. Ogilvy, M. J. Pascual, C. F. Pinzka, Michael Rosen, E. M. Scheuer, Nathan Schwid, Ya'akov Shima, Arnold Singer, Paul Slepian, Peter Treuenfels, Chih-yi Wang, R. H. Wilson, Jr., Dale Woods, David Zeitlin, and the proposer. Late solutions by Edward Barbeau, Julian Braun, A. E. Danese, R. H. Hou, and Vencil Skarda.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4788. *Proposed by D. J. Newman, AVCO Research, Lawrence, Mass.*

Let $f(z)$ be an entire function and consider the two statements: (1) If f, f', f'' never vanish, then $f(z)=e^{az+b}$. (2) If f, f', f'' each miss a value, then $f(z)=a+be^{cz}$. (1) is a known theorem. (2) is apparently stronger than (1). Show, however, that (2) is an elementary consequence of (1).

4789. *Proposed by Kurt Mahler and P. M. Cohn, the University, Manchester, England*

Let A be an associative algebra over a field of characteristic zero. If $(x+y)^n = \sum_r \binom{n}{r} x^r y^{n-r}$ for some $n \geqq 2$ and for all x and y in A, is it true that $(x+y)^{n+1} = \sum_r \binom{n+1}{r} x^r y^{n+1-r}$ for all x and y in A?

4790. *Proposed by Leonard Carlitz, Duke University*

Let p be a fixed prime >2 and put $\psi(x)=(x/p)$, the Legendre symbol. Evaluate the sum

$$N_r(a) = \sum_{x_1,\cdots,x_r=1}^{p-1} \psi\{x_1 \cdots x_r(a - x_1 - \cdots - x_r)\}.$$

4791. *Proposed by R. C. Lyness, Preston, England*

If α, β, γ are real and $\alpha^3+\beta^3+\gamma^3=0$, prove

$$\left[\sum (\beta-\gamma)^2\right]\left[\sum \alpha^4\right] \geqq \left[\sum \alpha^2\right]^3.$$

4792. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research, Lawrence, Mass.*

Show that the following operators are reducible:

(1) $\qquad x^n D^{2n}, \qquad\qquad$ (2) $\qquad x^{2n} D^n,$

and thus solve the differential equations

(1a) $\qquad [x^n D^{2n} - \lambda]y = 0, \qquad\qquad$ (2a) $\qquad [x^{2n} D^n - \lambda]y = 0.$

SOLUTIONS

Editorial Note. A. Makowski of Warsaw, Poland, has interested himself in the list of unsolved

MONTHLY Problems in the recent *Otto Dunkel Memorial Problem Book* (August–September 1957, Part II). We are indebted to him for the following comments and solutions. Similar assistance from other readers will be most welcome.

2798 [1919, 458]. *Proposed by S. A. Corey.* Prove that every positive integer is equal to the sum of at most four squares.

Comment by Andrzej Makowski, Warsaw, Poland. This is a well-known theorem of Lagrange (or Bachet). See, *e.g.*, Nagell, *Introduction to Number Theory*, New York, 1951, pp. 191–195.

2935 [1921, 467]. [Proposed in *Gazeta Matematică*, April 1921]. Find the integral solutions of the equation $x!+1=y^2$.

Comment by Andrzej Makowski, Warsaw, Poland. This is equivalent to Problem E 534 [1943, 261; 1950, 557; 1951, 193].

3057 [1924, 101]. *Proposed by A. A. Bennett.* Euler in a letter to Stirling stated (without any hint as to the method of proof) that $\pi/4$ might be represented as an infinite product where the nth factor is the quotient of the nth odd prime when divided by the integral multiple of 4 nearest to it. Prove that

$$\frac{\pi}{4}=\frac{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdots}{4\cdot 4\cdot 8\cdot 12\cdot 12\cdot 16\cdots}.$$

Comment by Andrzej Makowski, Warsaw, Poland. This result was proved by E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 2nd ed., 1953, Bd. 1, s. 449.

3133 [1925, 261]. *Proposed by A. A. Bennett.* Show that in every set containing but a finite number of elements and admitting an associative rule of multiplication, there must be an idempotent element, that is, one which is equal to its square. Show that this does not continue to hold if the set contains an infinite number of elements.

Solution by Andrzej Makowski, Warsaw, Poland. Let A be a set with the given properties, and let a be an element of A. By the associative rule, for every choice of the positive integers i, j, the equality $a^ia^j=a^{i+j}$ holds. Since A is finite, there exist positive integers n, k such that $a^{n+k}=a^n$. From this, we obtain $a^{n+sk}=a^n$ $(s=1, 2, \cdots)$, whence for $s=2n$, $a^{n(1+2k)}=a^n$. Finally, upon multiplication of both members by $a^{(2k-1)n}$, we have $(a^{2kn})^2=a^{2kn}$, so that a^{2kn} is an idempotent element.

If A is the set of integers $\geqq 2$ and multiplication is the ordinary multiplication of integers, then an idempotent element does not exist.

3951 [1940, 182]. *Proposed by V. V. Johnston.* Does the equation $m^3+3m^2+2m=2n^3+3n^2+n$ admit positive integral solutions in m and n other than $m=n=1$?

Comment by Andrzej Makowski, Warsaw, Poland. E. B. Escott (*L'Inter-*

médiaire des Math., vol. 8, 1901, p. 249) asked if the equation $n(n+1)(n+2) = m(m+1)(2m+1)$ has integral solutions.

Equations $m^3+3m^2+2m=2n^3+3n^2+n$ and $(m+1)^3-n^3-(n+1)^3=m-2n$ are equivalent. The last equation has no integral solution with $1<m<3164$, $1<n<3164$, $|2m-n| \leqq 100$. This follows immediately upon inspection of the tables of solutions of the equation $x^3+y^3+z^3=n$ which are given by Miller and Woollett (*J. London Math. Soc.*, vol. 30, 1955, pp. 101–110).

Vector-matrix Differential Equation, a Stability Theorem

4745 [1957, 437]. *Proposed by Marvin Marcus, National Bureau of Standards and the University of British Columbia*

Let $A(t)$ be a complex n-square matrix function, continuous on $[0, \infty)$. Let A^* be the conjugate transpose of A and let $X(t)$ be the real trace of $A(t)$. Assume that

$$\text{(i)} \quad A(t) + A^*(t) \geqq 0, \qquad 0 \leqq t < \infty; \qquad \text{(ii)} \limsup_{t\to\infty} \int_0^t X(s)ds < \infty.$$

Show that all solutions of the linear vector-matrix differential equation $\dot{y}=A(t)y$ are bounded on $[0, \infty)$.

Solution by Leopold Flatto, Reeves Instrument Corporation, New York City. $A+A^* \geqq 0$ is assumed to mean that the characteristic roots of $(A+A^*)$ are nonnegative. Let y be an n-dimensional vector, A an $n\times n$ matrix, and let $|y|^2=|y_1|^2+\cdots+|y_n|^2$. We have $dy/dt=Ay$, $dy^*/dt=y^*A^*$. Now

$$\frac{d(|y|^2)}{dt} = \frac{d(y^*y)}{dt} = y^*\frac{dy}{dt} + \frac{dy^*}{dt}y = y^*Ay + y^*A^*y = y^*(A + A^*)y.$$

Since $A(t)+A^*(t)$ is Hermitian, there exists a unitary matrix $U(t)$ which diagonalizes $A+A^*$. Let $y=Uz$; then

$$y^*(A + A^*)y = z^*U^*(A + A^*)Uz = \sum_{i=1}^n \lambda_i |z_i|^2,$$

where the λ_i are characteristic values of $A+A^*$. Hence

$$\frac{d(|y|^2)}{dt} = \sum_{i=1}^n \lambda_i |z_i|^2 \leqq \sum_{i=1}^n \lambda_i(|z_1|^2 + \cdots + |z_n|^2).$$

(The last inequality follows from the assumption $\lambda_i \geqq 0$), so that

$$\text{(1)} \qquad \frac{d(|y|^2)}{dt} \leqq \left(\sum_{i=1}^n \lambda_i\right)\sum_{i=1}^n |z_i|^2 = \text{trace } (A + A^*)|y|^2$$
$$= (\text{tr } A + \text{tr } A^*)|y|^2 = 2 \text{ Re } [\text{tr } A]|y|^2 = 2X(t)|y|^2.$$

Let y be a nontrivial solution of $dy/dt=Ay$, so that $|y|^2$ is never 0. Hence we

get from (1),

$$\frac{1}{|y^2|}\frac{d|y|^2}{dt} \leqq 2X(t). \tag{2}$$

Integrating (2) from 0 to t gives

$$\log\frac{|y(t)|^2}{|y(0)|^2} \leqq 2\int_0^t X(t)dt$$

or

$$|y(t)|^2 \leqq |y(0)|^2 \exp 2\int_0^t X(t)dt \leqq |y(0)|^2 \exp 2 \limsup_{t\to\infty} \int_0^t X(t)dt,$$

which implies that $y(t)$ is bounded over $[0, \infty)$ provided hypothesis (ii) holds.

Also solved by the proposer.

Number of Solutions of a Congruence System

4746 [1957, 437]. *Proposed by Leonard Carlitz, Duke University*

Let p be an odd prime. Show that the number of solutions of the system

$$x + y + z \equiv 3, \qquad xyz \equiv 1 \pmod{p} \tag{1}$$

is equal to $p-2-(-3/p)$, where $(-3/p)$ denotes the quadratic character of -3. If $p>3$, show that the number of solutions is the same for the system

$$x + y + z \equiv 3, \qquad x^3 + y^3 + z^3 \equiv 3 \pmod{p}. \tag{2}$$

Solution by W. J. Blundon, Memorial University of Newfoundland. We have

$$(x - y)^2 = (x + y)^2 - 4xy \equiv (3 - z)^2 - 4/z \equiv (z - 1)^2(z - 4)/z \pmod{p}.$$

If $x\equiv y \pmod{p}$, then $z\equiv 1$ or 4. This leads to exactly two solutions, namely $(1, 1, 1)$ and $((p-1)/2, (p-1)/2, 4)$.

If $x\not\equiv y \pmod{p}$, then $(z-4)/z$ is a quadratic residue modulo p. Let $(z-4)/z \equiv n^2$, where $n^2\not\equiv 0, 1, -3 \pmod{p}$. Thus z assumes $\frac{1}{2}(p-1)-2$ or $\frac{1}{2}(p-1)-1$ distinct values according as -3 is or is not a quadratic residue modulo p. It follows that the number of solutions with $x\not\equiv y$ is double this number, namely $p-4-(-3/p)$. The total number of solutions of this system is thus $p-2-(-3/p)$.

If we apply the transformation $2x=Y+Z$, $2y=Z+X$, $2z=X+Y$ to the system (1) we obtain

$$\begin{aligned} X + Y + Z &\equiv x + y + z \equiv 3 \pmod{p}, \\ X^3 + Y^3 + Z^3 &\equiv (X + Y + Z)^3 - 3(Y + Z)(Z + X)(X + Y) \\ &\equiv (x + y + z)^3 - 24xyz \equiv 3 \pmod{p}. \end{aligned}$$

Since the transformation is linear with nonzero determinant, the number of

solutions is the same for both systems. The condition $p>3$ excludes the exceptional case $(X, Y, Z)\equiv(0, 0, 0)$ which is a solution of the second system and yet is the transform of $(x, y, z)\equiv(0, 0, 0)$ which is not a solution of the first.

It is interesting to note that, upon replacing congruence modulo p by equality in the above, we may use the substitution $(z-4)/z=m^2/n^2$ to obtain an infinity of rational solutions for the Diophantine equation

$$X^3 + Y^3 + Z^3 = 3,$$

and hence an infinity of integral solutions of

$$a^3 + b^3 + c^3 = 3d^3,$$

namely

$$a = 4n^3 - 3mn^2 - m^3, \qquad b = 4n^3 + 3mn^2 + m^3,$$
$$c = -5n^3 - 3m^2n, \qquad d = n^3 - m^2n.$$

Also solved by Emil Grosswald, Emma Lehmer, D. C. B. Marsh, and the proposer. Late solution by Yoshio Matsuoka.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The World of Mathematics. Edited with commentaries and notes by James R. Newman. Simon and Schuster, New York, 1956. 4 volumes. 2536 pp. $20.00.

In recent months it has not been possible for the literate American to remain unaware of this anthology of James Newman's favorite essays on mathematical and scientific topics which has reached the book dividend stage as a gift offering with membership in the Library of Science. Mr. Newman's desire to share these literary treasures with the general reader deserves applause. For no more convenient, no more delightful, no less painful means of acquainting onself with some of the basic ideas of the mathematical-scientific elements in our culture is to be found in our scientific literature.

The collection will yield but little gain to the reader who seeks exclusively the exposition of specific mathematical techniques or the reprinting of the great landmarks in the development of the subject since few technical papers of renowned mathematicians or scientists have been included. The emphasis is on the *opinions* of other renowned mathematicians, scientists, logicians, philosophers and historians of science, and on the importance of the great mathematical and scientific landmarks. The anthology can be of great value to one who not only believes with Norman Campbell that "the only way to understand what

Einstein did is to look at the symbols in which his theory must ultimately be expressed and to realize that it was reasons of symbolic form, and such reasons alone, which led him to arrange the symbols in the way he did and no other" (p. 1829), but who also wishes to become acquainted with authoritative critical thought on the foundations, development, and philosophy of mathematics, on the applicability of its methods and conclusions to problems in the social, physical, and natural sciences, and on its essential character as a creative art.

Inasmuch as Mr. Newman has drawn on masses of diverse material to accomplish his several purposes, the cover-to-cover reader will be aware of repetitive themes that weave, each in its own way, threads of continuity throughout the anthology. Mr. Newman has done little in his editing to bind the whole together in terms of any one theme, but the reader who is biased in his preference for subject matter will quite naturally use that interest as the element of linkage.

The reader, for example, who shares the author's intellectual sensitivity to the intrinsic beauty of mathematical form, will highlight John Sullivan's statement that "The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds. . . . We are the law-givers of the universe; it is even possible that we can experience nothing but what we have created, and that the greatest of our mathematical creations is the material universe itself" (p. 2021). He will relish Sylvester's oft-quoted words: "May not Music be described as the Mathematic of sense, Mathematic as Music of the reason?" (p. 364). He will find pleasure in the opening sentence of Section 10 in Hardy's famous essay: "A mathematician, like a painter or a poet, is a maker of patterns" (p. 2027).

An anthologist must be granted the freedom to select the material that he feels is pertinent to his purpose and to ride his "prejudices," as Mr. Newman might express it. This anthologist has wished to include "material to suit every taste and capacity" and has offered enough material in each category to leave an indelible impression.

The general reader would have benefited still more by the inclusion of editorial criticism in the editorial commentaries which have been interspersed throughout the four volumes. These "settings" are mostly descriptive and biographical in character and often become little gems of characterization. But controversial viewpoints quite naturally appear in the essays and some guidance in evaluating the differing opinions seems necessary for the untutored reader.

There remain a few general observations. The dozens of quotations introducing the essays have been chosen for their aptness with great skill. The text seems singularly free of typographical errors. The two columns in the description of the Egyptian problems on p. 177 seem to be arranged in reverse order.

There is so much of general value in this "small library of the literature of mathematics"—there are 133 items in the collection—that it cannot be disregarded by the mathematics teacher on any level. Students in the various scien-

tific disciplines have already been dipping into it as an easily accessible and rewarding source of reference material. Perhaps the collection was not originally compiled "with the hope of encouraging the study of science" as Campbell declares of his own work on p. 1796, but it may well come to serve that worthy purpose. Reflecting colorful shafts of light from the mathematical firmament, *The World of Mathematics* is recommended for the pleasure and information it has to offer and should be shelved within easy reaching distance of every mathematical work-desk.

CAROLYN EISELE
Hunter College

The Computing Laboratory in the University. Edited by Preston C. Hammer. University of Wisconsin Press, Madison, 1957. xv+236 pp. $6.50.

This book is a compilation of papers presented at a conference with the same title as the book, held at Madison in August, 1955. The scope of the papers is very broad. Some discuss the philosophical and sociological impact of large, high-speed computers on society. Others give surveys of computing applications in specific scientific and engineering areas, such as weather prediction, fluid dynamics, chemistry, physics, and in the aircraft industry. Most of the papers are on the subjects of equipping, organizing, and financing a university computing laboratory, curricula in computing, and the acute shortage of personnel, unanimously agreed to get worse, in computational mathematics. The number of universities with modern computing equipment has more than doubled in the two years since this conference. It is unfortunate that these valuable papers were not available earlier, but since the same problems of organization, whether to build, buy, borrow or rent, and how to raise money without running a service bureau, are facing dozens of schools today, the book is still timely. These papers, by distinguished administrators and scientists in universities and in industry, should be read by most academic administrators as background for deciding how their schools can use the expensive, but probably the most versatile research tool yet devised—the digital computer.

WILLIAM VIAVANT
University of Oklahoma

BRIEF MENTION

Numerical Calculus. By William Edmund Milne. Princeton University Press, 1954. x+393 pp. $4.50.

The current interest in computing machines, numerical analysis, approximations, interpolation, finite differences, numerical integration, and curve fitting is sufficient to merit calling the attention of our readers to a book on the above subjects which is noted for its clarity, even though that book was first published in 1949.

Mathematics of Investment. By William L. Hart. Heath, New York, 1958. viii+494 pp. $6.75.

The fourth edition of Hart's well-known *Mathematics of Investment* needs no review other than to announce its availability in a highly legible new format. The book is designed to "meet the needs and the ability of the typical student in a college of business administration."

Calculus. By Edward S. Smith, Meyer Salkover and Howard K. Justice. Wiley, New York, 1958. xiii+520 pp. $6.50.

Another old friend in a new edition. The inclusion of chapters in solid analytic geometry, and elementary differential equations will be welcomed by many potential users.

From Zero to Infinity. By Constance Reid. Crowell, New York, 1958. 145 pp. $3.00.

Not a book on number theory, but a delightfully written light little book on the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. It can be read by anyone and enjoyed by most.

Van Nostrand's Scientific Encyclopedia (3rd Ed.). Van Nostrand, New York, 1958. vii+1839 pp. $30.00.

This beautifully illustrated one-volume encyclopedia contains a surprising amount of knowledge. Specialists may be somewhat disappointed to note that the discussion of topology is restricted to rubber sheet geometry rather than point set topology. One may question the desirability of including formulas for differentiation under the integral sign in a special paragraph of the same title without stating the conditions which must be satisfied. Still, it is picayune to criticize. A specialist should not expect a complete treatment of his pet subject in a one-volume encyclopedia. This welcome volume is superior to anything similar that this reviewer has seen.

How to Study. By Clifford T. Morgan and James Deese. McGraw-Hill, New York, 1957. vii+136 pp. $1.50.

Teachers may wish to call the attention of their students to this interesting little book. It contains many excellent hints and suggestions on the gentle art of study. Unfortunately, the section devoted to mathematics is one of the poorest in the book. The examples seem to have been taken largely from the sub-high school level rather than the college level. Nevertheless, teachers of mathematics may wish to recommend it to their students in the hope that the general organization which it suggests may help the student arrange his schedule so that there is sufficient time to study mathematics; and to suggest, also, the excellent "How to Study, How to Solve" by Dadourian (Addison-Wesley) for helpful hints on actual mathematical problems.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Lloyd J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1958.

Columbia University, July 7 to August 15: Professor Stein, introduction to linear algebra, fundamental concepts of mathematics; Professor Mendelson, differential equations; Mr. Gordon, probability; Professor Griffin, functions of a complex variable; Dr. Taft, higher algebra; Professor Taylor, general topology, theory of group representations.

Fordham University. July 7 to August 14: Summer Institute for high school teachers of mathematics offering courses in probability and statistical inference and basic concepts and structures of geometry; in the Graduate School, differential equations, integral equations, and foundations of analysis.

Massachusetts Institute of Technology. July 8 to July 18: Probability theory and its application to operations research, under the direction of Professor George P. Wadsworth assisted by Dr. James K. Knowles with lectures by Philip M. Morse, Professor of Physics, and Dr. Herbert P. Galliher, Operations Research Project. July 21 to August 1: Mathematical analysis and simulation in industrial operations, sponsored by the School of Industrial Management with the cooperation of the Operations Research Project and other departments, under the direction of Dr. Herbert P. Galliher, Operations Research Project; other participants include George E. Kimball, Visiting Professor of Chemistry, Edward H. Bowman, Robert B. Fetter, Dean Arden, Gregory Chow, William S. Jewell, Richard B. Maffei. August 4 to August 15: Design and analysis of scientific experiments, Professor Harold Freeman and Dr. Marian Krzyzaniak.

St. John's University, June 30 to August 8: Professor Tolle, foundations of mathematics, solid analytic geometry, material and methods of modern mathematics.

University of California, Berkeley, June 16 to July 26: Visiting Professor Besicovitch, almost periodic functions. June 16 to July 26 and July 28 to September 6: Professors Neyman, Fix, and Hodges, research seminar in statistical problems of health; Dr. David (University College, London), research seminar in statistical studies of structural relations in physical sciences; Professor Blackwell, individual research leading to higher degrees.

University of Minnesota, Institute of Technology, June 16 to July 19: Professor Wilcox, vector analysis; Professor Pope, numerical analysis, introduction to programming modern digital calculators; Visiting Professor Roy, elements of statistical inference—theory, elements of statistical inference—applications. July 21 to August 23: Professor Stenberg, advanced calculus.

A COMPUTING CENTER AT SOUTHERN METHODIST UNIVERSITY

Southern Methodist University announces the opening of a Computing Laboratory on its campus. A new building houses the Univac Scientific 1103 Computer, the Remington Rand Service Bureau and the SMU Computing Laboratory offices and classrooms. The computer is operated jointly by Remington Rand as a service to industry and by SMU as an academic service for research and teaching. The SMU operation is associated

with the University's new Graduate Research Center. Professors and students have free use of the machine for academic research and training in computer work. Training programs are available for faculty and students. Computing projects are now under way in the fields of engineering, mathematics, psychology, law, religion, management and others. SMU will make the computer available to other universities and nonprofit institutions on a cooperative arrangement involving only a nominal fee for overhead, and invites inquiries leading to such use of the machine. SMU regards its laboratory as a regional university computing facility.

NATIONAL SCIENCE FOUNDATION SUMMER INSTITUTES

The following are additions to the list (this MONTHLY, vol. 65, 1958, p. 303) of Summer Institutes for high school teachers:

Fordham University. The institute for high school teachers, announced on page 378 of this issue, is sponsored by the National Science Foundation. Alternates selected by other NSF Summer Institutes may apply by May 24, 1958.

University of Missouri, School of Mines and Metallurgy, Rolla, June 9–August 1: H. Q. Fuller, Director.

RUSSIAN TECHNICAL JOURNAL TRANSLATION

A leading Russian technical journal will soon be translated into English on a regular basis by the American Society of Mechanical Engineers. Under a $35,000 grant from the National Science Foundation, ASME will publish the bimonthly "Journal of Applied Mathematics and Mechanics." The Society has undertaken the task of translation "in an attempt to correct the present situation in which the Russians are familiar with the content of most, if not all, of our technical publications, while only a few of theirs are translated for use by the English-speaking world."

The magazine contains the latest theoretical and practical advances made by Russian scientists in mathematics, fluid dynamics and solid state physics. Copies will be sold, on a subscription basis, to any interested persons or groups at an annual rate of $35 for the six issues. Subscriptions may be ordered from the Order Department, The American Society of Mechanical Engineers, 29 West 39th Street, New York 18, New York. Publication begins with the first 1958 issue.

Professor George Herrmann of Columbia University will serve as editor of the translated journal.

PRELIMINARY ACTUARIAL EXAMINATIONS

For a number of years Educational Testing Service, the organization that assists the Society of Actuaries in the preparation of the Preliminary Actuarial Examinations, has participated in a program of Graduate Record Examinations given at a large number of schools in the United States and Canada. The tests are used as a measure of student achievement and ability to do graduate work, and are currently required of applicants for National Science Foundation fellowships.

The Graduate Record Examinations consist of an Aptitude Test that includes a verbal section similar to the Society's Part 1 and several Advanced Tests, including one in mathematics similar to Part 2. They are scheduled four times a year, in January, April, July, and November. Detailed information regarding the nature of these examinations, the centers where they are given, *etc.*, may be obtained by writing Educational Testing Service, 20 Nassau Street, Princeton, New Jersey, for a copy of the Bulletin of Information for Candidates.

The Board of Governors of the Society of Actuaries has approved the recommendation of the Education and Examination Committee that credit for Part 1 of the Preliminary Actuarial Examinations be allowed to candidates who take the Graduate Record Aptitude Test and who achieve a score on the verbal section determined by the Educa-

tion and Examination Committee to be equivalent to the Part 1 passing score. Similarly, credit for Part 2 will be allowed to candidates who take the Graduate Record Advanced Test in Mathematics. In either case, to be eligible for credit a candidate must have taken the Graduate Record Examinations while an undergraduate or graduate student in a college or university. An application to the Society for credit may be completed either in advance of taking the Graduate Record Examinations, or within two years of having taken them as an undergraduate or graduate student.

In the future, the Preliminary Actuarial Examination pass list published by the Society will include the names of students who have received credit for Graduate Record Examinations. They will also be listed appropriately in the Year Book of the Society.

PERSONAL ITEMS

Professor S. W. McCuskey of Case Institute served as the representative of the Association at a meeting of the Council on Documentation Research held at Western Reserve University on February 3 and 4, 1958.

Professor Abe Gelbart of Syracuse University has been appointed Editor of *Scripta Mathematica* and Director of Yeshiva University's Institute of Mathematics. He will succeed the late Professor Jekuthiel Ginsburg and will assume his new duties July 1, 1958.

Associate Professor A. R. Harvey of San Diego State College has received a Fulbright award to lecture at the College of Arts and Sciences in Baghdad, Iraq.

Mr. R. C. Jurgensen, Chairman of the Mathematics Department at Culver Military Academy, has been appointed to hold the first endowed chair in mathematics in the school's history.

Professor Emeritus Solomon Lefschetz of Princeton University has been elected a corresponding Member of the Academie des Sciences of Paris.

Catholic University of America: Assistant Professor R. W. Moller has been appointed Associate Professor; Dr. P. P. Saworotnow has been appointed Assistant Professor.

Lebanon Valley College: Mr. G. F. Heck has been elected the first president of the newly-organized Davis Chapter of the Industrial Mathematics Society. Other officers elected were: Mr. Earl Edris, vice president; Mr. E. A. Anderson, secretary; Mr. John Ray, treasurer; and Professor R. J. Wagner, faculty advisor.

Lincoln Memorial University sponsored its First Annual Symposium for the Improvement of Science and Mathematics Teachers of Secondary Schools on October 18 and 19, 1957. The program included an invited address by Professor F. L. Wren, George Peabody College, on *The teaching of mathematics in secondary schools.* The symposium will be repeated in October, 1958.

University of New Hampshire: Dr. S. L. Ross and Mr. F. J. Lorenzen, Jr. have been appointed Assistant Professors.

Mr. W. R. Abel, University of Nebraska, has been appointed Assistant Professor.

Dr. Brian Abrahamson of the University of Cape Town has been appointed Professor at Rhodes University.

Assistant Professor J. J. Andrews, St. Louis University has been appointed Associate Professor.

Professor M. G. Boyce, Vanderbilt University, on leave this semester, is a Visiting Fellow at Princeton University.

Mr. D. A. Breault, Sylvania Electric Products Co., Waltham, Mass., is on leave of absence with the U. S. Army Signal Corps.

Mr. H. H. Brown, Ramo-Wooldridge Corporation, has accepted a position as research mathematician for the Missile Systems Division of the Lockheed Aircraft Corporation, Palo Alto, California.

Miss Winifred K. Burroughs, Ohio Wesleyan University, has been appointed Instructor.

Dr. D. L. Clark, Oregon State College, has accepted a position as a member of the technical staff of Bell Telephone Laboratories, New York, N. Y.

Mr. J. G. Cox, formerly Head, Math. Services Dept., VITRO Corp., Eglin Air Force Base, Florida has been appointed Associate Professor at Alabama Polytechnic Institute.

Assistant Professor H. F. Cullen, University of Massachusetts, has been appointed Associate Professor.

Assistant Professor M. R. Demers, University of Nevada, has been appointed Associate Professor.

Professor Benjamin Epstein of Wayne State University is on leave and has been appointed Visiting Professor at Stanford University.

Dr. J. D. Esary of the University of California, Berkeley, has accepted a position as research engineer for the Boeing Airplane Company, Seattle, Washington.

Dr. Jean B. Feidner, University of Buffalo, has accepted an appointment as Assistant Professor at Hobart and William Smith Colleges.

Professor R. N. Festa of the Ministry of Education, Vienna, Austria, is on leave and has been appointed Visiting Professor at the State College of Washington.

Professor A. H. Fox, Union College, has been appointed Chairman, Department of Mathematics.

Professor J. E. Freund, Virginia Polytechnic Institute, has been appointed Professor at Arizona State College, Tempe.

Dr. R. D. Glauz of the General Electric Company has accepted a position as technical specialist for Aerojet-General Corporation, Sacramento, California.

Dr. Carl Hammer of Remington-Rand International Corporation has accepted a position as engineering specialist for Sylvania Electric Products, Inc., Waltham, Massachusetts.

Assistant Professor Aaron Herschfeld of Canisius College has been appointed Assistant Professor at Pennsylvania State University.

Dr. B. M. Ingersoll, City College of New York, has been appointed Assistant Professor at San Diego State College.

Professor J. B. Jackson, University of South Carolina, has been appointed Visiting Professor at Mary Washington College.

Assistant Professor H. T. Jones, Emmanuel Missionary College, has been appointed Associate Professor.

Assistant Professor R. J. Kasriel, Georgia Institute of Technology, has been appointed Associate Professor.

Assistant Professor L. H. Lange of Valparaiso University, as the recipient of a Danforth Foundation Teacher Study Grant, continues on leave at the University of Notre Dame.

Rev. C. J. Lewis, Fordham University, has been appointed Assistant Professor.

Assistant Professor R. W. MacDowell, University of Rochester, has been appointed Associate Professor at Antioch College.

Mrs. Ruth J. MacKichan, University of North Dakota, has been appointed Assistant Professor.

Dr. M. E. Mahowald, formerly with the General Electric Co., Cincinnati, Ohio, has been appointed Assistant Professor at Xavier University.

Colonel J. D. Matheson of the United States Army has been appointed a senior research analyst for Melpar, Inc., Falls Church, Virginia.

Mr. H. W. McCurdy, Graduate Student, Florida State University, has been appointed Instructor at Bradley University.

Dr. M. A. Medick, City College, has accepted a position as senior staff scientist at the Research and Advanced Development Division of AVCO Manufacturing Corporation, Lawrence, Massachusetts.

Assistant Professor Paul Meier, Johns Hopkins University, has been appointed Associate Professor at the University of Chicago.

Mr. V. H. Morrill, Tarleton State College, has been appointed Assistant Professor.

Dr. J. A. Nickel, Willamette University, has been appointed Assistant Professor.

Mr. Enuenwemba Obi, teaching fellow at the University of Kansas, has been appointed part-time Instructor at Park College.

Dr. J. B. O'Toole of Hughes Aircraft Company has accepted a position as senior engineer for National Cash Register Company, Hawthorne, California.

Dr. P. A. Penzo, University of Pittsburgh, has accepted a position as senior research engineer for CONVAIR Astronautics, San Diego, California.

Assistant Professor W. J. Pervin, University of Pittsburgh, has been appointed Assistant Professor at Pennsylvania State University.

Mr. C. F. Pinzka, Xavier University, has been appointed Instructor at the University of Cincinnati.

Assistant Professor L. E. Pursell, Grinnell College, has been appointed Associate Professor.

Associate Professor Gustave Rabson, Antioch College, has accepted a position as senior mathematician in the Research Center of American Optical Company, Southbridge, Massachusetts.

Assistant Professor J. B. Roberts, Reed College, has been appointed Associate Professor.

Associate Professor F. V. Rohde, University of Florida, has been appointed Professor at the University of Chattanooga.

Dr. D. W. Sasser, Yale University, has accepted a position as staff member of Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor Seymour Schuster, Polytechnic Institute of Brooklyn, has been appointed Associate Professor.

Associate Professor L. L. Scott, University of Mississippi, has been appointed Associate Professor at Southwestern at Memphis.

Professor M. E. Shanks, Purdue University, is on leave and is at the Institute for Advanced Study.

Dr. Abraham Spitzbart of the General Electric Company, Cincinnati, Ohio has been appointed Associate Professor at the University of Wisconsin in Milwaukee.

Dr. T. D. Sterling, University of Michigan Engineering Research Institute, has been appointed Assistant Professor at Michigan State University.

Associate Professor E. A. Sturley of Allegheny College has been appointed Associate Professor at the University of Southern Illinois.

Assistant Professor Patrick Suppes, Stanford University, has been appointed Associate Professor.

Mr. R. E. Thomas of North American Aviation, Inc., has accepted a position as assistant division consultant for Batelle Memorial Institute, Columbus, Ohio.

Associate Professor Henry Van Engen of Iowa State Teachers College has been appointed Professor at the University of Wisconsin.

Professor C. P. Wells, Michigan State University, is on leave and has been appointed a research associate at the California Institute of Technology.

Dr. F. J. Weyl, formerly Director of the Mathematical Science Division, Office of Naval Research, has been appointed Director, Naval Analysis Group, Office of Naval Research.

Professor S. S. Wilks, Princeton University, has been appointed Chairman of the Mathematics Division of the National Academy of Sciences-National Research Council.

Professor J. C. Wilson of Fenn College has been awarded a Danforth Teacher Study Grant for 1958–59 and will study toward his Ph.D. in mathematics at Case Institute of Technology.

Dr. E. M. Zaustinsky, University of Southern California, has been appointed Assistant Professor at San Jose State College.

Professor Israel Abrams, Drexel Institute of Technology, died January 4, 1958.

Mr. Nathan Barotz, Brooklyn College, died on August 16, 1957.

Professor Emeritus R. M. Deming, retired, Upper Iowa University, died January 13, 1958. He was a charter member of the Association.

Professor Carl M. Erikson, Eastern Michigan College, died on February 10, 1958. He had been a member of the Association for thirty years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1957

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1957. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

There was a surplus of $1,876 in the Current Fund for 1957. The balances in the regular funds of the Association are less because of the decrease in the value of the securities owned by the Association and because of printing expenses in some cases.

Assets of the Association	January 1, 1957	December 31, 1957
M & T Trust Co., Buffalo	30,857.02	25,703.17
Savings Accounts	19,297.28	75,999.51
Securities	111,828.64	100,974.12
	$161,982.94	$202,676.80
Funds of the Association		
Current Fund	$ 450.03	$ 325.80
Carus Fund	23,763.26	20,235.25
Chace Fund	12,369.13	8,808.28
Houck Fund	12,082.88	11,309.92
Chauvenet Fund	1,420.06	1,328.43
Dunkel Fund	19,917.92	15,889.55
General Fund	40,518.72	38,414.63
	$110,522.00	$ 93,311.66
Visiting Lecturers Fund	$ 42,942.44	$ 48,424.53
Fund for Committee on Undergraduate Program	8,069.77	55,669.95
Fund for Committee on Films	448.73	85.66
Fund for Committee on High School Contests	—	2,184.80
	$161,982.94	$202,676.80

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 175 persons have been elected to membership by the Board of Governors on applications duly certified.

IAIN T. ADAMSON, Ph.D. (Princeton) Lecturer, Queen's University of Belfast, Northern Ireland.

JULIA E. ADKINS, Ph.D. (Ohio State) Asst. Professor, Central Michigan College.

RALPH J. ANDREE, B.S. (Oklahoma S.U.) Student, Oklahoma State University.

MARY F. BAEURLE, M.A. (Columbia) Teacher, Central High School, Paterson, N. J.

ALEXANDER R. BEDNAREK, B.S. (New York S.T.C.) Teaching Fellow, University of Buffalo.

JOHN J. BENEDETTO, Student, Boston College.

RONALD J. BENICE, Student, University of Buffalo.

RICHARD J. BETTS, Student, Rice Institute.

CLAIR W. BLACK, Ed.D. (Columbia) Dean, School of Eng. & Sc., Fairleigh Dickinson University.

CLIFFORD D. BLACK, B.S. (Jacksonville S. C.) Teaching Fellow, University of Kentucky.

THOMAS BONNER, M.S. (Illinois) Grad. Asst., University of Illinois.

MRS. FLORENCE W. BORGESON, B.S. (Delaware) Teacher, Roosevelt Junior High, Westfield, N. J.

DUSHAN BOYANOVITCH, M.E. (Stevens) Grad. Asst., Stevens Institute of Technology.

MILTON BROOKS, M.A. (Pennsylvania) Head, Dept. of Math., West Philadelphia High School.

MICHAEL BUCKLEY, JR., M.S. (Purdue) Asst. Professor, Santa Clara University.

THOMAS W. CAIRNS, M.S. (Oklahoma S.U.) Grad. Asst., Oklahoma State University.

ALESSANDRO C. CALIANESE, B.S. (St. Peter's) Grad. Stud., University of Notre Dame.

JAYME M. CARDOSO, D.Sc. (Universidade do Parana) Professor, Universidade do Parana, Brazil.

MRS. BEVERLY N. CARRELL, M.A. (George Washington) Teacher, Woodrow Wilson High School, Washington, D. C.

PHILIP W. CARRUTH, Ph.D. (Illinois) Asso. Professor, Swarthmore College.

ALAN P. CLARK, M.A. (Columbia) Teacher, Parsippany High School, Parsippany, N. J.

ANDREW CONTI, JR., Student, Youngstown University.

HENRY C. COOKE, M.S. (N. Carolina S. C.) Asst. Professor, North Carolina State College.

DONALD T. COTTINGHAM, B.S. (Harvard) Teaching Asst., University of Washington.

THADDEUS F. CROMWICK, B.S. (Worcester Poly. Inst.) Vice President, Telemat Corp., Worcester, Mass.

DAVID R. CROSBY, M.S. (Harvard) Elec. Engr., Radio Corp. of America, Camden, N. J.

DONALD I. CUTLER, B.S. (Illinois) Asst. Math., System Development Corp., Santa Monica, Calif.

DAVID F. DAWSON, Ph.D. (Texas) Asst. Professor, University of Missouri.

JAMES E. DELANY, Student, San Diego State College.

WILLIAM H. DENT, JR., B.A. (Maryville Coll.) Teaching Fellow, University of Kentucky.

MRS. RUTH E. DILLAVOU, M.A. (Columbia) Instr., Gonzaga University.

ROBERT G. DOUGLAS, Student, University of Kansas.

KENNETH B. DUNHAM, Student, Georgia Institute of Technology.

ORVILLE E. ETTER, M.S. (Kansas S.T.C.) Asst. Professor, Fort Hays Kansas State College.

ARMSTRONG O. FARMER, B.S. (Lafayette Coll.) Draftsman, American Bridge Co., Trenton, N. J.

FAY FARNUM, Ph.D. (Cornell) Asst. Professor, University of Arizona.

RAY H. FARRIS, M.Ed. (Abilene) Asst. Professor, York College, Nebraska.

ROBERT C. FISHER, Ph.D. (Kansas) Asst. Professor, Ohio State University.

JOE W. FITZPATRICK, M.A. (Texas) Instr., Trinity University.

LOWELL C. FLETCHER, B.S. (Eastern Ky. S. C.) Teaching Fellow, University of Kentucky.

ROBERT J. FOPMA, A.B. (Hope) Asst. Professor, University of Cincinnati.

DAVID C. FRIED, B.S. (Arizona) Numerical Analyst, General Electric Computer Dept., Phoenix, Ariz.

CHARLES J. FREIFELD, Student, Bronx High School of Science.

MARSHALL L. FREIMER, M.A. (Harvard) Staff Member, Lincoln Laboratory (MIT).

FRED GALVIN, Student, University of Minnesota.

STEWART GAMMILL, III, B.S. (Millsaps) Grad. Asst., University of Mississippi.

PHYLLIS R. GANSZ, M.A. (Columbia) Teacher, Baldwin School, Bryn Mawr, Pa.

HARRY S. GAPLES, JR., B.A. (Minnesota) Teach. Asst., University of Minnesota.

MRS. MARTHA J. GARREN, B.A. (North Carolina) Instr., North Carolina State College.

WARREN B. GARRISON, M.S. (Tulsa) Asst. Professor, University of Tulsa.

FREDERICK R. GEORGIA, Ph.D. (Cornell) Supervisor of Water Works, Cornell University.

THOMAS GILMARTIN, M.S. (Harvard) Staff Geologist, Pan-American Petroleum Corp., Fort Worth, Texas.

GERALD GOERTZEL, Ph.D. (N.Y.U.) Technical Director, Nuclear Development Corp. of America, White Plains, N. Y.

RONALD N. GORDON, B.S. (Kentucky) Teaching Fellow, University of Kentucky.

LEON GREENBERG, M.A. (Yale) Res. Asst., Yale University.

CLIFFORD F. GREENE, JR., Student, Rutgers University.

FRANK J. HAHN, M.A. (Illinois) Grad. Asst., University of Illinois.

CARL W. HAMILTON, Student, West Plains High School, West Plains, Missouri.

R. W. HART, M.A. (Illinois) Professor, Kansas State Teachers College, Pittsburg.

JACK F. HECHT, SR., B.S. (West Coast U.) Design Specialist, Lockheed Missile System Div., Sunnyvale, Calif.

RALPH T. HEIMER, M.A. (Penna. State) Instr., Pennsylvania State University.

RICHARD C. HENRY, Student, University of Toronto.

JIMMY R. HICKEY, B.S. (East Texas State) Grad. Student, Baylor University.

AGNES M. HIGGINS, M.A. (Buffalo) Associate in Math., N.Y.S. Dept. of Education, Albany, N. Y.

WILLIAM F. HILL, Ph.D. (Baylor) Professor & Head of Dept., Tarleton State College.

ROBERT G. HOEHN, B.S. (Manchester) Asst. Professor, Indiana Technical College.

MRS. RUTH B. HONEYCUTT, M.A. (Duke) Instr., North Carolina State College.

CATHARINE HOWARD, M.S. (V.P.I.) Instr., University of Akron.

MRS. LUTRECIA A. HUNTER, M.A. (George Peabody) Instr., Armstrong College of Savannah.

ANTOINETTE K. HUSTON, Ph.D. (Chicago) Asst. Professor, Rensselaer Polytechnic Institute.

ZENNIA M. HYDUK, B.S. (Alberta) Lecturer, University of Alberta.

MERL E. ILES, A.B. (Northwest Nazarene Coll.) Instr., Middleton High School, Idaho.

GEORGE S. INNIS, Technical Staff Asst., Defense Research Lab., University of Texas.

JAMES T. JOICHI, M.S. (Illinois) Grad. Asst., University of Illinois.

KARL S. KALMAN, M.A. (Pennsylvania) Head, Dept. of Math., Lincoln High School, Philadelphia, Pa.

DAVID L. KATZER, Student, University of Chicago.

EDGAR P. KELLY, JR., M.S. (Florida S.U.) Grad. Asst., Oklahoma State University.

MARGARET J. KENNEY, B.S. (Boston) Teaching Asst., Boston College.

ROBERT R. KINKADE, B.S. (Northeastern S.C.) Grad. Asst., Oklahoma State University.

FREDERICK L. KNOWLES, B.S. (Memphis S.C.) Engineer, E. I. DuPont de Nemoirs Co., Charlestown, Ind.

DOROTHY I. KOEHLER, A.B. (Woman's Coll., U.N.C.) Teaching Fellow, University of Kentucky.

GEORGE H. KUBY, M.A. (Columbia) Analyst, Digital Computation, Bell Aircraft, Buffalo, N. Y.

DAVID C. LARSON, Student, Texas A. & M. College.

MRS. OLGA LATONI, M.S. (Miami) Coral Gables, Fla.

DEAN LAWRENCE, Student, Baker University.

WILFRED E. LAYMAN, Student, Baker University.

WILLIAM J. LEE, Student, Georgia Institute of Technology.

CONCEPCION LEONOR, M.A. (Univ. of Santo Tomas) Professor, University of Santo Tomas, Manila.

JANET W. LETT, Student, Baylor University.

CHARLES F. LEWIS, M.A. (Peabody) Asst. Professor, North Carolina State College.

RICHARD J. LIBERA, B.A. (American International) Instr., University of Massachusetts.

KATHERINE S. LIPPS, Student, St. Louis University.

PAUL E. LONG, M.A. (Missouri) Grad. Asst., Oklahoma State University.

FRANCIS L. LYNCH, JR., M.A. (Seton Hall) Teacher, Madison High School, Madison, N. J.

CAPT. ROBERT A. MACKERRACHER, M.A. (N. Carolina S.C.) USN (Ret.) North Carolina State College.

MRS. MATILDA MACNAUGHTON, B.A. (Adelphi) Teacher, Winchester-Thurston School, Pittsburgh, Pa.

W. S. MAHAVIER, Ph.D. (Texas) Instr., Illinois Institute of Technology.

ARMSTRONG MALTBIE, B.S. (Vermont) Asst. Professor, North Carolina State College.

HERNAN MATILLA, B.S. (Habana) Surveyman, Camaguey, Cuba.

REV. WILLIAM T. MATYAS, O.S.B., M.S. (Pittsburgh) Instr., Benedictine High School, Cleveland, Ohio.

WALTER H. MCCURDY, JR., M.S. (Florida S.U.) Instr., Bradley University.

CHANDLER L. MCKELVEY, B.S. (Northwestern) Consulting Actuary, George V. Stennes & Associates, Minneapolis, Minn.

JAMES C. MCLAUGHLIN, Student, University of Michigan.

F. L. MCMAINS, JR., B.A. (Oregon) Grad. Asst., University of Arizona.

WILLIAM EUGENE MCMILLEN, B.S. (Indiana S.T.C.) Technical Programmer, Standard Oil Co. of Ohio, Cleveland, Ohio.

BRUCE C. MCQUARRIE, M.A. (New Hampshire) Teacher, Wasatch Academy.

HERBERT AUGUST MEYER, M.A. (Nebraska) Asso. Professor, Concordia Teachers College.

JOHN P. MICHALSKI, M.S. (Northwestern) Student, Alabama Polytechnic Institute.

FRANK E. MILNE, B.Sc. (Dalhousie) Teacher, Queen Elizabeth High School, Halifax, Nova Scotia.

NICHOLAS C. MITROWSIS, M.A. (South Carolina) Instr., Fordham University.

MEHDI S. MOHEBAN, B.A. (Montana S.U.) Grad. Asst., Montana State University.

MAURICE L. MONAHAN, B.S. (S.D.S.C.) Asst., South Dakota State College.

WILLIAM L. MORSE, B.S. (Oregon S.) Teaching Asst., Washington State College.

ELIEZER NADDOR, Ph.D. (Case) Asst. Professor, Johns Hopkins University.

PAUL R. NEUREITER, Ph.D. (Vienna) Professor, Teachers College, Geneseo, N. Y.

MRS. VIRGINIA K. NEWELL, M.A. (N.Y.U.) Chm. of Dept., J. W. Ligon High School, Raleigh, N. C.

KAJ L. NIELSEN, Ph.D. (Illinois) Head, Math. Division, U. S. Naval Avionics Facility, Indianapolis, Ind.

JACK I. NORTHAM, M.A. (Michigan State) Statistician, Upjohn Co., Kalamazoo, Mich.

FRANCIS C. OGLESBY, M.S. (Lehigh) Instr., Lehigh University.

FLOYD R. ORR, Student, University of California.

MRS. CARLOTTA P. PATTON, B.S. (Coll. of Charleston) Instr., North Carolina State College.

DELBERT F. PENHALL, M.S. (So. Calif.) Teacher, Long Beach City Schools, Calif.

BLAISE PERLIC, Student, Carnegie Institute of Technology.

JOHN PETRENKA, JR., Student, Seton Hall University.

MYRNA H. PIKE, B.A. (Cornell) Grad. Asst., University of Illinois.

MRS. BEVERLY A. PODMOLIK, B.Ed. (Chicago T.C.) Grad. Asst., University of Illinois.

WALLACE A. RAAB, M.A. (South Dakota) Asst. Professor, California State Polytechnic College.

GEORGE N. RANEY, Ph.D. (Columbia) Asst. Professor, Pennsylvania State University.

SALVATORE J. RAPISARDA, Sc.D. (Calvin Coolidge) Professor, Merrimack College.

ARTHUR G. REINKE, B.A. (Arizona) Teaching Asst., University of Arizona.

RAINERIO O. REYES, B.S. (Mapua Inst. of Tech.) Grad. Asst., Lehigh University.

EDMOND F. REYNOLDS, A.B. (California) Actuary, Marsh & McLennan-Cosgrove & Co., San Francisco, Calif.

GERALD R. RISING, M.Ed. (Rochester) Instr., Brighton High School, Rochester, N. Y.

BRYAN W. ROBINSON, M.D. (Emory) Strong Memorial Hospital, Rochester, N. Y.

CHARLES H. SAMPSON, Student, University of Kentucky.

MRS. HELGA I. SCHWARTZ, M.S. (S.U. of Iowa) Minneapolis, Minnesota.

ROBERT N. SCHWARTZ, Student, University of Buffalo.

JOHN M. SHERWIN, B.A. (Eastern Wash. Coll. of Ed.) Grad. Student, University of Washington.

ALMA I. SHIPLEY, M.S. (Southern Calif.) Teacher, Southwest High School, Kansas City, Missouri.

ROBERT J. SILVERMAN, Ph.D. (Illinois) Asst. Professor, Illinois Institute of Technology.

SISTER HELEN MARIE FINN, R.S.M., A.M. (St. Louis) Instr., College of St. Mary, Omaha, Nebr.

SISTER MARY J. HELLMANN, M.S. (Catholic) Teacher, St. Francis High School, Morgantown, W. Va.

SISTER M. FRANCIS BORGIA STAUDER, S.S.N.D., Ph.D. (Notre Dame) Professor, Notre Dame College, Missouri.

CHARLES C. SMITH, M.A. (Eastern State) Instr., Sue Bennett College.

LEANDER W. SMITH, B.A. (Trinity) Teacher, Canton High School, Collinsville, Conn.

SYDNEY C. SMITH, Student, University of Kentucky.

SYDNEY STEPHENS, JR., B.S. (Eastern Kentucky S.C.) Inst., Eastern Kentucky State College.

DOUGLAS R. STOCKS, JR., Student, University of Texas.

GLEN R. STREVEY, Student, University of Kansas.

DONALD V. SWARD, B.A. (Montana) Grad. Asst., Montana State University.

RICHARD H. TALLMAN, B.A. (Saskatchewan) Actuary, Northwestern National Life Ins. Co., Minneapolis, Minn.

SELMO TAUBER, Ph.D. (Vienna) Instr., University of Kansas.

ANGEL H. TELLEZ, M.Ed. (Arizona) Instr., University of Arizona.

JOSEPH T. THOMAS, M.S. (Michigan) Student, University of Tennessee.

DANIEL P. THOMPSON, A.B. (Harvard) Res. Asst., Harvard Observatory.

RONALD S. TOCZEK, B.A. (Chicago) Grad. Asst., University of Utah.

JURIO TSUCHIYA, M.S. (Kyoto Imperial U.) Research Aerodynamicist, Mississippi State College.

THEODORE T. TYLASKA, II, Student, University of Texas.

JOSEPH J. VERRONE, Student, Seton Hall University.

THOMAS H. VICKERS, B.A. (Western Reserve U.) Teach. Asst., Rutgers University.

MARK VILLARINO, Student, Notre Dame High, Sherman Oaks, Calif.

CHARLES M. WALKER, B.S. (Western Carolina Coll.) Instr., University of Kentucky.

SARA C. WALSH, M.A. (Buffalo) Teacher, Bennett High School, Buffalo, N. Y.

EDWARD L. WALTERS, M.A. (Syracuse) Chm., Dept. of Math., William Penn Senior High School, York, Pa.

ELVIN S. WARRICK, M.A. (Illinois) Math. Librarian, University of Illinois.

WILLARD H. WATTENBURG, B.S. (Chico S.C.) Instr., Chico State College.

GEORGE K. WILLIAMS, A.B. (Kentucky) Grad. Asst., University of Kentucky.

THOMAS O. WILLIAMS, Student, Rensselaer Polytechnic Institute.

ROBERT W. WILLSON, M.S. (S.U. of Iowa) Instr., Wisconsin State College.

GLENN J. WIMBISH, JR., B.A. (Millsaps) Grad. Student, University of Mississippi.

MRS. MARY T. WOLBIER, B.A. (D'Youville) Teacher, Bishop O'Hern High School, Buffalo, N. Y.

JOHN W. WOLL, JR., Ph.D. (Princeton) Asst. Professor, Lehigh University.

ROBERT A. WONDERLY, B.A. (Emmanuel Missionary Coll.) Teach. Asst., University of Minnesota.

MARILYN J. WOODYARD, M.S. (Northwestern) Programmer, IBM Research Lab., Poughkeepsie, N. Y.

GAIL S. YOUNG, Ph.D. (Texas) Professor, University of Michigan.

THE OCTOBER MEETING OF THE INDIANA SECTION

A joint meeting of the Indiana Section of the Mathematical Association of America and the Mathematics Division of the Indiana Academy of Science was held at DePauw University, Greencastle, Indiana, on October 18, 1957. Professor I. W. Burr, Chairman of the Mathematics Division of the Academy, presided.

There were 69 in attendance including 37 members of the Association.

The following papers were presented:

1. *The mathematics of the future*, by Professor P. D. Edwards and Professor C. F. Brumfiel, Ball State Teachers College.

Professor Edwards stressed the increasing importance of having high school teachers of mathematics who are much more thoroughly trained in advanced mathematics than was considered necessary a relatively few years ago. This applies not only to the traditional topics needed by the engineer and the physical scientist but also to the needs of workers in other fields. It was pointed out that very definite improvements in the teachers' preparation may be made by drastic changes in the content of high school mathematics. Professor Brumfiel made a progress report on an experimental program now being supervised by Ball State which was made possible by a grant from the National Science Foundation.

2. *A characterization of n-adic equivalence relations*, by Professor J. L. Lawrence, Wabash College and International Business Machines, introduced by the Secretary.

The concepts of symmetry, transitivity, composition, and equivalence associated with dyadic relations were generalized to apply to the n-adic case. In the generalized scheme, symmetry and transitivity persist as necessary and sufficient conditions for an n-adic relation to be an equivalence relation. Furthermore, as in the dyadic case, an n-adic relation is transitive if and only if the relation contains the composition of the relation with itself.

3. *The concept of surface integral*, by Mr. L. H. Turner, Purdue University, introduced by the Secretary.

A continuous parametric surface (T, A) is a continuous mapping T from a subset A of E_2 into three space E_3, $p=T(w)$, $w=(u, v)\in A$, $p=(x, y, z)\in E_3$. The usual definition of area in terms of the Jacobians of the mapping is inadequate in modern analysis. An adequate definition was given by Lebesgue in terms of sequences of polyhedral mappings which approach the original mapping. When the area is finite, four measures ϕ, V_1, V_2, V_3 may be defined by means of these sequences on a certain ring of Borel subsets of A such that V_1, V_2, and V_3 are absolutely continuous with respect to ϕ. The Radon-Nikodym derivatives $\theta_1(w)=dV_1/d\phi$, $\theta_2(w)=dV_2/d\phi$, $\theta_3(w)=dV_3/d\phi$ exist and satisfy $\theta_1^2+\theta_2^2+\theta_3^2=1$ a.e. (ϕ). The vector $\theta(w)=(\theta_1, \theta_2, \theta_3)$ may be thought of as the directional normal to the surface (T, A) at $T(w)$. Then if $D=\{(\theta_1, \theta_2, \theta_3):\theta_1^2+\theta_2^2+\theta_3^2=1\}$ and $f(x, y, z, \theta_1, \theta_2, \theta_3)$ is any function which is Borel measurable and bounded on $T(A)\times D$, the integral $H(T, A, f)$ $=(A)\int f(T(w), \theta(w))d\phi$ exists and is the integral of f over (T, A).

4. *Periodic solutions of nonlinear differential equations*, by Professor W. R. Fuller, Purdue University.

In this paper, which was expository in nature, were indicated some types of nonlinear differential equations and systems of such equations, for which existence theorems for periodic solutions have been studied. This includes systems containing a small parameter, ϵ, which for $\epsilon=0$ have periodic solutions. In particular the study of systems of the form $x+\sigma^2x=\epsilon f(x, t; \epsilon)$ where σ^2x and f are n-vectors has been very fruitfully attacked by a method of L. Cesari which has been applied to a wide class of problems by Cesari, Hale, Gambill, Bailey and the author (see, *e.g.*, Atti Acad. Italia, vol. 11, 1940, pp. 633–692; Bull. AMS 60, 1954, pp. 64–66, 367; 62, 1956, p. 567; 63, 1957, p. 271).

5. *Localization experiment for teaching geometry*, by Professor A. D. Hummel, Ball State Teachers College, introduced by the Secretary.

A method of determining the x, y, and z coordinates of points within a body by means of X-rays was reviewed. Projected images of scales are used as measuring sticks in two radiographs. The source of the X-rays has different positions for the two radiographs. Elementary geometry is used to compute the coordinates of any point appearing in both. In the experiment described for teaching, a small source of visible light is substituted for the X-ray source. Data obtained from shadows is used to calculate the length of the object. Verification of the result by direct measurement should strengthen the students' faith in geometry.

6. *Undergraduate curricula—some brave experiments and cogent lessons*, by Professor A. E. Ross, University of Notre Dame.

There are many critical problems which confront one in the task of implementing an effective undergraduate program in mathematics. Today such a program must not only appeal to the people who have traditionally relied upon mathematical tools, but it must also appeal to the users of new mathematics, most of whom come from the humanities, from commerce, and from the "preprofessional" groups. The novelty of the new undergraduate curriculum lies not only in the new content but also in the growing recognition that mathematical manipulative skills alone do not develop the capacity for intelligent application nor do these skills alone justify considering mathematics as one of the liberal arts. To design a proper blending of the new and the old, of ideals and of skills, and to make this blend accessible to the very young brings us close to the fundamental questions in the art of communication. The purpose of this talk was to discuss and to illustrate some of these questions.

7. *A graphical solution for a particular finite series*, by Dr. R. H. L. Howe, Eli Lilly and Company, Lafayette, Indiana, introduced by the Secretary.

There are problems in engineering and the physical sciences which require the evaluation of y given as a finite series of the form $y=1/x_1+\cdots+1/x_n$, where $x_1, \cdots, x_n$ are positive or negative real numbers. Let $y=1/R$. Then R can be found graphically using a simple geometric principle. When a large number of terms is involved, this graphical method is particularly time saving. It is thus very useful in checking problems such as those concerning resistance of resistors in parallel, capacity of condensers in series, focal length in optical systems, and total resistance or conduction coefficient of materials in heat transfer and transmission.

8. *On the inter-relationship of applications and mathematical research*, by Dr. K. L. Nielsen, U. S. Naval Avionics Facility, Indianapolis, Indiana.

Emphasizing the constant increase and breadth of scope in the utilization of mathematics in contemporary nonmathematical fields, the author concentrated primarily on the inter-relationship between mathematical research and some technological developments, electronic calculating machines, and the philosophy of education. He discussed the role of the mathematician in industry, industry's utilization of mathematics, the development of new mathematics, and the need for closer cooperation between the educators and those engaged in research and the application of mathematics.

J. C. POLLEY, *Secretary*

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The second annual meeting of the New Jersey Section of the Mathematical Association of America was held at Fairleigh Dickinson University, Rutherford, New Jersey, on November 2, 1957. Dean A. E. Meder Jr., retiring Chairman of the Section, presided at the morning session; Dean C. W. Black, Fairleigh Dickinson University, presided at the luncheon and during the address by Dr. Morris Meister; and Professor B. E. Meserve, newly-elected Chairman of the Section, presided at the afternoon session. 104 persons registered.

At the business meeting, Professor B. E. Meserve of the State Teachers College at Upper Montclair was elected Chairman of the Section for 1957–58. Dr. J. D. Daugherty, Eastside High School, Paterson, was elected a member of the Executive Committee to serve for three years. Mr. R. S. Lockhart, Chairman of the Contest Committee, reported that the Committee recommended that: (a) the New Jersey Section endorse the national contest as a means of promoting a maximum of local participation (in secondary school mathematics contests); (b) the national contest results be used as a guide in selecting participants to enter contests such as the New Jersey State Mathematics Day at Rutgers; and (c) the Chairman of the Section appoint a committee to administer the national contest in New Jersey. These recommendations were adopted.

The following papers were presented, all by invitation of the Program Committee:

1. *The theory of braids*, by Professor Emil Artin, Princeton University.

The meaning of the postulates of group theory can be easily explained by the example of the group of all braids with a given number of strings. The composition consists in tying one braid to another, the unit element is the braid with unentangled strings, and the inverse is a certain reflection of a given braid. The word problem solves the topological question of classification of braids.

2. *Numerical calculation of flows in rivers and reservoirs*, by Professor J. J. Stoker, Institute of Mathematical Sciences, New York University, introduced by the Secretary.

The development of calculating machines and methods of numerical analysis make it possible to compute flows in rivers over hundreds of miles for periods of weeks. The present paper reports the successful outcomes of such calculations in three cases: 1) 375 miles of the Ohio River; 2) the junction of the Ohio and the Mississippi; and 3) Kentucky Reservoir at the mouth of the Tennessee River.

3. *A new program in the teaching of physics*, by Dr. Morris Meister, Bronx High School of Science.

The course in physics for secondary schools being developed by the Physical Science Study Committee at MIT stresses major developments of physics as a logical and integrated whole. Designed to meet the needs of the top twenty-five percent of students, it aims to give them intellectual and cultural insight into present-day human activity and achievement. Teaching aids, including a textbook, laboratory manual, teacher's manual, films and film strips, monographs related to the course, demonstrations, and laboratory experiments and apparatus kits are in process of development. The course is now being tried out in seven schools in various parts of the country. Next summer there will be at least four institutes for physics teachers interested in the new program.

4. *The work of the Commission on Mathematics, College Entrance Examination Board*, by Dean A. E. Meder, Jr., Rutgers University.

The Commission seeks the modernization, modification, and improvement of the college preparatory secondary curriculum so that it may be oriented to the needs of mathematical, physical, and social science, and of industry and technology as these needs exist in the second half of the twentieth century. The basis for change is to be found in the point of view of modern mathematics, which looks for patterns, not tricks (Sawyer). Specific suggestions for changes in school algebra, geometry, and trigonometry were given, as well as proposals for elective work in elementary analysis, statistical inference, and other fields.

I. L. BATTIN, *Secretary*

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The third annual meeting of the Northeastern Section of the Mathematical Association of America was held at Dartmouth College, Hanover, New Hampshire, on Novem-

ber 30, 1957, with Professor Stanley Bezuszka, S.J., of Boston College, Chairman of the Section, presiding. There were 75 people in attendance, including 59 members of the Association.

A brief welcoming address was delivered by President John Dickey of Dartmouth College. At the business meeting, the following officers were elected: Chairman, Professor D. E. Richmond of Williams College; Vice-Chairman, Professor N. H. McCoy of Smith College; Secretary-Treasurer, Professor Anne F. O'Neill of Wheaton College.

Professor Richmond presented the report of the Committee that had been appointed to consider the matter of Section participation in the National Contest for High Schools. It was voted to refrain from participation in the National Contest for the next year. It was also voted to appoint a committee to study the details of setting up the contest for the year after this, and to investigate alternatives to participation in the contest as well as the work done in other countries in contests of this kind.

The following program was presented:

1. *In retrospect*, by President C. V. Newsom, New York University.

Dr. Newsom opened his remarks by pointing out the fact that in his experience as a mathematician and educator he had been embarrassed frequently by his inability to describe in somewhat popular fashion the precise role of the mathematician in the total field of knowledge. Such an explanation is essential, for example, in any attempt to justify the inclusion of mathematics in a general education. Then he indicated his concurrence with many of the viewpoints expressed by John von Neumann, in the article entitled "The Mathematician," published in the newsletter of the Canadian Mathematical Congress, sixth issue, May, 1957. In particular, he commented upon von Neumann's observation that, "The most vitally characteristic fact about mathematics is, in my opinion, its quite peculiar relationship to the natural sciences, or, more generally, to any science which interprets experience on a higher than purely descriptive level." Dr. Newsom stated that the theoretical scientist is studying a nature that is basically inscrutable, but he has been successful in developing patterns of many types that correlate known data. Any such pattern becomes of great utility if it serves to cover other phenomena which were not considered or even not known at the time when the pattern was evolved. He said that the most sophisticated of all such patterns is the collection of propositions, called axioms, which characterize a situation under consideration. Such a scheme possesses values with which mathematicians generally are familiar. If the role of the mathematician is described essentially as that of pattern-maker, he takes on a stature that is comprehensible to nonscientists as well as scientists. The speaker culminated his remarks by indicating that the adoption of such a viewpoint would have a desirable effect upon the curriculum, and it might serve to accomplish the very desirable end of drawing pure and applied mathematicians more closely together.

2. *Statistical number geometry*, Professor H. J. Zassenhaus, McGill University.

Pack as many as possible convex bodies that are congruent under translation to a given convex body into a given Jordan region. This basic problem of statistical number geometry, a new mathematical discipline, was solved in two dimensions by N. Oler recently, proving a conjecture of Zassenhaus. Thus a new proof of C. A. Rogers' 1951 result that the critical lattices of Minkowski's geometry of number in the plane form the densest pointsets admissible for any given Minkowski-distance was obtained. Generalizations to other dimensions by Zassenhaus and distance functions by N. Smith and M. Rahman were also discussed.

3. *Operational meaning in mathematics*, by Professor Reinhard Korgen, Bowdoin College.

Professor Korgen discussed the problem of keeping mathematical machinery from obscuring operational meaning in the applications of mathematics. Thus an "average" may be defined as the number G which leaves invariant the value of some prescribed function of a set of measurements when G is substituted for each and every one; the context of a given problem will determine what

function to prescribe as the one to be left unchanged in value. The theory of measurement itself was involved in other examples adduced: it was pointed out that certain numerical laws in the natural sciences seem to report on nature when they are used instead to introduce scale-form in the measurement of quantities for which scale-form has not otherwise been shaped.

4. *Mechanics for undergraduates*, by Professor Garrett Birkoff, Harvard University.

Professor Birkoff described an ideal course in mechanics for undergraduates.

5. *University of Illinois School mathematics program*, by Professor Max Beberman, University of Illinois.

The purpose of the University of Illinois Committee on School Mathematics is to create a four-year program in college preparatory mathematics which treats mathematics in the manner of contemporary mathematicians and which stimulates interest among young people in the continued study of mathematics. In carrying out this purpose, the UICSM has written texts for students and teachers. The topics covered in these texts include the distinction between numbers and numerals; algebraic manipulations based on arithmetic generalizations [commutativity, *etc.*]; graphing equations and inequalities; a postulational development of euclidean geometry in which geometric objects are taken to be sets of points; the idea of a deductive theory in which models are constructed for various sets of postulates; mathematical induction; induction proofs of the laws of exponents; complex numbers; integral rational functions; polynomial equations; circular functions.

ANNE F. O'NEILL, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958.
ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.
INDIANA, Ball State Teachers College, Muncie, May 3, 1958.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, St. John's University, Collegeville, May 17, 1958.
MISSOURI, University of Missouri, Columbia, May 3, 1958.
NEBRASKA
NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.
NORTHEASTERN
NORTHERN CALIFORNIA, University of California, Berkeley, June 17, 1958 (joint meeting with ASEE, mathematics division).
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.
PHILADELPHIA, Lehigh University, Bethlehem, November 29, 1958.
ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, University of Montreal, Montreal, Quebec, Canada, May 10, 1958.
WISCONSIN, Carroll College, Waukesha, May 3, 1958.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

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Ralph D. James, *Editor*

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THE USE OF FILMS AND TELEVISION IN MATHEMATICS EDUCATION*

F. A. FICKEN, University of Tennessee

ORGANIZATION AND ACTIVITY OF THE BOARD

Educational institutions are facing a tremendous challenge in maintaining high standards and quality of instruction while accommodating vastly increased enrollments. Able and inspiring teachers are urgently needed at all levels of instruction. Far greater use will be made of various technological aids to education.

Films and television are already widely used for educational purposes and their use is increasing at an accelerated pace. The number of operating educational television stations, for example, has increased from one in 1950 and three in 1953 to 27 in 1957.

The Advisory Board on Education (ABE) of the National Academy of Sciences—National Research Council (NAS-NRC) took cognizance of this activity, and decided as a first step to study these media as they may be used in mathematics education. A Film (and Television) Evaluation Board (FEB), appointed through the Division of Mathematics of the NAS-NRC, was charged with the responsibility of developing criteria on content and objectives for production and for use of filmed or televised mathematical lectures. The FEB was also requested to recommend a course of action appropriate for the ABE as part of a continuing program within the NAS-NRC, relating to the use of such media in education in mathematics, and in other areas of science.

The FEB met at the Pennsylvania State University on August 21–24, 1957. The appointed members consisted of F. A. Ficken (University of Tennessee), chairman, and A. M. Gleason (Harvard University), T. H. Hildebrandt (University of Michigan), G. Hochschild (Institute for Advanced Study, Princeton, and the University of Illinois), J. D. Mancill (University of Alabama), and B. E. Meserve (State Teachers College, Upper Montclair, New Jersey). The meeting was also attended by P. S. Jones (University of Michigan), who is Chairman of the Committee on Educational Films of the Mathematical Association of America, by P. A. Smith (Columbia University), who was then Chairman of the Mathematics Division of the NAS-NRC, and by R. M. Whaley, who is Executive Director of the ABE of the NAS-NRC. The FEB also enjoyed the helpful counsel of C. R. Carpenter and A. W. VanderMeer, who are well acquainted with the extensive activity in film research and production at the Pennsylvania State University.

* This paper consists of a précis of the report of the Film Evaluation Board (FEB) of the Advisory Board on Education (ABE) and the Division of Mathematics of the National Academy of Sciences-National Research Council (NAS-NRC). The complete report is Publication 567 of the NAS-NRC [Library of Congress Card No. 58-60009], and may be purchased for $1.00 from the Publications Office, NAS-NRC, 2101 Constitution Avenue, Washington 25, D. C.

The Board viewed over thirty-five recorded films, selected for variety of content, mathematical level (rudimentary arithmetic to college algebra and trigonometry), intended audience, and general objectives (motivation, auxiliary to a course, or part of a course).

The work of the FEB was supported by a grant from the Fund for the Advancement of Education.

The report of the FEB has been accepted by the ABE of the NAS-NRC, and action is being taken to carry out the recommendations.

GENERAL OBSERVATIONS

By a "presentation" we shall mean a single showing of a film (in one place or over television) or of a television program (live or recorded, closed-circuit or broadcast).

A presentation in which an outstanding teacher achieves his best performance can offer a receptive student more skillful exposition than a "normal" classroom lecture; teachers, moreover, can learn from such a presentation some ways in which to improve their own work.

The new media make it possible to offer academic instruction beyond the technical capacity of conventional means.

The new media, especially television, provide an opportunity to offer public presentations in which the nature of mathematics and mathematical activity can be clarified and attractively illustrated; enlightenment can be made entertaining in the best sense of the word.

The new media make it possible, finally, for the viewer to witness objects, scenes, and occurrences at distant times and places. Motion and change in visible and audible data can be displayed in ways not otherwise possible. Phenomena invisible or inaudible to unaided senses can be brought above the thresholds. The possibility of producing a permanent record justifies elaborately contrived demonstrations, perhaps otherwise prohibitively expensive or destructive.

One unfavorable possibility, on the other hand, lies in the wide dissemination of erroneous ideas and unfortunate pedagogical stereotypes. Mass media entail a heavy responsibility; a single misunderstanding communicated in a presentation to a large group of students can handicap the efforts of all the teachers who must deal with the students personally. Large investment in equipment and recordings moreover, may strengthen resistance to expensive improvements.

Serious problems are raised by recorded sequences, in which a very substantial share, at least, of the expository burden of an academic course is allocated to presentations. It will not be easy to arrive at a favorable division of responsibility between presentations and various more personal methods of instruction. The new media should be used to supplement and improve instruction rather than to displace teachers. Another problem centers on proprietary interests; a recorded sequence is a substantial product.

RECOMMENDATIONS

A. Recommended Action.

1. *Creation of a Standing Committee on Mathematical Films and Television.* The Film Evaluation Board recommends the establishment, within the Division of Mathematics of the National Academy of Sciences—National Research Council but fully connected with and perhaps reporting to the Advisory Board on Education, of an adequately-staffed Standing Committee on Mathematical Films and Television. It appears to the Board that the functions of the proposed Standing Committee should include these:

a. Collecting, classifying, storing, and disseminating reliable data on existing mathematical presentations (both domestic and foreign, at all levels), including trustworthy indications of content, duration, purpose, appropriate audiences, and likely value for each purpose to each audience, in such a way as to facilitate advantageous choice and use of these presentations.
b. Promoting well-designed research on means of evaluating mathematical presentations and of principles for guidance in their effective preparation and use, research on the technical problems of feedback, and on other emerging problems.
c. Promoting the active participation of professional mathematicians in planning, refereeing, presenting, and reviewing presentations having significant mathematical content or significantly related to our profession.
d. Assisting in the development of equitable and realistic economic policies with reference to the proprietary interests of those engaged at any stage in presentations, and with reference to the intellectual and economic impact of films and television on teaching and on teachers.
e. Providing liaison with the mathematical community, with related professional groups, and with interested commercial enterprises; in particular, collecting and distributing information on emerging needs for presentations and on advantageous uses of them.
f. Publicizing its own existence and activities.

The Committee should avoid any action which would give it the status of a certifying agency.

The Film Evaluation Board estimates that such a Standing Committee* can be of workable size and still reflect regional differences within the nation, and a wholesome variety of professional experience and attitudes, as well as the views and interests of the American Mathematical Society, Mathematical Association of America, National Academy of Sciences, American Association for the Ad-

* Following acceptance by the ABE of the FEB report, the Division of Mathematics of the NAS-NRC has appointed the following standing Committee on Mathematical Films and Television: C. B. Allendoerfer (University of Washington), Chairman, R. A. Good (University of Maryland), G. A. Hedlund (Yale University), P. S. Jones (University of Michigan), J. R. Mayor (American Association for the Advancement of Science), E. J. McShane (University of Virginia), P. C. Rosenbloom (University of Minnesota).

vancement of Science, National Council of Teachers of Mathematics, and other organizations having a basic interest in mathematical education.

2. *Action for Other Areas of Science.* The Film Evaluation Board recommends that the National Academy of Sciences—National Research Council consider steps to ensure that the functions listed above be performed also in other areas of science, perhaps wholly or partially by similar standing committees.

B. Recommended Policies.

1. *Qualitative Considerations in Planning and Producing Presentations.* The presentations viewed by the Film Evaluation Board embodied various mathematical and pedagogical values which were a prime concern of the Board. Our views are set forth as supporting argument. The FEB recommends that these and similar considerations be kept in mind in planning and producing mathematical presentations.

2. *Proprietary Interests in Presentations.* The Film Evaluation Board recommends that whenever a presentation is exhibited an equitable dividend should accrue, as a matter of right, to all persons who participated in any phase of the presentation. Participation in some phase of a live educational presentation on television by a member of the staff of an educational institution, with no thought of later presentation, should be treated as an appropriate fraction of his academic duties; if a kinescope is made, an understanding should be reached, prior to his participation, regarding his interest in the event of later repeated presentation. The foregoing proprietary interests, as well as those relating to supplementary materials, professional acknowledgments, and so on, should be embodied in a negotiated contract before work is begun.

3. *Policies Regarding the Use of Presentations.* The Film Evaluation Board recommends that presentations be used in such a way as both (a) to promote frequent direct association between students and active scholars and (b) to promote healthy growth of younger teachers toward mature ability and professional dignity. With reference to an extended series of presentations offered in connection with a more or less conventional academic course, it is the unanimous opinion of the Board, pending further experience, that not more than half the time allotted to formal group instruction should be used for presentations, that such presentations should be devoted primarily to the exposition of basic ideas and principles (with a few well-chosen illustrations but no repetitive manipulation), and that the rest of the time should be used in accordance with the recommendation in the preceding sentence.

SUPPORTING ARGUMENT

A. Recommended Action (omitted from précis).

B. Qualitative Considerations in Planning and Producing Presentations.

Many principles of effective teaching, and even the very spirit of mathe-

matics, were violated in some of the presentations viewed by the Film Evaluation Board and we can offer only a brief summary of our observations. We emphasize at the same time our opinion that the very substantial outlay required to produce a program dictates extremely attentive preparations, and the further opinion that costly technical features often add less educational value to a program than the comparatively inexpensive services of able collaborators.

Our statement follows the chronological order in which a presentation is produced and used. With certain *data* in mind, the production is *planned.* After suitable *preparations*, a *live performance* occurs before film or television equipment. Whether or not a recording is made, certain *further steps* may be desirable.

1. *Data.* Certain data must be fixed, by choice or circumstance, before intelligent planning is possible. The presentation has an aim, an audience, and certain technical resources for use in its production. The primary aim of an educational presentation is to arouse and nourish thoughtful interest and activity. The intended audience may vary widely in vocabulary, information, experience, interest, susceptibility, motive, reaction-time, duration of attentiveness, and so on. The technical resources include the capabilities of the equipment and its operators. Other technical variables which may be fixed before planning begins include auxiliary audio-visual equipment and such parameters as the duration of the program, time and context of intended presentation, and so on.

2. *Planning.* Planning is essentially a collection of choices among varieties of substances and treatment, of emphasis, and of talent. These choices should take full account of the data. Language and substance should be correct and accessible to the audience, whose preparation should permit it to respond to most of the amount offered at the pace at which it is presented. The level during most of a program should be low enough for the estimated "average" of the audience, but high enough in occasional clearly labeled digressions to interest more sophisticated viewers as well.

Certain principles regarding substance and treatment apply to all presentations. Correctness must be impeccable. Mere freedom from obvious blunder is not enough; the reflective viewer who tries to push his thinking further must not be led astray by erroneous implications. The material must give a sound general impression, in harmony with the most enlightened professional understanding. The treatment should bring to light the fundamental intrinsic significance of the facts and relationships presented.

Any presentation should attempt to communicate basic principles and ideas, along with genuinely illuminating illustrations. The capacities of the media to portray motion and change in visible and audible data should be exploited fully. The Board was particularly impressed with animated drawings, and feels that their wise exploitation would yield valuable educational dividends.

Public presentations should emphasize motivation and entertainment, while maintaining significant contact with genuinely mathematical ideas. The aim should be to convey information of cultural interest and conceptual relationships

rather than technical skills; extensive manipulations should be avoided.

Presentations auxiliary to an academic program should have substance and treatment appropriately related to the material they supplement, serving objectives somewhat similar to those of visiting lecturers, and should be used only if they convey the intended idea in a superior way. Routine manipulations should be kept to a minimum.

An extended series of presentations, bearing a significant part of the primary expository burden of an academic course ("recorded sequence"), should improve on standard presentations, amplifying and elucidating the text. While some manipulations will be unavoidable, no attempt should be made to replace practice by the student. New courses and curricula present special opportunities for the experimental development of original discussions which may be very suitable, after modifications indicated by experience, for presentation on film or television.

The primary aim of a presentation is educational, to stimulate active mathematical thought, or at least a discriminating interest in it. The emphasis will therefore be on challenging the viewer, on stimulating his curiosity. All presentations should appeal to active reason, thus enhancing the viewer's confidence in his own rational processes. A supreme effort should be made to convey the intrinsic interest and value of the subject, and the pleasant excitement of seeking and gaining understanding for its own sake.

The speaker should display genuine authority with a lively but not exhausting personality. Especially, perhaps, in a public presentation, he should be identified as an active scientist with a recognized gift for communication. A speaker of unusual ability should ordinarily be engaged even for elementary material.

Unlike commercial presentations, whose principal goal is acceptability to the audience, an educational program strives for acceptability only secondarily, in order to advance the quite different primary aims indicated above. It lies in the nature of organized education that the knowledge and appreciations of the teacher shall surpass those of the viewer, whether student or layman. The value of an educational program is therefore determined only in part by its acceptability to the viewer; to be of educational value it will ordinarily have features to which his standards of acceptability do not apply, but which are subject to the considered judgment of experts. For this reason the aid of an expert consultant or referee is indispensable both in planning and in the ensuing preparations. He can detect glaring defects in substance and treatment not noticeable to most viewers, nor even to experts in the media. He should be rewarded accordingly; like the expert speaker, an able referee will cost little compared with the total outlay. Presentations making significant contact with other areas (*e.g.*, physics, engineering, social studies) should be monitored also by specialists in those areas.

3. *Preparations*. Motivations must be honest. There are many problems,

even some with quite practical aspects, having genuine intellectual appeal, capable of conveying to many viewers in an exciting way the power and elegance of our subject.

Illustrations should be clean. An example may leave a very foggy impression if it involves many features whose relation to the ideas at hand is not clear, thus leaving the viewer in some doubt as to just which features are illustrating which ideas.

Response to viewer's reaction must be judicious. The reactions of a "guinea pig" representing the "average" viewer may supplement helpfully those of the expert referee.

A very great amount of time may be required for thoughtful adjustment of content, style, and displays to the particular aim and audience intended.

Auxiliary preparations include suitable written materials, audio-visual aids, forewords to class and collaborating teachers, and so on. Technical preparations must ensure that sight and sound and esthetic appeal will be above the thresholds of acceptability.

In both preparation and live performance it is important to observe the difference between entertainment and distraction. Both films and television have been accepted by most viewers, before witnessing an educational presentation, as pastimes yielding in many instances mere distraction or, at best, entertainment of a somewhat frivolous kind. It is thought that sensitive judgment can maintain an entertaining atmosphere in an educational presentation while avoiding features which will divert the viewer's attention from the fundamental aims of the program.

4. *The Live Performance.* Visual displays must be visible, adequately lighted, and not obscured by perspective or by the speaker's shadow. The camera should not dwell at tedious length on the speaker.

The speaker should be lively and enthusiastic but relaxed in voice and manner, should avoid both apologetic and patronizing manner and language, and should expect and merit the respectful attention of the viewer.

5. *Further Steps.* The need for research on methods of evaluation indicates that thoughtfully prepared follow-up studies of individual presentations will be essential.

An instructional presentation will presuppose certain well-defined responsibilities of students and of collaborating teachers. It is essential, in particular, that a student have ample means, after a presentation, for determining (perhaps for himself) what he has understood and can deal with confidently, and what he has not understood. He must have ready access, in time of need, to clarifying personal guidance.

C. Proprietary Interests.

The novelty of presentations, especially of television programs and recordings of them, implies tentative and negotiable economic arrangements. One

factor determining a collaborator's reward is clearly the extent of his responsibility; is he, for example, composer or performer, or both? Economic forces, left to themselves, would ultimately produce decisions on many of these questions; indeed, they have yielded mutually-acceptable relationships between, for example, authors and publishers. We believe that the proposed Standing Committee should take cognizance of the evolution and stabilization of the economic rights of collaborators in presentations, in order that equitable and realistic policies may be reached as quickly and easily as possible.

D. Intellectual and Economic Impact of Recorded Sequences on the Teaching Profession.

The FEB holds firmly to the belief, which admittedly still lies beyond objective demonstration, that a student's growth toward professional stature is enormously stimulated by personal contact with scholars of professional ability engaged in professional activity. We are also concerned to secure the healthy development of further generations of teachers and scholars, observing that our society now offers permanent support, with minor exceptions, only to the *teaching* activity of a pure scholar in any field of learning.

We may distinguish between personal and group instruction according as the teacher can or cannot adjust his instruction, deliberately and immediately, to the specific personal characteristics of individual students. Presentations are essentially means of indirect group instruction.

Carefully prepared public presentations and occasional presentations supplementing and enriching conventional direct group instruction can clearly have great value. These uses of presentations are comparatively well-understood, at least in principle, although the need persists for improvement in the production and use of presentations. Extensive systematic use of presentations as a primary means of group instruction, while in a sense possible before the advent of television, has since then been enormously stimulated.

We shall speak of a recorded sequence whenever a substantial part (not necessarily all) of the mission of group instruction is allocated to an extended series of presentations produced for the purpose. We are thinking primarily of indirect group instruction in high school and at the lower-division collegiate level.

A fully recorded course makes feasible indirect group instruction not possible by conventional direct means; for example, television can reach sparsely distributed students, the bedridden, the snowbound, and so on.

A recorded sequence makes it possible to entrust one teacher (the speaker in the recordings) with the corresponding part of the expository responsibilities toward a large group of students normally borne by each teacher toward his own students. It is then possible to meet a given commitment for group instruction with fewer live teacher-hours in the classroom. Example: 12 sections, 3 hours weekly, require 36 teacher-hours in the classroom; if all sections view a presentation simultaneously during one of the three hours then, quite crudely,

only 25 teacher-hours in the "classroom" are required.

There is little doubt that the more specific, more tangible needs of group instruction can be met acceptably by recorded sequences of sufficiently high quality. This generally favorable estimate raises in some minds an appealing fantasy of "best" teaching, in which the unique definitive exposition is installed on film and tape and distributed widely, with a revised version appearing once every decade or so.

It must be remembered, however, that even the most successful expositor fails to reach some entirely worthy students, and in any case will not arouse uniformly the enthusiasm felt by those with whom he is most successful.

Pending the development of really effective and convenient feed-back devices, moreover, a live lecture has an undeniable advantage over an equally good presentation, in that interruption and explicit interchange between class and teacher is in fact possible, whether or not this facility is used advantageously being another question.

Even without interruption, finally, an experienced lecturer who can see and hear his audience, even a large one, can often sense their attitude and can adjust to it, stimulating them, so to speak, at a resonant frequency.

Insofar as presentations may be less responsive than direct group instruction to the needs of an individual student, and insofar as presentations may achieve less fully the tangible and intangible goals of group instruction, to that extent each student may need more and better personal instruction than he would need if all group instruction were direct.

Personal contact between students and teachers is important at all levels of instruction. Such contact is already regrettably scanty in many situations and threatens to deteriorate further as mounting enrollments outstrip the supply of qualified teachers. Recorded sequences should be used in such a way as to promote personal association between students and teachers; some of the eleven "released" hours in the example above should be devoted to strengthening the program of personal instruction.

Recorded sequences may not only increase the need for personal instruction, but may also change its nature and the qualifications of those entrusted with the responsibility for it. The personal teacher must understand the presentation fully (and may therefore need to know more) and may have to respond more sensitively than now to differences between individual students. Perhaps special training will be needed in giving personal instruction accompanying recorded sequences.

A recorded sequence may decrease the amount of group instruction required at a given time in a given institution. Unless the program of personal instruction is correspondingly expanded, as would indeed seem most desirable, a temptation may be felt to continue employing prospective teachers, in order to help meet their economic problems, but to use them primarily for menialities such as marking papers, taking attendance, calculating grades, and so on. Such a policy

would deny prospective teachers a rewarding share of the actual conduct of direct group instruction.

The Film Evaluation Board is firmly convinced of the present and future value to students and to the community of an alert, well-informed, proud corps of teachers. Scholars and teachers nourish and transmit our culture. Their economic rewards are comparatively small. Further decline in their intellectual or moral position would be disastrous at the present time. Policies denying genuine intellectual responsibilities to younger teachers would degrade the demands and attractions of the profession and would lead, in the long run, to serious deterioration of its standards and of the attainments of its members.

On turning from the intellectual to the economic aspects of recorded sequences, we may note that the question of proprietary interests takes on added urgency from the sheer magnitude of the investment. The fact, moreover, that a comparatively large outlay is required both for the production and for the use of recorded sequences may have two educationally undesirable consequences: first, a reluctance to change, thus tending to perpetuate the maladroit along with the praiseworthy (though new techniques permitting "erasure" and correction may alleviate this difficulty); and second, a compulsive tendency to use presentations merely because they are available and were costly, and operating costs are low, regardless of their detailed suitability for the specific purpose at hand.

We are more concerned, however, with the apprehension, apparently widely felt among teachers, that the use of recorded sequences may displace contact instruction to such an extent as to cause the demand for well-qualified teachers to decline seriously.

On the contrary, we believe, recorded sequences will lead to more rather than less demand for personal instruction. There is no prospect that the "average" student will be able to learn fully and efficiently without having some access to an agency that can respond to his individual needs. There is no prospect that any agency will surpass a human teacher in responsive capabilities. Except under very special conditions, recorded sequences seem unlikely to replace direct group instruction entirely.

The same rising enrollment, moreover, which lends special attraction to recorded sequences, does so in large part because of the growing shortage of qualified teachers. If our schools and colleges are to discharge their responsibilities effectively in coming years they will need to use every genuinely helpful teaching aid, including presentations, along with every qualified teacher who can be persuaded to serve under circumstances then prevailing.

The recorded sequence is the newest fruit of a new technology. Its ultimate effects cannot be reliably predicted, and the preceding tentative opinions are subject to adjustment in the light of accumulating experience. The principal question, perhaps, concerns the most advantageous distribution of responsibility, all things considered, between personal instruction, direct group instruction, and the newly feasible recorded sequence. The proposed Standing Committee

could exert a most beneficial influence in assuring that sound educational considerations are kept in the foreground as policies governing the use of presentations are developed.

NOTES ON THE STATUS AND FUTURE OF FILMS AND TV IN COLLEGIATE MATHEMATICS EDUCATION*

PHILLIP S. JONES, University of Michigan

The Committee on Instructional Films was appointed in September 1955 as a result of a suggestion to the Board of Governors of the Mathematical Association of America that the Association consider producing, for classroom use in colleges, a sequence of films based on an outstanding freshman course.

The Committee has (1) corresponded with itself and others; (2) interviewed producers of films and kinescopes, and persons at the Educational Television and Radio Programming Center; (3) viewed films; (4) consulted with persons from the Department of Audio-Visual Education of the National Education Association and from the Eastman Kodak Company; (5) consulted with other mathematicians; (6) studied various reports on the teaching effectiveness of films and television, and (7) met for discussions in Washington November 15–17, 1956 and in Ann Arbor July 18–21, 1957. These meetings were made possible by a grant of funds from the National Science Foundation.

The Committee found such an interest in films and television as teaching aids and such a rapidly changing situation as to experimentation with and use of these media that it submitted its report in two parts. Part I was its recommendations to the Board as to the activities in this area which it thought desirable for the Association to undertake. Part II, on which this note is based, was designed both to elaborate and to document the specific recommendations of Part I and also to summarize what seemed to it to be the present status and immediate future of (1) research in the use of these media; (2) current and recently completed projects using films and television in teaching mathematics; (3) the needs and possible audiences which it considered.

That there is an increasing number of mathematical television projects is revealed by the list in Appendix A. A survey of these projects will show that they can be roughly classified as: (1) "cultural," where the chief objective is

* This is a revised version of Part II of the report submitted to the Board of Governors of the Mathematical Association of America dated July 30, 1957 by its Committee on Instructional Films composed of Howard H. Campaigne, Marguerite Lehr, Phillip S. Jones (Chairman).

to interest a general audience, give it some idea of a small segment of mathematics, and perhaps by this device give it a better appreciation of what mathematics is, or at least of wherein some of its fascination lies; (2) "public relations," and/or "guidance," where various groups, but largely public schools, have tried to show taxpayers and parents how their children are taught and to interest students (and their parents) in electing more mathematics; (3) "course," where the object is to teach a definite body of mathematical knowledge, often, but not always, for high school or college credit.

If one does this classification on a chronological basis one perceives very clearly that whereas programs of types (1) and (2) predominated in the earlier years, the preponderance of mathematical television programs are now of type (3) and are frequently "for credit." The courses range from high school algebra taught for persons often out of school for some time who are working toward a high school diploma, through courses in solid geometry and trigonometry to make up college entrance deficiencies or for employed persons seeking extension courses, to full fledged college freshman courses, and to a course in calculus on color TV for the "in-service" training of secondary school teachers.

The Committee has done no research itself (although two of its members have had TV experience), and has not read all of the material on the subject available (Appendix B contains a rather randomly selected list of references). However, it does believe that the following conclusions are fairly valid. It is not clear to us, however, that all the results obtained from studies of teaching by films and TV can be automatically transferred to the teaching of mathematics. Further we are not certain that the tests which frequently show that students learn as well by television as with standard instruction have actually tested all of the important ideas, attitudes, insights which we hope we teach in addition to facts and techniques. Finally, in many statistical studies the differences between instructional groups were not very significant—a sort of negative endorsement of film and TV instruction. With these reservations we list the following conclusions.

1. A topnotch teacher can extend his effectiveness through the use of television (or films).

2. Pupils learn at least as much in television classes as with conventional instruction.

   (These two statements were attributed to Dr. Alvin C. Eurich in a Fund for the Advancement of Education press release, but they, especially the second, are supported by evidence in studies involving the teaching of the slide rule [**1**], general science, chemistry [**30**], army and navy skills [**30, 41**], where the numbers in square brackets refer to items in the bibliography of Appendix B).

3. In some cases pupils have learned more by television than by standard instruction, and in a few they learned less [**39**]. Able students profited more from high school algebra on TV than did slower ones [**11**], but the

army found TV more effective than live instruction for persons of low ability.

4. Disadvantages of television instruction are
   a. Lack of teacher-class discussion and interaction. (There are some devices to offset this—*e.g.*, two-way communication, discussion group follow-up of the film, devices built into the film such as asking questions, pauses for thought or discussion, repetitions of parts of the film or TV presentation.)
   b. Dislike of television instruction: Actually students frequently prefer it to other instruction, especially where they see demonstrations better on TV than in a lecture hall. On the other hand, a group of TV-instructed psychology students did substantially as well as others but *liked* the subject less well. In one study, superior students liked TV instruction less than the less able students, but preferred it to large classes.

5. Advantages claimed for film and TV instruction
   a. Good teachers and mathematicians on film may do a better job than poor teachers with a small class, or than good teachers in large classes.

      In this connection readers may be interested in the following comment from C. B. Read of the University of Wichita which has *not* tried teaching by television, "Prior to establishing instruction by television the University of Wichita decided to experiment with live teaching, using large classes and the lecture method. Conference hours were set up under the supervision of experienced faculty members (not graduate assistants) trying to duplicate as far as possible the situation which would be used with instruction by a lecturer over television. We have used three different professors in giving the lectures. The course has been restricted to intermediate algebra. After evaluating the results which have been tried for three semesters both the department and the University Administration is thoroughly convinced that this method of instruction is definitely inferior to our customary small-sized classes with personal contact by the teacher for each group. Our experience has been that under this plan students do not take advantage of conference hours to any great extent. More than 50 per cent of those enrolling in the course have failed to complete either through withdrawal or failure. For this reason the University has abandoned any attempt to utilize television or films as a method of teaching mathematics classes."
   b. Experts and famous scholars may be seen and heard in more places.
   c. Models, special devices, "animation" may be used to enrich instruction.
   d. Instructors viewing good films may see how to improve their own teaching.

e. Instructional time may be freed and made available for individual instruction and help, or for teaching more advanced courses.
f. The ease of repeated viewing of a filmed lesson may help adapt instruction to the slow or ill-prepared student.

6. Supplementary materials, outlines, reading lists, *etc.*, increase the effectiveness of film instruction, as do preliminary and follow-up discussions.

7. Kinescope film quality is a little more variable than that of regular movies and is never as good as a very superior movie, but is better than poor movie films and is very adequate for teaching purposes.

8. Planning, and sympathetic, intelligent, and cooperative direction including adequate walk-through rehearsals are essential to production of good films or kinescope. Were it possible, trial runs on closed circuit TV might help to produce better kinescopes.

In its discussions, particularly those held in Washington in which a number of additional persons participated, the Committee listed the following groups or "audiences" for which films might serve a useful and somewhat unique purpose and for which the Association and its members should have some real and immediate concern. (The committee report which stresses *films* more than *TV* is still significant here because kinescope films, made relatively inexpensively at the time a "live" TV program is produced, may then be available for classroom showing elsewhere or for repeated showings on television.) These audiences somewhat correspond to objectives for films. It is important that the objectives be clearly defined before embarking upon the production of TV programs or films.

A. *Audiences for single films or short film sequences.*

1. *Superior underclassmen*, persons who had been good high school students. Purposes: Enrichment of their course work, giving them an opportunity to see "a mind in action" embodied in a well-known mathematician-expositor who might use an approach different from standard texts, add supplementary materials, try to give them "insights" and "appreciations."
2. *Upperclassmen who are not necessarily mathematics majors*, but who are interested and able. Purposes: Similar to those of audience 1 but with material presented at a higher level of maturity.
3. *Upperclassmen, mathematics majors.*
4. *Secondary School Teachers*—especially those in the field who might be reached by extension courses and workshops.
5. *Adults* who are interested in continuing their "general" or "cultural" education.
6. *Superior high school students*—"enrichment" films.

7. *High School students, college freshmen and their parents*—"guidance" or "career" films.

B. *Audiences or populations for "course" films or kinescopes.*

1. *The general freshman course* where the objective of the film would be to conserve and extend the effectiveness of a teaching staff which is in short supply. The Committee does not believe that a film will ever be a better teacher than a good teacher with a small class. It may be better than a poor teacher or a good teacher in a large lecture hall. Washington University and several others have made beginnings here. (See Appendix A and the articles in this issue.)

2. *The class for really superior freshmen.* They need inspiration and contacts with persons and ideas not always available on all campuses, but their backgrounds vary and the number of schools which could and would offer a particular course for this group is limited, hence it is doubtful if TV or films for them is desirable except for the enrichment of regular instruction.

3. *A nontraditional first year course*, such as "Finite Mathematics," "Universal Mathematics," or "Allendoerfer and Oakley." Kinescopes for this might serve to "spread the gospel," but the problems of whose gospel would be accepted by whom seem to make such a project an uncertain place to begin experimentation in TV and film teaching.

4. *Analytic geometry and calculus.* The growing popularity of this combination as a freshman course makes it a good candidate for film.

5. *Trigonometry.* The Committee felt that there might be value in one or more film series in this area. Several schools have worked on this (see Appendix A). The objectives of this series might be (a) to improve the quality of teaching by making an outstanding teacher available to more students in this one course, and thereby at the same time improving instruction in other courses which would be better staffed as a result of the lightening of the elementary teaching load carried by experienced staff members; (b) to improve the quality of teaching by showing to all instructors an outstanding teacher at work (this could even be viewed by and have an effect upon high school teachers); (c) to improve the quality of teaching by showing the use of charts, graphs, oscilloscopes, and other devices such as animation; (d) to improve or modernize the content of courses now taught by giving a modern emphasis (or de-emphasis) to the solution of triangles, the role of angles, the nature of periodic functions. Some single films of such a series might be usable for enrichment purposes by persons not using the entire series; (e) to improve the "pace" of the course without handicapping the ill-prepared student who may get his added help from repeated viewings rather than from a slowed pace which may dull the interest of better students.

6. *College Algebra.*
7. *Analytic geometry.*

The problems of adaptation to local situations and texts suggest that local development of course films may be preferable to Association- or foundation-supported projects. Several schools have been working on this (See Appendix A).

Note that the Film Evaluation Board appointed by the Division of Mathematics of the National Academy of Sciences—National Research Council has recommended the establishment of a standing Committee on Mathematical Films and Television. The functions of this proposed committee and the policies recommended by the Board to producers of mathematical films and television programs are discussed in [12] in Appendix B.*

APPENDIX A—MATHEMATICS ON TELEVISION

(The Committee's Chairman has gathered this data from many sources, including newspaper accounts. He has written to every person or group mentioned, but has not had confirming data from all of them. There are probably inaccuracies and incompletenesses in what follows, but the interest in the topic seemed to him to justify the publication in spite of this.)

1958–1959 Projects
(Added in proof)

Martin T. Wechsler of Wayne State University, Detroit, is making 64 30-min. kinescopes to be used to teach *intermediate algebra and trigonometry* in the fall. This is intended for prospective freshmen with high school deficiencies, to be used on both closed-circuit and open broadcasts.

The University of Detroit will teach *analytic geometry and calculus* for 4 hours credit in 4 30-min. lectures per week in the fall.

Emil Berger presented 12 30-min. programs for in-school viewing by high school students over KCTA, St. Paul, Minnesota, early in 1958.

1957–1958 Projects

	Station-Sponsor-Personnel	*Topic and/or Purpose-Length*
1.	Closed circuit—Massachusetts Institute of Technology. Hartley Rogers, Jr.	Elective freshman course for nonmathematics majors. 6 1-hr. TV lectures. Remainder of course given in large lecture room.
2.	Closed circuit—color. Walter Reed Army Medical Center, Washington, D. C. and University of Maryland, R. A. Good.†	*Foundations of Analysis.* For teachers. 26 50-min. lectures. Sponsored by National Academy of Science-National Research Council's Advisory Board on Education and A.A.A.S. Kinescopes and an evaluation supported by the Fund for the Advancement of Education.
3.	Educational TV stations. N.B.C. and the Educational Television and Radio Programming Center.	13 30-min. programs planned by J. R. Newman in spring of 1957. Second series, fall, 1957 planned by Howard Fehr.
4.	Closed circuit. Stephens College.	8 25-min. lectures by Adele Leonhardy followed by small-group student discussions.

5. The Fund for the Advancement of Education has given $986,000 to schools in Atlanta, Cin-

* See the article by F. A. Ficken in this issue, pp. 393–403. This committee has now been appointed. With C. B. Allendoerfer as chairman, it includes R. A. Good, G. A. Hedlund, P. S. Jones, E. J. McShane, J. R. Mayor, P. C. Rosenbloom.

† See this issue, pp. 426–427.

cinnati, Detroit, Miami, Norfolk, Oklahoma City, Philadelphia, Wichita, and the states of Nebraska, and Oklahoma to begin regular classroom instruction to large classes in September, 1957. Algebra is among the subjects to be taught to junior and senior high school students. There will be accompanying studies in methods of teaching large classes.

6. Alabama Educational Network.* WBIQ Birmingham, WTIQ Mumford, WAIQ Andalusia, University of Alabama.	*Mathematical Potpourri.* Popular lectures by various staff members, $\frac{1}{2}$ hr. weekly, 16 weeks, fall. *Topics for high school students.* Designed for in-school viewing, fall and spring. 15 members of mathematics staff have participated.
7. Closed circuit. University of Alabama* Holland C. Filgo, Jr.	*Trigonometry.* 3 groups, TV and control. 30-min. lesson followed by 20-min. discussion.
8. WTTW. Chicago City Junior College, Jerome M. Sachs and Florence M. Miller.	*Introduction to Mathematics.* 3 hrs. credit, $\frac{1}{2}$ hr, 3 days a week, 46 lessons. Live, spring 1957, by kinescope in fall of 1957. See Appendix B, [**42**], and projects **9**, **36**.
9. WTTW. Chicago City Junior College, Jerome M. Sachs, Henry Patin.	*Slide Rule.* 1-hr. credit, $\frac{1}{2}$ hr., 1 day a week, 14 lessons. Fall 1957 and spring 1958. Kinescoped.
10. WFBG-TV. Pennsylvania State University, Steven A. Adler.	*Topics in elementary algebra and recreational materials.* 27 $\frac{1}{2}$-hr. programs.
11. Closed circuit. Pennsylvania State University, George N. Rancy, L. T. Dunlap, Isador M. Sheffer.	*Algebra for non-technical students.* 3 hrs. credit. 3 hrs. a week, 15 weeks, control classes, tutorial sessions, and talk-back system. Spring 1958.
12. WFBG-TV. Pennsylvania State University.	*Cultural Program.* 13 one-hr. sessions. Scheduled for summer 1958.
13. KUHT. University of Houston.†	*Trigonometry.* 27 lectures of 44 min. were filmed in summer of 1957 to be shown at rate of 2 a week on both open circuit TV (twice) and by projectors (three times). Students also required to attend a one-hour conference each week.
14. Kinescopes. Westinghouse Broadcasting Co.	*Adventures in Numbers and Space.* 9 $\frac{1}{2}$-hr. programs devised and supervised by H. F. Fehr with script written by Bill and Cora Baird and using their puppets. To motivate junior high school students.
15. Kinescopes. University of Michigan, A. J. Lohwater.	*The Meaning of Geometry. Modern Geometry. The Element of Chance.* 3 $\frac{1}{2}$-hr. programs distributed to a number of TV stations.
16. WEWS. Western Reserve University, Robert Carson.	*College Algebra.* 44 $\frac{1}{2}$-hr. programs, 3 days a week. Credit. Scheduled for fall 1958.
17. University of New Mexico and Albuquerque Public Schools.	Kinescopes distributed to five schools pending completion of TV broadcast facilities.
18. WQED. Adult School of the Air, Pittsburgh, Harry A. Snyder, Director.	*General Mathematics and Business Arithmetic. Algebra* is scheduled for 1958–1959. This program began in 1954. See also projects **27**, **28** and **51** listed below. Courses prepared for army and State of Pennsylvania exams leading to high school diplomas.

* See the article by Ayrlene McGahey Jones in this issue, pp. 421–424.

† See the article by Martin Wright in this issue, pp. 425–426.

19. Closed circuit. Purdue University,* John Dyer-Bennett, William R. Fuller.
Freshman College Mathematics. 5 hr. credit. 5 days a week. 30 min. on TV followed by 20-min. session with graduate assistant in each room.

20. KETC. St. Louis, Washington University,† H. M. MacNeille, H. Margaret Elliott, J. K. Goldhaber, I. I. Hirschman, Jr., Jack Indritz, Ross R. Middlemiss.
College Algebra and Trigonometry. 5 units. 45 min. 5 days a week on film. Twice daily on open circuit supplemented in first semester by non-TV lecture twice daily. Film only in second semester, twice daily on screen.
Analytic Geometry and Calculus. 5 units. 45 min. 4 days a week, second semester; live on open circuit, once daily. Supplemented by two non-TV lectures daily.
Intermediate Algebra. 2 units. 30 min. 3 days a week, live on open circuit, to supplement instruction for about 500 University College (evening) students who meet with an instructor one night a week.
Theory of Games. One unit. 30 min. one day a week. (See project **33** also.)

21. KUON-TV. University of Nebraska, David Wells.
General Mathematics.
First Year Algebra (High School).
High School Geometry. 5-hr. high school credit per semester. 2, 3, and 5 TV presentations a week, respectively, each 30-min. long. Correspondence study is also involved, supported by the Fund for the Advancement of Education. Begun in 1956–1957.

22. WOI-TV, Ames, Iowa. Iowa State Teachers College, Cedar Falls, Irvin Brune.
The Teaching of Arithmetic. 10 90-min. programs for college credit.

1956–1957 Projects

Station-Sponsor-Personnel	*Topic and/or Purpose-Length*
23. WATV, Newark, N. J. Ernest R. Ranucci.	*Spotlight on Mathematics*, 6 30-min. popular-interest programs. See project **61.**
24. WTVS. Detroit Public Schools, Franklin Frey and Geraldine Dolan, Cass Technical High School.	30 min. on *conic sections* and their applications.
25. KCTS, Seattle. University of Washington, Carl B. Allendoerfer.‡	*Intermediate algebra.* 5 credits; 20 weeks, 30 min., 3 times a week.
26. KCTS, Seattle. University of Washington, Carl B. Allendoerfer.‡	*Plane trigonometry.* 3 credits. 3 $\frac{1}{2}$-hr. programs a week, 10 weeks on new material, one reviewing home work.

* See this issue, pp. 430–439.
† See this issue, pp. 440–443.
‡ See this issue, pp. 444–446.

27. WQED, Pittsburgh. Pittsburgh Public Schools, Alvin Stuart.	*Fifth-Grade Arithmetic.* 28 classes in 23 schools, 20 min. on TV 5 days a week with 20-min. classroom instruction. 175 school days. Fund for the Advancement of Education. See Appendix B [**10**].
28. WQED, Pittsburgh. Jack Roush.	36 $\frac{1}{2}$-hr. kinescopes on *algebra* were made under an Armed Forces contract for use by the United States Armed Forces Institute. See projects **18**, **51** also.
29. Closed circuit.* Purdue University, John Dyer-Bennet, Merrill E. Shanks.	*Calculus.* 4 hr. credit. 3 hrs. a week on TV, 1 hr. conventional instruction.
30. Closed circuit. Mount Pleasant High School, Schenectady, N. Y. Edward Sherley.	*Intermediate Algebra and Trigonometry.* 2 classes. Supported by the Fund for the Advancement of Education.
31. KETA-TV, Oklahoma City.	*High School mathematics.*
32. KETA-TV, Oklahoma City. University of Oklahoma, Arthur Bernhart.	*Intermediate Algebra.* 3 hrs. college credit. Three $\frac{1}{2}$-hr. lessons a week for 18 weeks.
33. KETC, St. Louis. Washington University,† Ross Middlemis, Holbrook MacNeille, Margaret Elliott.	*College algebra and trigonometry.* 5 credits. 45 min., 5 days a week, twice on open and 3 times on closed circuit. See project **20** also.
34. WTTV. Indiana University, Philip Peak.	*High school trigonometry.* $\frac{1}{2}$ unit. 18 30-min. shows to provide the course for small high schools.
35. KUON-TV. University of Nebraska, David Wells.	*Beginning high school algebra.* Credit. 20 min. a day, accompanying correspondence study material. Fund for the Advancement of Education.
36. WTTW. Chicago Public Schools, Jerome Sachs.	*Ten experimental lessons on quadratics for high school students.* See Appendix B, [**11**]. See project **8**.
37. Alabama Educational Network.‡ University of Alabama, Ayrlene Jones, Susie Lee Ward.	*College Preparatory Algebra.* No credit. 12 weeks, 30-min. lessons. Summer 1956 and 1957.
Ayrlene Jones.	*Trigonometry.* Credit. 16 weeks, 30-min. lessons. Fall.
Ayrlene Jones.	*College Algebra.* 16 weeks, 30-min. lessons. Spring.
	Mathematical Potpourri. Popular lecture by various staff members. 15 weeks.
38. Hagerstown, Md.	*Plane geometry.*

1955 Projects

Station-Sponsor-Personnel	*Topic and/or Purpose-Length*
39. WHA. University of Wisconsin, Professor Sara Rhue.	*Arithmetic.*

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* See this issue, pp. 430–439.
† See this issue, pp. 440–443.
‡ See this issue, pp. 421–424.

Station-Sponsor-Personnel	Topic and/or Purpose-Length
40. KUHT. University of Houston, School of Business Administration, Rolland Crouch.	*Mathematics of Finance.* 3 credits, 45 min. twice (?) a week.
41. KTTS-TV, Springfield, Mo. Springfield Public Schools, Robert C. Glazier.	2 *arithmetic demonstration lessons.* 30 min. each. 2 *Why Mathematics?* 30 min. each. See Appendix B, [**10**].
42. WCET, Cincinnati. E. Havlish, Cincinnati Milling Machine Co., and public school students, arranged by Mildred Keiffer.	*Measurement in Industry.* 3 25-min. programs designed for junior and senior high school classes to show the importance of mathematics in industry.
43. WOI-TV, Ames, Iowa. Iowa State Teachers College, Cedar Falls, Irvin Brune.	*The Teaching of Arithmetic.* 10 90-min. programs for college credit. See Appendix B, [**10**].
44. WKAR-TV, East Lansing. Michigan State University, B. M. Stewart.	*Nomography and graphical methods.* 12 30-min. programs.
45. WQED, Pittsburgh. Pittsburgh Public Schools, Mrs. Anita Seewald.	*Fifth Grade Arithmetic.* 20 classes in 10 schools. TV teaching for 25 min. of the 40-min. arithmetic classes. 5 days a week for 178 days. Supported by Fund for the Advancement of Education. See Appendix B, [**10**].
46. WUNC-TV, Raleigh, N. C. North Carolina State College, Henry C. Cooke.	*Solid geometry.* Taught in summer. Noncredit but to fulfill college entrance requirements. 30 min. once a week, 16 weeks. This has been presented three times as of February 1958.
47. WRC, Washington, D. C. Louise R. Grover, Faith F. Novinger, Carl Hansen, pupils from public schools.	*Reading, Riting and Rithmetic on the High School Level.* 2 15-min. programs to show modern teaching techniques.

1954 Projects

Station-Sponsor-Personnel	*Topic and/or Purpose-Length*
48. WKRC-TV, Cincinnati. Teachers, students, Mildred Keiffer, Supervisor, Mathematics 7–12.	*Demonstration lessons for N.C.T.M. meeting.* 3 30-min. programs. Twelfth, eighth, fifth, and second grades. See Appendix B [**8**].
49. Kinescopes. University of Michigan, Phillip S. Jones.	*Understanding Numbers.* 7 30-min. kinescopes made for The Educational Television and Radio Programming Center and widely distributed by them. Available for audio-visual use from Indiana University Audio-Visual Center. See also projects **63**, **67**, and Appendix B, [**4**], [**5**].
50. WREX-TV, Rockford, Ill. Northern Illinois University, DeKalb, Ill. Herbert Miller.	*Magic Squares.* One 15-min. program. Appendix B, [**9**].
51. WQED, Pittsburgh. Jack Roush.	*High School Algebra.* In preparation for state high school credit examinations—18 lessons, 30 min. each. See also project **18**.

52. KETC, St. Louis. Panel discussion: Dr. Harold Fawcett and Mr. Eugene P. Smith of Ohio State University and students from Ohio and Missouri high schools, arranged by Dr. Margaret Willerding.	*Should I Study Mathematics?* One 25-min. kinescope. Originally made in connection with the Dec. 1954 meeting of the N.C.T.M. in St. Louis. This is program 1 in Series 1 of "High Time, a Panel Discussion Program for High School Students," designed for in-school viewing and accompanied by a published "High Time Study Guide."
53. Seattle, Washington.	*Demonstration classes* for public relation purposes.
54. WGAL-TV, Lancaster, Pa. Elizabethtown College, Carl Heilman.	4 30-min. programs: Contributions of Pythagoras, Fermat, and Pascal, Descartes, and an Introduction to Nomographs.
55. WFIL, Philadelphia, Pa. Bryn Mawr College, Marguerite Lehr.	*Invitation to Mathematics.* 15 27-min. programs. *E.g.*, "Regular Patterns and Symmetry," "Regular Shapes," "Products and Primes," "Matching and Counting, Big and Infinite," "Maps, Terrestrial and Otherwise," "On Measure of How Likely," "Questions on Least and Most." See Appendix B, [**6**].
56. WTAV, Newark. Rutgers University, Fred G. Fender.	*This is Mathematics.* 13 programs. Number systems and number theory, greatest and least, transfinites. See Appendix B, [**7**].
57. Marquette University. William A. Golomski, H. P. Pettit.	*Statistics, Geometry.*

1953 Projects

Station-Sponsor-Personnel	*Topic and/or Purpose-Length*
58. Illinois Institute of Technology. Karl Menger.	Two programs.
59. WGAL-TV, Lancaster, Pa. George R. Anderson.	*Let's Look at Mathematics.* 16 programs. Appendix B, [**1**], [**2**].
60. Texas State College for Women. Harlan C. Miller.	One noon-hour show—Moebius strip, *etc.*
61. WATV, Newark. Ernest R. Ranucci.	*Spotlight on Mathematics.* 4 30-min. programs. See also project **23**.

1952 Projects

Station-Sponsor-Personnel	*Topic and/or Purpose-Length*
62. WBEN-TV, Buffalo, N.Y. Buffalo teachers, 10–14 students, commentator Louis F. Scholl, Director of Mathematics.	*Demonstration lessons.* Chiefly grades 1–8. Public relations. 30 min. See Appendix B, [**3**], [**14**]. These lessons were continued through 1955–1956.
63. WWJ-TV, Detroit, Michigan. Phillip S. Jones, University of Michigan.	*Understanding Numbers: Their History and Use.* 7 18-min. programs, "The Earliest Numbers," "Bases and Places," "Big Numbers," "Fundamental Operations," "Short Cuts and Computing Devices," "Fractions," "New Numbers." See also project **49** and Appendix B, [**4**], [**5**].

64. WOW-TV, Omaha. Creighton University, John Englund.	*Counting—Then and Now.* 15-min. program. Also shown over this station in 1955 and over KOLN-TV in 1954. See Appendix B, [**24**].
65. WGAL, Lancaster, Pa. State Teachers College, George R. Anderson.	*The Slide Rule.* 6 30-min. programs. See Appendix B, [**1**], [**2**] and program **68**. *The Conic Sections.* *Visual Devices in Mathematics.*
Lee E. Boyer.	*Logic in Mathematics.* *Mathematics of a Lampshade.*
66. WOOD, Grand Rapids, Mich., University of Michigan, Phillip S. Jones.	*Mathematics Around Us:* I Numbers, II Geometry (Conics, Parallelograms). Two 27-min. programs.

1951 Project

Station-Sponsor-Personnel	*Topic and/or Purpose-Length*
67. WEWS, Cleveland, Ohio. Cleveland teachers, 8–10 students. Commentator. Herschel Grime, Directing Supervisor of Mathematics.	*Demonstration lessons.* Primary, upper elementary, grade 7 (Mental Arithmetic), grade 8 (Informal Geometry), grade 10 (Plane Geometry, Micrometer), grade 10 (Shop Mathematics Class, slide rule in a Trigonometry class). 15-min. programs. 2 each year to the present time.

1949 Project

Station-Sponsor-Personnel	*Topic and/or Purpose-Length*
68. WGAL-TV, Lancaster, Pa. State Teachers College, George R. Anderson.	*The Slide Rule.* Six 15-min. programs. See Appendix B, [**1**], [**2**].

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SPIES, ELECTRIC CHAIRS, AND HOUSEWIVES

PAUL L. CHESSIN,* Institute for Fluid Dynamics and Applied Mathematics, University of Maryland

In this article I write of my personal experiences arising from three television shows given on WFIL-TV, Philadelphia, January 8, 15, 22, 1957. Whereas many of the papers in this issue of the MONTHLY are devoted to various campus studies and are statistical in nature, my experience differed considerably. Three reasons for writing this chronicle are: (a) to acquaint the readers of the MONTHLY with my activity and thereby encourage them to enter this field; (b) to set forth a plan for a use of television in mathematical promotion; and (c) to learn from those who may respond to this article and therefore benefit doubly. The title of this paper will be explained presently.

Professor Francis A. C. Sevier, lately chairman of the mathematics department, College of South Jersey (Camden), having successfully presented a series of fifteen-minute programs over WFIL-TV, 1955–1956, was asked to supervise, for the academic term 1956–1957, that station's "University of the Air Series: A Survey of the Sciences." Throughout the semester guest speakers discussed topics in a variety of physical sciences and it was my lot to conclude the series with three talks on mathematics.

During the summer of 1956 I prepared scripts for the three talks. My subject was to be "Mathematics in Our Western World" wherein I hoped to talk about "The History and Development of Western Mathematics," "Mathematics and Human Institutions," and "Mathematics *Is* Fun."

Early in September, the participants in WFIL-TV's "Series" were assembled at the studios to be briefed on the mechanics of the show. We were told that the "University of the Air" had been telecast for seven years; that between 9 and 10 A.M. each weekday morning three "classes" are held; that the 9:20–9:45 class "A Survey of the Sciences" was chosen for rebroadcast by kinescope films

* Science Faculty Fellow, National Science Foundation.

over affiliated stations: WFBG-TV (Altoona), WNBF-TV (Binghampton), WNHC-TV (New Haven), and WLBR-TV (Lebanon); that approximately two or three million people would view the class (at least at the outset—this was *our* challenge!). This last bit of information caused me some personal discomfort.

Specific instructions dealt with what type of stage properties could be used (slides, charts, models, film strips, blackboards, *etc.*), dress, and stage presence. With respect to the last item, the director evidently felt that a natural, unpretentious attitude was necessary if we were to hold our audience. To this end, we were given no further rehearsals, makeup tests, or plan of action regarding camera movement. The director emphasized that unless informality prevailed, the viewer could instantly replace us with a mere twist of the wrist. Since I was to appear four months later, I promptly prepared strenuously for my "extemporaneous" talks. In this respect I was perhaps more fortunate than the others, who, lacking adequate time, resorted mainly to reading from prepared texts.

At this point I should like to indicate my aim in presenting such an austere sequence. Briefly, I hold that mathematics and the current well-known situation in mathematical education (let's face it, I mean apathy on the part of students and public in general) can be improved if a well-planned promotional scheme can be brought to fruition. I feel that if mathematics—its history, its development, its *raison d'être*—could be properly presented to parents of high school students, say, then a more friendly attitude towards the subject would develop in the home. Any attempt to "sell" mathematics to students during school hours will fail if the after-school sentiment at home is ignored. I hope to awaken a interest by showing the adult lay public that mathematicians are human, that work in mathematics can be rewarding and gratifying, and that hard work and willingness rather than genius are the necessary ingredients. With this awareness, perhaps parents and teachers will be able to motivate the new generation of students.

At any rate, with this hypothesis in mind, I gave my first talk on "The History and Development of Western Mathematics." I spoke about the "reckoning" phase of early mathematics, the specific knowledge of Egyptian society and the "new" concepts introduced by Greece. I attempted to trace the threads of the mathematical cloth through the dark ages, the Renaissance, and classical times. From *Scripta Mathematica* I obtained the well-known portfolio of portraits which were viewed at appropriate intervals (thus I hoped to add some interest to an otherwise lengthy talk).

(Part of this paper's title may be explained at this point. As an illustration of the sort of thinking sometimes involved in mathematics, I posed the following problem (referring to a picture which showed a man, question mark over his head, gazing at two identical electric chairs behind which stood two executioners):

Unless this man can ask a suitable question and act properly, he is doomed;

for one of these chairs is wired while the other isn't. Of course, the executioners know which it is. Also, one of these men is a truth-teller and the other a persistent liar. Again, each executioner knows who is the liar. The victim is allowed to ask of either man (but not both) just one question whose answer is either "yes" or "no." What question should he ask?

I received an unexpected volume of mail in response to this puzzler and to an offer to send a bibliography and other material. I'll discuss this later.)

Of course, without a rehearsal, it was bound to happen: some portraits did not agree with the individuals under discussion. Throughout the talk, the emphasis was on the men and ideas rather than the techniques developed (I hoped to show in the second talk how the rigor and beauty of mathematical thinking affected men's ideas in politics, economics, art, literature, and music). Unfortunately time ran out before I could demonstrate that twentieth century mathematics exists and is flourishing!

Two points ought to be made now: rehearsals are necessary to get the timing set perfectly (I needed two minutes to run out of material); it is too ambitious to reduce twenty-five centuries of history to twenty-five minutes. Possibly, in talking with professionals in this field, I would have come to realize this.

Unexpectedly, mail came in to the studio which reflected a mild interest in what I had said. However, it was suggested that I review my scripts with an eye to toning down the level. To this end I decided to discuss mathematics and art (demonstrating with the use of slides from the Philadelphia Museum of Art the idea of Greek simplicity, Latin ornateness, pre-renaissance "flat" paintings, perspective and the development of "depth," and finally the ultra-modern movement), mosaics and line drawings (illustrated by the well-known charts of Colonel R. Beard), mathematical influence in architecture (I showed photographs of the Hotel Waikiki with its hyperbolic paraboloidal roof), and mathematics and music. To demonstrate simple applications of Fourier's work, I prepared on tape, recordings of various musical instruments and combinations. An oscilloscope was used to present sound visually. Various harmonic syntheses were performed with the purpose of demonstrating the usual elementary notions.

All this was planned to replace "Mathematics and Human Institutions" since I was told (secretly) to think of it now as "The Kindergarten of the Air Series." Now, about five minutes before going on, I was informed that only four of the dozen and a half slides could be used (the slides, being in color, wouldn't be telecast successfully, I was told)! More evidence that rehearsals are necessary, in spite of the admonition to be "natural."

I found that it was not too difficult to ad-lib half a program; but I would have preferred more advanced notice. Certainly the program content was out of balance. Generally, the views of the oscilloscope were poor, there being insufficient contrast on the 'scope face. Significantly, Professor Sevier, who introduced me each time and concluded each performance, noted publicly that that was the first class that ended "on time."

Regarding this hastily revised second program, I feel that the viewing public showed more interest in the light of its response and the general sentiment expressed by professional TV men. At any rate I decided to conclude the series as planned, demonstrating that "Mathematics *Is* Fun."

Once again poor program planning interfered with my plans. Just before going on, I was told to cut short my talk by five minutes in order that concluding credit announcements might be made and the moderator for the succeeding series might be introduced to outline his forthcoming fifteen week series. In this last program I discussed a few elementary algebraic puzzles, the Mobius strip with some demonstrations, another elementary topological pleasantry, and some paradoxes. I hope I astonished some people by calculating the thickness of a sheet of paper .001 inch thick folded in half fifty times. Some of Colonel Beard's art work was shown and explained and wax-paper folding exercises were demonstrated. The wax-paper creases came through with brilliant clarity, surprising the studio engineers. All told it was (for me at least) a pleasant romp through the field of mathematical recreation. Since many of the "answers" to the electric chair problem received during the past two weeks were incorrect, I repeated the problem and pointed out the inadequacy of such simple answers as "ask either executioner if he is the liar," "ask either if his chair is wired," *etc.* Time, however, did not permit my giving suitable answers.

I tried to avoid giving the impression in this last talk that mathematics is "just a collection of tricks," but does in fact have its lighter side. It was an attempt on my part to motivate an interest in mathematics by way of its entertainment value. The mail indicated that to a limited degree I succeeded.

This, then, is a chronicle of some seventy minutes I spent on television talking mathematics to an "adult" audience. To this of course was an aftermath: my "fan" mail. I received many letters and cards from people of diverse backgrounds ranging from truck drivers who "caught the show" in diners to housewives who were "relaxing after sending the children to school"—explaining another part of this paper's title. Most of the letters contained answers to the electric chair problem. Almost all letters requested the bibliography and more information about recreational mathematics. A few offered suggestions for improvement. Several hoped that the math series might be continued.

The following illustrations should be self-explanatory:

"Thank you for having thrown a new *light* on mathematics. I would appreciate your sending me further information on 'mathematical curiosities.'" (Philadelphia housewife).

"University of the Air is doing an excellent job and I want to thank all of the participants." (Philadelphia homemaker).

"Would it be possible to obtain a leaflet or paper listing some of items for a Girl Scout troop?" (Ventnor, New Jersey scoutmaster).

"Would asking one of the men to try the chair out first solve the problem? Could you send me a copy of some of the mathematical designs to use for scout work (girls) . . . I thoroughly enjoyed your program." (Upper Darby, Pennsylvania mother).

"Thank you for the mathematical puzzles. My husband and I are both brain weary after attempting to solve them." (Philadelphia).

" . . . I would ask the men to sit on the chairs. The man who refuses to sit down is standing behind the wired chair. Am I right . . . I am curious and interested." (Kutztown, Pennsylvania woman).

" . . . My husband would like the solution to the electrocution puzzle. I am enclosing a self-addressed stamped envelope. Thank you for the enjoyable minutes we spent while you were on the air." (Gibbstown, New Jersey).

" . . . I have a son twelve years old who enjoys mathematics and would appreciate your sending him some mathematical quizzes." (Philadelphia).

"Please send me the information about geometric math puzzles." (Seventh grade teacher, New Jersey State Teachers College).

"I enjoyed the last lecture of the Survey of the Sciences (and wish there were to be more of the same type later). Is there a book I can get on the subject discussed?" (Moorestown, New Jersey viewer).

" . . . Thank you for the information and entertainment your program has provided." (Glenside, Pennsylvania boy).

" . . . Will you please send me these designs (Col. Beard's). I am an amateur artist and I teach children in my home. These children would enjoy drawing these designs." (Collegeville, Pennsylvania woman).

"If I asked you the following question would your answer be 'yes.' Is your chair the one that is wired?" (Engineer at WFIL-TV studio—devised during show). Explanation of solution and correct procedure outlined.

I have quoted some of the letters received to show what interest the recreational approach can achieve. Purists may snicker at this view of "mathematics," but a promotions man is more interested in ends, not means. Interest *was* evoked. Parents *are* becoming interested. How long will it be before their children are infected?

One or two other points ought to be made now. Many solutions to the electric chair problem depended upon elaborate devices (voltmeters, cut-out switches, *etc.*) and overlooked the point entirely. Some letters followed the pattern " . . . I didn't get the folding of the paper that would measure to miles and miles, but the figures are tricky and perhaps my hubby and son would enjoy trying to make some of them (in fact I know they will). So sorry I can't figure out the problem of which chair is wired and what one isn't. But I do believe the one who pulls the switch isn't a good Christian." (Holmes, Pennsylvania mother.) Several other letters posed problems for *me* to answer! Most gratifying were the replies that said essentially:

" . . . It seems to me that the prisoner could choose the non-lethal chair by asking the question: does the liar have the wired chair?" (Riverton, New Jersey housewife). This lady went on to give a neat and complete solution indicating her reasoning graphically and logically. She concludes by writing "If this is the correct answer to the problem, the credit goes to Dr. Sevier, who can teach anyone mathematics—even housewives." Can more be said?

Though it took time, I replied to each letter. *Scripta Mathematica* was sent a mailing list of people interested in knowing more about the philosophical and recreational sides of mathematics. With each response, aside from bibliography, I included an additional set of posers (without answers), one of which, to explain the last item in this article's title, was the following: A spy was

caught behind enemy lines one Sunday morning. Placed in a cell, he was told that late one night he would be executed in his sleep—in fact he would never see the following Sunday afternoon. Moreover, he would never know which evening was to be his last. Question: On which night will he die?

I received almost as many letters in response to this latter set of puzzlers and in particular in reply to the spy question as I did originally. Thus my correspondence did not drop off until the end of the second cycle.

Very significant, I have been told, is the fact that no derogatory letters were received. It appears to be well known to workers in the field that the ratio of negative to positive mail is enormous. Here then is a situation that contradicts the usual expectation.

This narrative then describes my experiences as an amateur television educator. In trying to stimulate an awareness of mathematics in the minds of adult laymen, an area of utmost importance if we are to be successful in motivating the young students of today and tomorrow, I have made an attempt. I hope I have convinced some of the readers that this is a worthy goal and that television can be used successfully as a means for bridging the gulf between the academicians and the laity. That experience in television acting is necessary to me now seems trivially obvious. Perhaps, as others have suggested, scripts should be written by mathematicians with professional actors employed to perform. I feel that with sufficient rehearsals, however, non-professionals can do a good job.

That care is necessary in preparing scripts is quite clear. With competent advice from others, I would never have put on that first telecast. I prefer a historical to a topical matrix in which to delineate mathematics. But in truth one must realize that each program, independent of the others, must be self-contained and concentrate on a single item.

It is true that such a series as mine is not altogether unique in its wide-spread audience, yet most of the current work in this area lay in closed-circuit, educational broadcasts for and by academic people. I invite others to join those of us who are currently trying to promote mathematics over open television. The kinescopes of my three talks are available on loan for any reader who wishes to view them. I would appreciate in return having the opinions and suggestions of those who do see them.

TELEVISION ACTIVITY, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA

AYRLENE McGAHEY JONES, University of Alabama

The Department of Mathematics of the University of Alabama has maintained a wide and varied program of television activity since it began its first series of telecasts in the summer of 1956. This program has included a series of

popular lectures on several topics in mathematics, review courses in high school mathematics, college courses in algebra and trigonometry, an experiment in teaching trigonometry by closed-circuit television and a series of topics in high school mathematics for enrichment. With few exceptions all members of the department have participated in these programs. Not only has the medium been used to reach larger and more varied groups than the campus classroom allows, but it has afforded an opportunity of using and experimenting with visual aids which are not usually available for classroom use.

Alabama Educational Television Network. All of the programs and courses offered in mathematics by the University of Alabama (except the closed-circuit campus experiment) have been broadcast over the Alabama Educational Television Network consisting of three transmitting stations: WTIQ, Munford, Channel 7; WBIQ, Birmingham, Channel 10; and WAIQ, Andalusia, Channel 2. These stations, comprising the first educational television network in America, are owned by the state of Alabama, operated by the Alabama Educational Television Commission and reach approximately 80% of the state's population. There are four program agencies under contract with the Commission: University of Alabama, Alabama Polytechnic Institute, Alabama College and Birmingham Area Educational Television, Incorporated. Studios are linked in a statewide network via micro-wave facilities maintained by the Commission.

Popular lectures. This series of programs on various aspects of mathematics was planned primarily for gifted high school students and their teachers, college students and the general public. Fifteen members of the department participated and discussed such topics as: "Fallacies and Paradoxes," "Concepts of Infinity," "Electronic Computers and the Binary Number System," "Voting Power," "Zero is Not Nothing," "Orbital Missiles," "Alabama Paradox," "Number Congruences," "Professional Opportunities in Mathematics," "Games and Recreations," "What is Mathematics?" and others.* Since this series was well received during the spring semester of 1956–57 it was continued throughout the fall semester of 1957–58. The series was called *Mathematical Potpourri* and was telecast weekly.

College courses. Two departmental courses, college algebra and trigonometry, were broadcast on the network during the fall and spring semesters of 1956–57. They were also placed on campus closed-circuit to afford an additional opportunity of hearing lectures for those taking the courses on campus and others wishing to review the material. Trigonometry was offered for credit to qualified viewers off campus but only auditors registered for the course. These received outlines and lecture notes. Teachers in the several colleges and high schools in Alabama (and one across the line in Georgia) have indicated that some of their students supplemented their own courses in trigonometry and

* A complete list of topics with abstracts is available upon request. Kinescopes of several of these programs are also available for loan.

algebra with these telecasts. Also, many students enrolling as freshmen at the University have stated that they followed the courses at home. These were broadcast three times each week during the evening.

Closed-circuit experiment in teaching trigonometry. This experiment involves one control class and two television classes. The television teacher lectures approximately thirty minutes and graduate students conduct discussion of the lecture for the remainder of the period. Items under study include: the role of the graduate student in instruction by television and the effectiveness of this type of teaching apprenticeship, and the feasibility of using this medium to insure the instruction of large numbers of freshmen by experienced members of the department. It is expected that an evaluation of the experiment which is being conducted during the spring semester of 1957–58 will be made during the summer of 1958. This experiment is a part of the Study and Planning Program of the University of Alabama financed by a Carnegie grant.

Review courses in high school mathematics. During the summers of 1956 and 1957 a course in college preparatory mathematics was offered three times each week. The course was planned primarily for high school graduates preparing to enter college, for high school students continuing to study mathematics and for mature people wishing to review or study mathematics as a part of their general education. Arrangements were made for students to report to various centers throughout the state at the conclusion of the telecourse to take entrance examinations or examinations to remove high school deficiencies in mathematics. An outline of the course and problems were sent in installments, without charge, to all who requested them. Mail from throughout the state indicated an interest in the course by people of varied backgrounds and occupations, *e.g.*, a laboratory technician, a doctor, a housewife, parents and a teen-age boy. Some sent in homework papers to be corrected.

High school enrichment programs. Weekly throughout the academic year 1957–58 various members of the department are lecturing on topics selected by high school teachers in the areas of geometry, algebra and general mathematics. These are a part of the In-School Program offered by the Alabama Educational Television Network and are telecast during school hours. Topics include: "Transition from Arithmetic to Algebra," "Fundamentals of Geometry," "Problem Solving," "Compound Interest," "Functions and Graphs," and others.* Surveys indicate that 230 schools in the state are viewing some of these in-school programs.

Organization and production. The departmental television committee consists of the fifteen members participating in the various television series with the committee chairman acting as coordinator of all television activity and pro-

* The study guide for these programs is available on request. Kinescopes of some of these lectures are also available for loan.

ducer of the various programs.

Since the Broadcasting Services of the University include a complete workshop with personnel interested in creating new visual materials, it has been possible to experiment with many kinds of production aids. This has resulted in greater interest and pleasure in working with this new medium.

Tentative evaluation. Members of the department are unanimous in the opinion that teaching techniques are improved as a result of the careful planning television requires; also, one becomes increasingly aware of his students when they are replaced by the red eye of the television camera.

There is a strong feeling that television is best used for enrichment or as a supplement to conventionally-taught courses rather than a substitute for them.

Professor Julian D. Mancill, Head of the Department of Mathematics, states: "We are certain that television presents a challenging new medium of education and that it will experience a rapid development. However, we are just as certain that it will not, since it cannot, replace the classroom teacher in the sciences and especially in mathematics. There seems to be an unlimited potential for educational television in the areas of continuation education; enrichment of traditional courses through access to outside experts, animation and art; and general informational type of instruction."

THE GEOMETRY OF FOUR DIMENSIONS

A. ZAWROTSKY, Universidad de los Andes, Mérida, Venezuela

The following is a brief description of a film* that was made in order to facilitate the visualization of the geometry of four dimensions.

A tessaract was chosen as a representative polytope of four dimensions. The film represents 988 successive intersections of a tessaract by parallel hyperplanes (each photographed three times). The hyperplanes, however, are not perpendicular to a main diagonal, but form with any set of four concurrent edges of the polytope, angles of

$$\sin^{-1} 0.384, \qquad \sin^{-1} 0.48, \qquad \sin^{-1} 0.512, \qquad \sin^{-1} 0.6,$$

respectively. This method, different from that of Dunchian described in Coxeter's well-known book *Regular Polytopes*, was chosen in order to give the pictures greater variety.

* This film has been offered for exhibition at the International Congress of Mathematicians to be held in Edinburgh, August 14–21, 1958.

A NEW PLAN FOR INSTRUCTING LARGE CLASSES IN BASIC MATHEMATICS

MARTIN WRIGHT, University of Houston

In January of 1957 the Mathematics Department of the University of Houston began preliminary study of an experiment designed to attack the problem of teaching an increasing number of freshman mathematics students without a corresponding increase in faculty and physical facilities while maintaining high academic standards in these basic mathematics courses. All known plans of instruction at other Universities that utilized video in any form were investigated before a decision was made on the details of the proposed experiment. Plane trigonometry was selected as the subject to be taught because of the large number of students enrolled therein. The main features of the plan used in the fall semester of 1957 to teach plane trigonometry to 350 freshmen students are listed below:

1. The course was divided into twenty-seven lectures of forty-four minutes each. All lectures were carefully checked by the Mathematics Department and filmed by the University Film Center. Six members of the Faculty participated as lecturers while others contributed as critics.

2. Lectures were given to the students at the rate of two per week. Each film was shown twice over open circuit television (morning and evening) and three times by projectors in viewing rooms of the Audio Visual Center (afternoons and Saturday mornings). Thus a student was given an opportunity to view each lecture several times if necessary to master the theory and problems.

3. All students enrolled for credit were supplied with a television supplement which contained routine instructions for the course, assignments, additional explanations and *an incomplete set of notes on all lectures*. The student completed the notes in this supplement as he viewed the lectures.

4. All credit students were divided into sections of 25–36 students and each section was required to meet one hour per week with a regular member of the Mathematics Department. These conference sessions were used to discuss lectures that had been viewed recently by the students, to answer any questions arising from these lectures, to distribute and collect homework assignments, and to review for examinations.

5. Ten hours per week of voluntary help sessions were scheduled in the afternoons and evenings, with a student assistant in charge, to answer questions about the solution of specific problems.

6. The examination schedule included two major examinations (midterm and final) and two minor examinations. The major examinations were two hours long and were given at an open period to the entire group. The

minor examinations were one hour long and were given in the weekly conference sessions.

Each major examination was preceded by a live television review, the material for the reviews being obtained from questions most frequently asked in conference sessions. Practice tests were distributed to students and discussed in weekly conference sessions prior to all examinations.

This plan of instruction deviated so radically from conventional procedure that an extensive orientation program was necessary for both students and faculty counsellors in order to assure a smooth operation.

Three hundred forty-three students participated in this program in the fall semester of 1957 and three hundred ninety-four are presently enrolled. No conclusion can be drawn at the present time as to the effectiveness of this plan of instruction. However, data is being collected for an evaluation of the program.

CLOSED-CIRCUIT COLOR TELEVISION

RICHARD A. GOOD, University of Maryland

The University of Maryland and the Advisory Board on Education of the National Academy of Sciences announce the experimental use of color television in the teaching of a graduate course, Foundations of Analysis, offered by the University. The closed-circuit color facilities at the Television Division of Walter Reed Army Medical Center are employed in teaching this course to a group of inservice high-school teachers of mathematics and science in the Washington, D. C. area. Associate Professor R. A. Good of the University is lecturer for the course and Dr. J. R. Mayor, Director of Education, AAAS, is serving as consultant. A grant from the Fund for the Advancement of Education of the Ford Foundation has made possible the production of color kinescopes which will be used in the comparison of various techniques. These kinescopes will later be made available to other suitably equipped institutions for further evaluation tests.

Twenty-six weekly lectures of approximately 50 minutes each (usually rehearsed at least twice on camera) are presented via television. About every third week the instructor meets the students in a regular class for examination, evaluation, recitation, or answering student questions. The course, carrying three credits, is still (at the time of this writing) in progress. The class is small in size. For these two reasons no significant evaluation of the project is presented

here. An independent agency will continue the evaluation during several classes via kinescope proposed for the 1958 summer session and next year.

The course stresses modern ideas concerning the background and basic concepts of the calculus. Set, relation (transitive, order, and many other kinds), function (as a type of relation), partition, continuity (defined by neighborhoods), limit, derivative, integral are highlights in the syllabus. To graphically portray relations, we have studied the fundamental notions of analytic geometry. Thus the course embraces more than just calculus concepts. In order to devote full justice to foundations, physical applications are not emphasized.

A major contribution to the teaching efficiency is being made by the very cooperative staff of the Television Division, particularly in connection with the art work. Many graphs, diagrams, charts are prepared in advance. Instead of the customary blackboard, we currently use colored chalk on a continuous roll of heavy paper which can be turned forward or backward. Whenever suitable, we use peg board, flannel board, magnetic blackboard, teleprompter (on camera, for lengthy definitions or statements of theorems). Color affords an excellent method to contrast, to emphasize, to distinguish. During a recent graphical portrayal of ϵ and δ, three studio assistants were employed. On a cardboard of unobtrusive color, dark coordinate axes and a brightly colored curve represented the function. Colored light from a projector gave a horizontal band 2ϵ units wide; variable shutters permitted ϵ to be arbitrary. Another projector with movable shutters beamed a vertical light zone of appropriate width 2δ. The third assistant could, whenever needed, overlay the graph with a transparent sheet having a vertical black-out line. We continue to experiment with any reasonable idea for making more understandable the delicate nature of our subject matter.

TV AND FILMS AT MONTCLAIR

B. E. MESERVE AND E. M. MALETSKY, State Teachers College at Montclair, New Jersey

The New Jersey State Teachers College at Montclair was among the first to experiment in the preparation of classroom lessons involving the use of television. Working with equipment furnished by the Allen B. DuMont Laboratories, Inc. and later with funds made available from the Allen B. DuMont Foundation, the college began in 1950 to explore and test the possibilities of television in education.

In 1952 the F.C.C. allotted UHF Channel 77 to be located at Montclair and

initial plans were made to utilize this facility. There was extensive closed circuit experimentation to college classes and to classes in the demonstration high school. On April 30, 1952 there was a full day of UHF broadcasting of planned lessons to thirteen schools in Bloomfield and Montclair, New Jersey. This was the first UHF broadcast into classrooms. It was made over the New York educational channel assigned to DuMont.

In the spring of 1954 the New Jersey State Legislature failed to authorize funds to continue educational television station WTLV which the State Department of Education had operated in New Brunswick for the previous six months. Much of the equipment and staff of this station were transferred to Montclair. The college then embarked on a detailed plan of experimentation with the aid of a grant from the Fund for the Advancement of Education of the Ford Foundation. However, after some preliminary work on the plan, it became apparent that the objectivity inherent in the college's concept of research was not compatible with the point of view of the administrators of the Fund for the Advancement of Education and the project was abandoned.

Montclair staff members prepared a two-week series of daily broadcasts of fifth-grade lessons to the public schools of Long Branch and Red Bank, New Jersey in May and June 1954. This was done over UHF station WLTV at Asbury Park assigned to Walter Reed Theatrical Enterprises.

The college first offered college credit courses in the use of television in education in 1950, when an educational television workshop was organized. The primary purpose of the workshop is to encourage students to explore the new medium and to become aware of its potentialities and inherent difficulties when used as an aid to the teacher. The workshop serves both undergraduate and graduate students in all academic areas. There are two elective courses for undergraduates. Two additional courses are available for seniors and graduate students. Each course is for two semester hours. The students gain experience in planning programs, handling television studio equipment, and producing programs for the classroom. Each student develops a program in an academic area. Thus the students (future teachers) gain an opportunity to experiment with materials and actually to produce and evaluate programs on closed circuit TV.

There has been a modest amount of work in mathematics. For example, one instructor presented a lecture-demonstration on vectors to the freshmen mathematics majors via television. Students have presented lessons on geometric solids for high school classes. A demonstration on the construction and use of surveying instruments has been televised.

Last semester undergraduate students presented two closed-circuit TV programs to a history of mathematics class. These were visualized as typical programs under the title *Mathematics, Old and New.* A twenty-minute program, *Computing Machines, Old and New,* was designed to trace the development of computers, discussing in detail the sand table, abacus and desk calculator. Production difficulties made it necessary for the students to design special models of a sand table (done as a plaster mould) and an abacus. The other

student program, *Inventions of Napier*, included a discussion with models of Napier's rods and a development of his spherical triangle formulas. Evaluations of the programs indicated that both shows brought close-up views of the models to the class that could not have been attained in a normal classroom situation.

There have been many audio-visual activities associated with Montclair. Among the films are: (1) *Not By Chance*, a National Educational Association film on teacher training which has been shown throughout the country; (2) *Together for Children*, filmed for the New Jersey Educational Association showing the history of education in New Jersey; and (3) *Resources Limited*, dealing with conservation in New Jersey. A series of four films on *Patterns of Good Teaching* is now in progress supported by a grant from the Danforth Foundation of St. Louis, Missouri. This project is concerned with the filming of outstanding teachers in action.

Exploration with films and TV is continually going on at Montclair even though at a modest rate. There is a trained staff. There are opportunities for students to gain considerable experience. Should expanded closed-circuit or UHF transmission facilities become available in this area, these students would be ready to initiate and make effective use of TV programs.

The history of TV at Montclair seems to reflect the problems of many efforts at educational TV—an impressive start, inadequate funds, very modest achievements in specialized subject areas such as mathematics, continued hard work and achievements by a small group of capable local staff members.

BASIC HIGH SCHOOL MATHEMATICS

JOSEPH SPEAR, Northeastern University

An engineering corporation located near Boston, Massachusetts, desired to have certain of its employees from the machine shops and the drafting and engineering departments review some basic mathematics in order to prepare themselves for further study in mathematics and to upgrade themselves in their work. As a result, Northeastern University is now giving a weekly half-hour program reviewing basic high school mathematics over WGBH-TV, the educational broadcasting station in Cambridge, Massachusetts.

This is the first time that a regular course in mathematics has been presented live over TV in this area. It is entirely an experiment and, to our great satisfaction, we are finding tremendous interest throughout the listening audience. This is our twelfth week* and already we have received comments and questions from about 1200 persons. These include teachers and principals of secondary schools, high schools, and colleges; lawyers, doctors, ministers; in-

* This article was written in January, 1958.

dustrial firms; engineers, soldiers, college graduates; and many mothers and school students.

We have found that the half hour is too short a period to discuss the topics without hurrying a bit. The course was originally intended as a review for those who needed it in their work, but we have found that many others, out of school for many years, are enjoying it, listening to it regularly, and interested in refreshing their memories. Much attention to the broadcasts is paid by students presently in high school and by their parents. Many other adults, perhaps for the first time, are recognizing that mathematics is a live and interesting subject. In many cases, fears traditionally associated with mathematics are being dispelled.

Since this present series seems so satisfactory, we expect to give, early in the summer of 1958, a regular high school mathematics review course for teachers who are not teaching mathematics, but who have at some time studied it, and who desire to refresh their knowledge and may thus be able to fill the vacancies now open for mathematics teachers. These teachers will enroll in the regular course and follow the lectures over TV one hour a night, four nights a week. They will do assigned work and attend classes at the University on Saturday mornings for discussion, classroom work, and examinations. The course will run for six weeks and will carry some credit towards the Master's degree in Education.

TEACHING CALCULUS BY CLOSED-CIRCUIT TELEVISION*

JOHN DYER-BENNET, WILLIAM R. FULLER, WARREN F. SEIBERT, AND MERRILL E. SHANKS, Purdue University

Among the many questions concerning instructional television that have been raised in the past two or three years are several that deal with the feasibility of televised mathematics instruction. Related to this, the progress of Washington University's program of televised mathematics instruction is familiar to many teachers of mathematics. Some may also know of the calculus course currently being offered in the Washington, D. C., area under the auspices of the National Academy of Sciences. Both of these programs are of considerable interest, but neither has yet provided information to permit an empirical test of the effectiveness of such instruction.

We suggest that the following report is of interest for two main reasons. First, it supplies information gathered under conditions of adequate control

* This article contains much of the information to be found in the following mimeographed report: W. F. Seibert, A Brief Report and Evaluation of Closed-Circuit Television Instruction in the First Semester Calculus Course. Purdue University, Lafayette, Indiana, July, 1957. Copies of this somewhat more detailed report are available on request.

and helps to answer questions concerning the effectiveness of televised mathematics instruction. Second, both the methods of the present study and its findings are similar in broad outline to the majority of other studies of televised instructional effectiveness. Thus, the report can serve as an introduction to this area of educational research. Those with time and interest to devote to further examination of TV research studies can secure report copies from such widely distributed universities as Pennsylvania State [**1**], [**9**]; New York [**8**]; Toronto [**13**]; Iowa [**12**]; Miami, Ohio [**5**], [**6**]; Houston [**3**]; and Purdue [**10**], [**11**]. An inventory of many instructional TV research studies is also available [**4**].

The following note of caution may not be necessary but it reminds each reader that infallible criteria for measuring educational outcomes are so rare as to be practically nonexistent. In the present study, criteria have been used for which some validity can be claimed, but other, equally valid criteria might have been used and it is possible that their use would have had a noticeable effect upon the findings of the study. For example, an argument can be made for using measurements of long-term student retention of knowledge and skill rather than learning immediately measured, as in the present study, and something less than a perfect correlation can be assumed to exist between measurements of these two types. These ideas should be borne in mind as one examines the material to follow.

The Study

Purposes. The main problem to be studied is the comparative effectiveness of televised and conventional first-semester calculus instruction. In addition, results will be examined to determine if televised instruction has proved either more or less appropriate for students whose past academic records were superior, average, and inferior. Attitudes will be measured to determine student feelings toward TV instruction and the ways in which these feelings are influenced by experience in a televised course.

Instructional Conditions. At Purdue, as at many other institutions, first-semester calculus is a four credit-hour course normally taken during the student's second year in college. Students in the course are tested periodically (in this case, six tests were given) and homework is assigned regularly. Homework assignments are not carefully graded because of the time demands upon the instructor which this would entail but records are kept of the number of assignments that each student submits. Students are expected to attend four 50-minute class meetings per week.

Calculus is normally taught by means of small-class teaching methods, but information in the *Encyclopedia of Educational Research* [**7**], pp. 212–215) and experience in the use of the large-class (predominantly lecture) methods suggest strongly that they are as effective as the more customary small-class methods. In this study, large-class methods were employed three periods per week for instruction of the approximately 90 students in the control group. The fourth period of instruction was devoted to small-class meetings of about 23 students

each, under the direction of a graduate assistant. Use of the large-class method of control-group instruction made it possible to achieve experimental controls which would not have been possible otherwise. All students received the same examinations, administered simultaneously in experimental and control sections. The two principal teachers spent equal time teaching conventionally (large-class) and by television.

The four experimental sections each numbered approximately 25 students who met three times per week for instruction by television. A fourth meeting for these sections utilized the same instructional procedures and teaching personnel as were employed for the fourth class meeting of the control group. Each television student was assigned to one of four viewing classrooms. Each room contained an ordinary 24-inch television receiver mounted on a moveable pedestal with the center of the picture tube approximately $5\frac{1}{2}$ feet above the floor. Receivers were connected with the campus television studio by coaxial cable. No proctors or supervisors were regularly assigned to oversee the television classes but faculty visitors were occasionally present.

Instructors. Two teachers of professorial rank in the Department of Mathematics were responsible for the large-class and televised instruction. Both teachers have the Ph.D. in mathematics and many years' experience teaching the course under consideration. One was responsible for control-group instruction during the first half (eight weeks) of the semester while the other taught the experimental group by television. At mid-semester, the instructors exchanged teaching responsibilities.

Students. The group of experimentally-taught students numbered 103 and the control group numbered 91. Of these two groups, all but three were male, 90% were first semester sophomores, 69% were in the 18–20 year old age range, and 90% were engineering students. Eighty-seven TV students and 73 control students had taken the *Purdue Mathematics Training Test* prior to enrolling for the calculus course. The respective average raw scores of these two groups were 48.7 and 49.2. Cumulative grade-point indexes (used as a basis for matching or equating students from the two groups) were available for 91 TV and 77 control students. The respective averages were 4.37 and 4.54 (4 = C, 5 = B, *etc.*). These indexes were based on work that students had completed by June, 1956, and for the majority of students represent their average performance in courses taken as freshmen.

Criteria. Two types of criteria were used in the analyses of this study. Student achievement was measured by six tests administered during the semester. Student reactions or attitudes were measured by a ten-item questionnaire administered to most participating students both before and after the semester's instruction.

The six tests were written by the course instructors and graded by them and their assistants. The grading procedure was one which should minimize irrele-

vant or inconsistent marking methods. In advance of grading, standards were agreed upon by the graders and each grader accepted responsibility for one portion of the test to be marked. They then scored an assigned portion of each test paper, without regard to the student group from which it had come.

Achievement Comparisons. Sixty-one pairs of matched students served as subjects for the comparisons of achievement of students in the experimental and control sections. Each of the 61 students from the experimental group was matched with a counterpart from the control group, using the student's cumulative grade-point index as the matching variable. In the TV group, this grade-point index correlated .65 with students' summed scores on the six calculus tests. The correlation was .55 in the control group. None of the achievement-test score data was available or used during this matching operation.

At Purdue, students' cumulative grade-point indexes carry two places to the right of the decimal, *e.g.*, 3.47. Twenty-three perfect matches were obtained, *i.e.*, grade point differences of 0.00 existed between the paired students. Eleven differences were 0.01, ten were 0.02. No matched pairs exhibited differences greater than 0.07 and there was only one such difference. The grade-point indexes of these matched students ranged from 5.40 (almost midway between an A and a B) down to 3.27 (just above a D). Average values describing characteristics of the matched groups are presented below.

	Matched Groups	
	Experimental	Control
Age	20.6	21.3
Cum. gr. pt. index	4.14	4.15
Purdue Math. Trng. Test*	44.9	42.8

The matched groups may be accepted as fairly representative of the original groups from which they came except that matched groups are composed only of students who had attempted all six calculus tests and had established a cumulative grade-point index at Purdue. The original experimental group missed 6%, the control group 4% of the calculus tests.

Scores on each of the six calculus tests and the summed scores from all six tests have been compared statistically by means of the t-test between correlated means ([2], pp. 278–282). For results of these comparisons, see Table 1.

Student Reactions Analysis. As mentioned earlier, most students in experimental and control sections answered a ten-item questionnaire both before and after the semester's instruction. Each of the ten items provided students with four response alternatives: Agree; Undecided, probably agree; Undecided,

* The correlation between the *Purdue Mathematics Training Test* and students' summed scores on the six calculus tests was found to be .30 in the TV group and .28 in the control group. These coefficients may be compared with the .65 and .55 which were found to exist between students' cumulative grade-point indexes and the same summed scores.

probably disagree; and Disagree. For analysis purposes, these responses were dichotomized with the "agrees" and "probably agrees" separated from the "disagrees" and "probably disagrees". Also, the analysis of student reactions treats all available pretest and posttest student responses, rather than responses from matched students only. A summary of student responses to the questionnaire is presented in Table 2.

Results. Table 1 presents matched-group test-score averages for each of the six calculus tests and for the summed score from all six of these tests. It also presents the *t*-ratio for the observed differences between comparable averages, standard deviations of the score distributions, and test-score averages for students whose past academic records were superior, average, and inferior. The superior subgroups are composed of students whose cumulative grade-point indexes range from 4.33 to 5.40 ($N=20$, each group), the average subgroups are those with indexes from 3.95 to 4.31 ($N=21$, each group), and the inferior subgroups are those students who had indexes from 3.27 to 3.93 ($N=20$, each group). No statistical tests were applied in connection with these subgroup performance comparisons.

TABLE 1

COMPARISONS OF MEAN TEST SCORES, TV AND CONTROL GROUPS, AND RELATED INFORMATION

Student Achievement Variable	Mean Scores and Standard Deviations		"*t*"	Remarks	Sub group Comparisons		
					Past Academic Record	Mean Scores	
	TV Group	Control Group				TV	Control
Test 1	56.49 18.19	57.20 17.70	0.253	Not statistically significant	High Medium Low	64.7 59.0 45.7	66.3 53.4 52.1
Test 2	52.46 16.12	65.54 16.69	5.032	Significant beyond the .01 level of confidence	High Medium Low	64.1 48.6 44.9	77.0 63.2 56.6
Test 3	51.92 19.34	48.07 15.95	1.350	Not statistically significant	High Medium Low	59.4 54.0 42.4	59.5 44.7 40.2
Test 4	62.30 17.13	68.02 11.94	2.115	Significant beyond the .05 level of confidence	High Medium Low	70.2 63.3 53.4	70.5 69.4 64.2
Test 5	58.25 20.63	55.02 22.64	0.977	Not statistically significant	High Medium Low	63.7 61.6 49.2	63.7 55.4 45.9
Test 6	59.10 13.86	58.77 15.37	0.129	Not statistically significant	High Medium Low	63.0 58.1 56.2	66.8 57.1 52.5
Sum of 6 tests	340.51 74.52	352.61 75.95	1.079	Not statistically significant	High Medium Low	385.0 344.5 291.8	403.7 343.2 311.4

Referring to Table 1, it will be noticed that, of the seven tests of significance, two provide information sufficient for rejecting the null hypothesis. Both of these statistically significant differences favor the control group. Of the five remaining nonsignificant differences, two relate to the occurrence of higher scores in the control group and three relate to higher scores in the experimental group. It should be noted that the average summed calculus test score of the control group is 352.61, while the corresponding score for the experimental group is 340.51. The difference is not statistically significant.

An inspection of the subgroup averages presented in the last two columns of Table 1 reveals nothing to suggest that these averages require separate interpretation. If there is any consistent pattern in them, it would seem to be that the test performance of comparable subgroups follows the same pattern as that of the larger groups. Among the 21 pairs of comparable subgroup averages, eight show at least slight differences favoring the experimental group, one is a "tie", and twelve show at least slight differences favoring the control group. These findings are discussed later.

Procedures employed in the analysis of student questionnaire responses were described earlier. Findings pertaining to the questionnaire responses are summarized in Table 2.

In response to item 1 of the questionnaire, from two thirds to three quarters of the students indicated at least some feeling that televised classroom lectures did not instruct as well as conventional lectures. Differences between pretests and posttests and between TV and control students are generally small. On question 2 pretest, over half of the TV students felt that televised demonstrations instructed as well as conventional demonstrations; however, on posttest slightly less than half continued to feel this way. On question three, about two thirds of the TV student pretest responses indicated at least a tendency to believe that televised instruction is not as interesting as conventional instruction. More than three quarters of the group made this response on posttest. The responses of the control group to question 3 were very similar to those of the TV group.

On question 4, more than two thirds of the TV group pretest responses favored the view that television-taught students are more comfortable than conventionally-taught students. This percentage climbed to 83 on posttest. In the control group, only slightly more than half of the students gave similar responses. Question 5 can be summed up quickly. An overwhelming majority of students feel that the absence of discussion opportunities in TV classes is a serious limitation of the TV method used. Question 6 provides some interesting responses. On this item, about 40 per cent of the experimental group feel that TV students can take better notes than conventionally-taught students. In the control group, less than 20 per cent shared this view. Question 7 is another one which can be rapidly summarized. Most students agree that TV students do not get to know their instructor as well as students who are conventionally taught.

Item 8 produces some interesting responses. On both pretest and posttest, the experimental group was about evenly divided on the question of whether or not TV students are inclined to do more homework than conventionally-taught students but only a quarter to a third of the control group made similar responses. Item 9 again shows an interesting difference between responses of the experimental and control groups. Almost two thirds of the TV students began

TABLE 2

ATTITUDE ITEMS AND RESPONSE PERCENTAGES FOR EXPERIMENTAL AND CONTROL GROUPS

ITEM	RESPONSE	Experimental Group		Control Group	
		Pretest %	Posttest %	Pretest %	Posttest %
1. A televised classroom lecture does not instruct as well as the "conventional" classroom lecture.	Agree or prob. agr.	64	76	70	78
	Disagree or pr. dis.	36	24	30	22
2. A televised classroom demonstration (of materials, procedures, *etc.*) instructs as well as the "conventional" classroom demonstration.	Agree or prob. agr.	61	45	55	59
	Disagree or pr. dis.	39	55	45	40
3. Instruction received over classroom television does not capture and hold student interest as well as instruction received in the "conventional" class.	Agree or prob. agr.	65	78	68	82
	Disagree or pr. dis.	34	22	32	18
4. Students are more comfortable, more at ease, when receiving televised instruction than when attending a "conventional" class.	Agree or prob. agr.	68	83	58	52
	Disagree or pr. dis.	32	17	42	48
5. The inability of students to discuss questions with the instructor during class is a serious limitation of TV instruction.	Agree or prob. agr.	97	98	90	93
	Disagree or pr. dis.	3	2	10	7
6. Students can take a better set of notes when attending TV classes than when attending a "conventional" class.	Agree or prob. agr.	36	41	19	18
	Disagree or pr. dis.	64	59	80	82
7. Students who attend a TV class do not get to know their instructor as well as they might if they attended a "conventional" class.	Agree or prob. agr.	94	96	84	91
	Disagree or pr. dis.	6	4	15	9
8. Students who attend a TV class will be inclined to do more homework and extra reading than those who attend a "conventional" class.	Agree or prob. agr.	47	50	36	24
	Disagree or pr. dis.	53	50	64	76
9. Classroom discipline is a bigger problem in the TV classroom than in the "conventional" class.	Agree or prob. agr.	64	54	62	74
	Disagree or pr. dis.	36	46	38	26
10. Television may be all right for entertainment, but it does not belong in the classroom.	Agree or prob. agr.	37	61	49	61
	Disagree or pr. dis.	63	39	51	39

the semester with some feeling that TV classroom discipline would be a larger problem than discipline in conventional classes. This was scaled down somewhat to 54 per cent by the end of the semester. However, in the control group, 62 per cent began the semester with some doubts about discipline in TV classes and this rose to 74 per cent on the control group posttest.

Posttest responses to question 10 show that 39 per cent of the students in each group tended to accept television as a device for classroom use. In the TV group, there was a significant decline in their acceptance of this idea since, at the beginning of the semester, 63 per cent had indicated a similar view.

Discussion. In considering the evidence of student achievement in the experimental and control sections, it is necessary to think of practical as well as statistical significance of the observed differences in group performance. It is also necessary to consider students' total performance as well as their performance on each of the several achievement tests. With these things in mind, it would seem that there is little reason for believing that the televised instruction was not as effective as the instruction received by the control groups. This remark rests heavily upon the fact that, in terms of total (summed) performance on the six calculus tests, the difference between the averages of the two student groups was small and nonsignificant. Differences this great and greater can undoubtedly be found between the performances of students taught by different but still qualified instructors. Even the effectiveness of a single qualified instructor may fluctuate to produce differences of comparable magnitude.

There is little evidence to suggest that the television method operates either in favor of or against students whose past academic records were above average, average, and below average. In Table 1, it can be seen that the interaction of students' past performance and methods of instruction is, at best, inconsistent and probably nonexistent.

Student attitudes toward televised instruction are not easily summarized. The following things, however, seem to be true. Students in the television sections did alter some of their feelings toward televised instruction during the course of the semester and, predominantly, these changes resulted in a view which was less favorable toward TV than the one initially held. Some exceptions to this exist in the slight TV group response changes on items 6 and 9 of the questionnaire. It also seems true that the negative reactions of the TV group amount to considerably less than complete rejection of such instruction. At the end of the semester, about four tenths of the students in both groups tended to feel that television had some place in the classroom and about the same percentage of TV students felt that they could take better notes than students in conventional classes. Half of them believed that TV classroom discipline was not a larger problem than in conventional classes.

Posttest responses of the control group present some interesting contrasts with those of the TV group on four of the ten questionnaire items. The control students did not perceive the TV classroom as being nearly as "comfortable"

as did the television students. Neither did any significant portion of the control group students feel that TV classes permitted the taking of notes superior to those from a conventional class. Only a fourth of the control group students felt that TV students would do more homework than normal, but half of the TV group felt this was true. About three quarters of the control group students had some doubts about the discipline situation in TV classes. This compares with 54 per cent making the same responses in the TV group.

Some of the contrasts noted above may be within the limits of sampling error or might be traced back to initial biases in the groups; however, it would seem that there are some apprehensions held by students with no experience in a TV class situation which are not shared by students with such experience.

Conclusions. Information gathered and examined in the course of this study would seem to support the following conclusions:

1. Student achievement in calculus, as measured by six classroom tests, is very nearly the same for conventionally- and TV-taught students. Although two statistically significant differences were found when comparing performance of the experimental and control groups, none was found when comparing total semester's performance of these groups.

2. There are no consistent indications that televised instruction operated either in favor of or against students whose past academic records were above average, average, or below average.

3. Television students' reactions to instruction by means of TV tend to be somewhat less favorable following exposure to this form of teaching than they were initially.

4. The final reactions of television-taught students, though negative, still give some indications of noticeable acceptance.

5. Although not directly supported by the evidence, there is reason to believe that the level of achievement of television-taught students can be gradually raised as experience with and understanding of this form of instruction are increased. Such belief stems primarily from the fact that all personnel in the present study (teachers, students, and TV production) had, at best, limited experience in televised classroom instruction.

6. An initial apprehension over the limited amount of blackboard information which could be transmitted via television was diminished. Apparently the television picture can include a sufficient quantity of such information for transmission to the class, or else such visual information plays an insignificant part in the determination of student achievement.

Addendum. During the first semester of 1957–58 a second experiment on the teaching of college mathematics over closed-circuit television was carried out at

Purdue. It differed from the one described above in several ways. The subject matter was roughly that of *Fundamentals of College Mathematics* (Johnson, McCoy and O'Neill), without the calculus. About 160 students were divided in a random manner into two groups. The students in one of these (called the test group in that which follows) were divided into four smaller groups alphabetically by last names. These groups met five times a week in separate rooms. They were taught for about 30 minutes each day over closed-circuit TV. For the remaining 20 minutes of each period they were taught by graduate assistants, who concentrated on answering questions on the lectures or on assigned problems.

The second group of eighty students (the control group) met five times a week in a large lecture room. It was originally planned that they would receive each day a lecture of about 30 minutes in length, after which they would be helped by two graduate assistants. However, the instructor in charge during the first half of the term felt that these last twenty minutes were largely wasted and, after about the first two weeks, instruction was by informal lectures, during which a special effort was made to obtain student participation.

A detailed analysis of the results has not yet been made but it appears that they were very similar to those in the first experiment. It is interesting to note, however, that there were some apparently significant differences in accomplishment between different TV groups, where the breakdown was alphabetical.

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TEACHING FRESHMAN MATHEMATICS BY TELEVISION

H. MARGARET ELLIOTT, Washington University

1. Introduction. In the Fall Semester, 1956–57, Washington University initiated a program of teaching its freshman course, Mathematics 115, College Algebra and Trigonometry, by televised lectures and accompanying help sections. Mathematics 115 is a five-hour course normally taken by science majors and students in engineering and architecture in the first half of the freshman year. Incoming students who can pass a qualifying examination on the material of Mathematics 115 may go directly to a course in analytic geometry and calculus. However, at the present time very few of our students are able to pass this qualifying examination. The prerequisite for Mathematics 115 is three or more years of high school mathematics, including at least one and one-half years of algebra.

2. Plan of operation. The material of the course was divided into 58 lessons with a text and problem assignment for each lesson. There was a 45-minute lecture on each lesson. (The lectures were kinescoped in advance during the preceding summer by H. M. MacNeille, R. R. Middlemiss, and myself.) The students had five opportunities a day to hear the lecture: it was televised to the community at 3 and 7 P.M. by the St. Louis educational television station, and was given over closed circuit in rooms on campus at 9, 10, and 12 in the morning. Students could view the 3 and 7 P.M. lectures either in rooms on campus that had been equipped with television sets or in any other place a set was available to them.

Three two-hour examinations and a three-hour final examination were held in the evening on campus. In addition to the kinescoped lectures on the 58 lessons, the day before each examination there was a live review lecture and a few days after each examination there was a live discussion lecture on the examination (both given via television).

Help sections staffed with assistants were maintained daily from 9 A.M. to 3 P.M. and from 8 to 10 in the evening for students who had questions on the lesson and needed personal help. Students were given complete freedom in regard to hearing the lectures and attending the help sections.

At registration each student in the course was given a detailed set of instructions and a lesson schedule containing the text and problem assignments. The scheme for computing mid-term and semester grades was set out explicitly in the instructions.

3. Help sections. The help sections formed an integral part of the program. They were conducted on an informal basis. There were no lectures in them; help was on a person-to-person basis. Some students would come only for a few minutes to ask one or two particular questions; others stayed for three or four hours working on the assignment and asking for help as they needed it.

There seemed to be little passive copying of problems. Attendance fluctuated with the difficulty of the assignment. The help sections proved especially valuable for weak students and for students who had been out of school for a number of years before entering the course.

The assistants in the help sections were, for the most part, junior and senior students who had had at least 18 hours of college mathematics with A's and who were taking more advanced mathematics courses at the time. In addition, there were several faculty assistants, and Professor Middlesmiss and I worked in the help sections as much as we could. The number of assistants needed per hour varied from two in the evening hours to as many as eight or nine during the most crowded daytime hours. One valuable aspect of the program was the training given the undergraduate assistants.

4. Homework and grades. Semester grades were computed on the basis of 1000 points. Each two-hour examination counted 175 points, the three-hour final counted 275 points, and homework counted 200 points. Counting homework this much forced a student to do it regularly, yet no student could pass on homework alone. As an added incentive to do the homework, we deducted five points for each homework paper a student failed to hand in. (Provisions were made for late homework papers.)

The careful handling of the homework papers and grades for the course involved an enormous amount of work for those of us who administered the course. However, we felt this to be essential with the type of student we had. Though it may be deplored, most of our students work for grades rather than for any love of mathematics. Hence we set up a system in which the student saw clearly the relationship between his grade and his performance.

5. Use by other colleges. The lectures were also followed by groups at two other colleges in the local area: Harris Teachers' College and Lindenwood College. The students in these groups took our examinations but had their own help sections. In addition, the lectures were used by a group of specially selected high school students at one local high school. These latter students averaged higher on the examinations than did our own students.

Qualified persons in the St. Louis area could take the course for credit through the evening school division of Washington University. We had 37 people registered for it in this manner. In addition, the course was followed informally by many high school students and teachers, as well as by other adults in the community.

6. Evaluation of the program gradewise. We deliberately raised the standards in the course by using more difficult, searching examinations and by raising the average required for a given letter grade. It was felt that the level of performance which was given a D this year would have been given a C last year. Despite this, the grades rose sharply, as illustrated below.

Grade	Televised Course, Fall 1956–57		Traditional Course, Fall, 1955–56	
	Number	Per Cent	Number	Per Cent
A	95	20.0	47	12.2
B	122	25.7	69	17.8
C	102	21.5	99	25.6
D	56	11.8	17	4.4
Incomplete	52	10.9	84	21.7
F	22	4.6	44	11.4
Nx	0	0.0	4	1.0
Withdrawn	26	5.5	23	5.9
TOTAL	475	100.0	387	100.0

7. Evaluation in other ways. The adoption of this new method of instruction was in large measure prompted by the necessity for finding a solution to the problems of inadequate classroom space and the impossibility of obtaining enough competent instructors to handle the rising enrollment in small sections. Besides meeting these problems, the program had other things to recommend it. One unexpected dividend was the change in the attitudes of many of the students toward mathematics; students in the new plan showed far more interest and enthusiasm for the material of the course and worked much harder than had former students in the course. The new method placed the responsibility for learning the material squarely on the student; it seemed to shift the emphasis from sitting in class to doing the homework and learning the material.

The new program served the heterogeneous freshman group which we have better than did the old one. The bright students were not forced to listen to repeated explanations of what they already knew; the slower students could get all the help they needed. The latter could, and frequently did, hear the lecture more than once.

One of the big gains we hope to achieve from the program is an improvement in the mathematical backgrounds of our incoming students, a large number of whom are from the surrounding area. We hope that in the future many of the local high schools will arrange to use our televised lectures in a course for their interested better students so that these students will not need to take the course in college, but can start immediately with analytic geometry and calculus. We feel that our lectures are also of value in informing the high school mathematics teachers as to the kind of mathematics courses for which their students should be prepared.

Because of the rather elaborate machinery necessary to operate a course in this manner and the large initial expenses, such a program is probably not suitable for courses with a small enrollment. On the other hand, the cost of operating the program—including the number of faculty members needed for it, classroom space, and other costs—increases only slightly with a substantial

increase in the number of students in the course, a factor which would recommend it for large basic courses.

Two pertinent questions are: (a) just what role does the use of television play in such a program, and (b) to what extent could the program be used by institutions that do not have access to the facilities of a television station. We feel that the closed circuit lectures could well be replaced by large (non-television) lectures to groups of 100–200, provided suitable lecture halls are available. We were unhappy with the closed circuit, partly because, despite continued efforts of technicians, it never worked very well. In addition, the noise it created was unpleasant for other classes in the building. A lecture over the closed circuit viewed in a classroom is probably not as effective as an actual classroom lecture. On the other hand, a televised lecture watched by an individual at home or by a small group in a fraternity house or dormitory is perhaps superior in many respects to an average classroom lecture. In practice, most of our students chose to watch the lecture in the evening at home or in their living group.

We felt that the success of the program was due for the most part not to the use of television but rather to the help sections and to the set-up of the course which forced the students to work.

8. Cautions. This method of instruction is no panacea for the problem of handling large enrollments with a small staff and limited classroom space. If the program is not very carefully run, we feel that it could easily fail. The task of administering such a program, particularly during its initial stages, should not be underestimated. The help sections, if they are to function efficiently, cannot be turned over completely to undergraduate assistants. At least one faculty member should be present most of the time. It should also be noted that lecturing via television requires a different technique from classroom lecturing and vastly more preparation time.

9. Continuation of the program. In the Spring Semester, 1957, the course which follows Mathematics 115, Analytic Geometry and Calculus, was given in a similar way, but without the use of a film. Two lectures a day were given on campus via the closed circuit, and one lecture was given in the evening over the educational television station. We found the live televised lectures to be far better than the kinescoped ones, both as regards clarity of the picture and the greater flexibility afforded.

The program is being continued in the present academic year (1957–58) with a few changes. The hours of the help sections have been curtailed somewhat; the examinations have been changed in form to facilitate grading. The closed-circuit lectures have been replaced by two live lectures a day given in a large auditorium. The film used last year (remade in part) is again being shown over the educational television station both in the afternoon and in the evening.

EXPERIENCE AT SEATTLE

CARL B. ALLENDOERFER, University of Washington

1. General description. Instruction in mathematics by television began at the University of Washington in January, 1956. The first course offered was Intermediate Algebra and was conducted over a period of 20 weeks through the facilities of KCTS-TV, the community educational television station for Seattle and King County. The course was officially offered through the Correspondence Department of the Division of Adult Education of the University of Washington and carried correspondence credit for five quarter-hours. On Tuesday and Wednesday evenings of each week lectures on new material were presented from 7:30 to 8:00 P.M. A help session was available at the University on Saturday mornings. Weekly homework assignments were due on Saturday morning and were returned for delivery on Monday. On Monday evening a review lesson was broadcast on which answers were given for all assigned problems and common errors in the homework were discussed. Two examinations at the University were given to registered students; grades were based on these with some credit for homework. Normal charges for correspondence courses were imposed.

A course on trigonometry was offered in a similar fashion during the winter quarter of 1957. Essentially the same pattern was followed, except that three lectures a week were given, the help session was on Thursday evening, three quarter-hours of credit were awarded, and the course ran for only ten weeks. In both courses the lectures were given by me, and the homework, help sessions, review broadcasts, and examinations were handled by Granville McCormick, a predoctoral associate. Kinescopes were made of the trigonometry lectures for possible future use.

2. Broadcast Technique. For a variety of reasons including financial stringency and lack of time for preparation, the technical aspects of the program were kept as simple as possible. Because of previous difficulties at KCTS in obtaining suitable contrast with an ordinary blackboard and chalk, a new method of presentation was devised. The lecturer was seated to the left of an easel which carried sheets of off-white newsprint. The lecturer wrote on these with a black grease pencil. One camera was trained over the lecturer's right shoulder onto these sheets, and the second camera viewed him from his left side. Excellent definition was obtained in this fashion, even in fringe areas, and the fixed position of the lecturer prevented him from wandering about the studio to the consternation of the director and camera men.

Variations in this technique were used for special purposes. Slides were prepared for the statements of word problems and for announcements; a wall chart of logarithmic and trigonometric tables proved successful in teaching the use of such tables; an oscilloscope was helpful in presenting sine curves of various periods and amplitudes; predrawn graphs of various functions were used

to improve legibility and to save time; and simple apparatus was borrowed from the Physics Department to illustrate applications of trigonometry.

There were essentially no rehearsals except for brief tests of camera angles when new devices were to be used. After a few lectures the director and the lecturer were so well coordinated that each could anticipate the actions of the other in time to prevent any serious confusion. Except for the director, the other station personnel (such as camera men and floor director) were students who were at the same moment taking a course in TV techniques. Their own enthusiasm made up for any lack in experience, and I often found myself talking to them as if they were students in my class. No classroom audience was provided and I did not find this a handicap.

3. **Preparation.** The outline of the course was carefully prepared in advance, and was distributed at cost to the registered students and to many others in the community as a "Viewer's Guide." After this was written, I spent about 30 minutes preparing each lecture. Most of this time was devoted to making careful notes on the solutions of problems to be presented on the air. No fumbling about for the best method of attack can be permitted (it wastes time), and numerical errors are a gross disaster, for there is no bright-eyed boy on the front row to point out a mistake. Correct timing is essential, of course; but after a little practice an experienced teacher gets a feel for 28 minutes as accurately as he does for 50 minutes.

4. **Effectiveness.** Comparative studies were made of the performance of the TV students as contrasted to those taught by conventional means. In addition the TV students were questioned in person and through a questionnaire on their own reactions. The conclusion we reached is that instruction on TV is just as effective as classroom instruction provided that help sessions are available and that homework is graded regularly. The students must also have access to a TV set under circumstances free from distractions. The performance of the TV students was remarkably better than that of ordinary correspondence students. In presenting theoretical developments and problems, the lecturer must be conscious of the small size of the picture and the relatively few equations that can be viewed at one time. These mean that TV is most effective in simple mathematical situations and in short problems. Algebra and trigonometry can be broken up into sufficiently small bits of this kind, but I have doubts about the effectiveness of TV in teaching analytic geometry and calculus where most situations require simultaneous viewing of a graph and fairly extensive calculations. The TV screen is just too small for this sort of thing.

5. **General Conclusions.** Based on this limited experience, I have reached the following conclusions—which are subject to immediate revision as my knowledge of this medium increases:

(a) The main use of TV in mathematics instruction is in Adult Education, for by this method we can reach people who are otherwise unable or unlikely

to come to the campus. Instruction in elementary subjects by this means is quite satisfactory. The effectiveness of such broadcasts upon the general public (apart from the registered students) is one of their most important features. Indeed, the public interest in our programs in Seattle was very gratifying, and they proved to be a most useful public relations device for mathematics as a subject and for the Mathematics Department of the University. It has been my experience that the public is more interested in the concrete mathematics of algebra than it is in cultural programs on the appreciation of mathematics.

(b) TV is not well suited as a replacement for normal classroom instruction of regular students. A far better method of "canning" our instructors is to make regular films to be projected on conventional movie equipment. These films, of course, can be kinescopes made in TV studios. Films can be shown at convenient hours without tying up air space, their images on the screen are much larger and clearer than on a TV set, and the equipment needed is less complicated and expensive than TV—either open or closed circuit. There is no reason why carefully prepared films presenting experienced lecturers should not be more effective than classroom teaching by young assistants. The shortage of mathematics teachers makes preparation of such films a most urgent matter for consideration by bodies such as the Mathematical Association of America.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1321. *Proposed by C. W. Bostwick, Riverdale, Maryland*

Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other.

E 1322. *Proposed by J. W. Andrushkiw, Seton Hall University*

Multiply the first p terms of the harmonic series by $(-1)^k$, the next q terms by $(-1)^{k+1}$, the next p terms by $(-1)^k$, the next q terms by $(-1)^{k+1}$, and so on, alternately, thus forming a series denoted by $H(p, q, k)$. Show that $H(p, q, k)$ is convergent if and only if $p=q$.

E 1323. *Proposed by Harry Goheen, Oregon State College*

Prove that all roots but one of the equation

$$nx^n = 1 + x + x^2 + \cdots + x^{n-1}$$

have absolute value less than 1.

E 1324. *Proposed by F. E. Clark, Rutgers University*

Suppose tic-tac-toe is turned into a game of pure chance as follows. Designate the squares by $1, \cdots, 9$, numbering them from left to right by successive rows, starting with the top row. Place a set of nine chips, labeled $1, \cdots, 9$, in a bag. The first player, A, draws a chip at random and enters an X in the corresponding square. The second player, B, draws at random from the remaining chips and enters an O in the corresponding square. The game ends, of course, as soon as three like entries are obtained which lie along a line (horizontally, vertically, or diagonally). Find the probability that A will win, draw, or lose the game. Also find the probability, for each drawing, that the game will end when that chip is drawn.

E 1325. *Proposed by Peter Treuenfels, Brookhaven National Laboratory, Upton, L. I., New York*

Let A, B, X denote $n\times n$ matrices. Show that a sufficient condition for the existence of at least one solution X of the matrix equation

$$X^2 - 2AX + B = 0$$

is that the eigenvalues of the $2n\times 2n$ matrix

$$R = \begin{bmatrix} A & I \\ A^2 - B & A \end{bmatrix}$$

be pairwise distinct. Here I denotes the $n\times n$ identity matrix.

SOLUTIONS

An Inequality

E 1291 [1957, 741]. *Proposed by G. B. Thomas, Jr., Massachusetts Institute of Technology*

Let a, b, c, d be nonnegative numbers such that $c+d \leqq \min(a, b)$. Prove that $ad+bc \leqq ab$.

Solution by Underwood Dudley, Carnegie Institute of Technology. We have $ad+bc \leqq (c+d) \max(a, b) \leqq \min(a, b) \max(a, b) = ab$.

Also solved by J. L. Alperin, A. G. Anderson, Winifred Asprey, Philip Bacon, D. W. Bailey, Edward Barbeau, Robert Bart, B. H. Bissinger, Guy Blondeau, Julian Braun, D. A. Breault, D. R. Brillinger, E. W. Brown, J. L. Brown, Jr., George Campagne, P. L. Chessin, W. J. Cody, Jr., Ronald Colling, A. E. Danese, J. E. Darraugh, C. W. Dodge, D. B. Dorsey, P. L. Duren, E. S. Eby, Irma Esrig, Eugene Famolari, Jr., D. P. Flemming, D. A. Freedman, M. L. Freimer, W. B. Fulks, John Gedeist, A. M. Glicksman, Michael Goldberg, L. K. Grodman, A. J. Gross, J. W. Haake, J. D. Haggard, Dunstan Hayden, Vern Hoggatt, J. R. Holdsworth, R. Holt, A. R. Hyde, Carlton Johnson, N. S. Kandalgaonkar, Geoffrey Kandall, A. F. Kaupe, Jr., M. S. Klamkin, Sandra Kohlenberg, J. D. E. Konhauser, Andrew Kraus, Lorraine Lavallee, A. I. Lieberman, Joe Lipman,

R. L. London, L. H. McFarlan, R. T. J. Mahoney, Wallace Manheimer, E. W. Marchand, D. C. B. Marsh, J. C. Mathews, William Moser, D. L. Muench, A. A. Mullin, J. B. Muskat, E. L. Nath, E. N. Nilson, C. S. Ogilvy, M. J. Pascual, D. J. Peterson, R. L. Pierce, J. L. Pietenpol, C. F. Pinzka, B. E. Rhoades, L. A. Ringenberg, Jeff Ritterman, F. E. Rote, H. D. Ruderman, Paul Schillo, D. L. Shell, Paul Slepian, G. S. Stoller, W. B. Stoval, Jr., D. R. Sudborough, Lawrence Van Cura, K. D. Ware, F. L. Wolf, David Zeitlin, and the proposer. Late solutions by C. D. Anderson, Roger Bachmann, J. Gallego-Diaz, R. H. Hou, Morton Kupperman, Jack Macki, Leo Moser, and J. B. Oenning.

The Pirate's Treasure

E 1292 [1957, 741]. *Proposed by Jose Gallego-Diaz, Vanderbilt University*

A pirate decided to bury a treasure on an island near the shore of which were two similar boulders A and B and, farther inland, three coconut trees C_1, C_2, C_3. Stationing himself at C_1, the pirate laid off C_1A_1 perpendicular and equal to C_1A and directed outwardly from the perimeter of triangle AC_1B. He similarly laid off C_1B_1 perpendicular and equal to C_1B and also directed outwardly from the perimeter of triangle AC_1B. He then located P_1, the intersection of AB_1 and BA_1. Stationing himself at C_2 and C_3, he similarly located points P_2 and P_3, and finally buried his treasure at the circumcenter of triangle $P_1P_2P_3$.

Returning to the island some years later, the pirate found that a big storm had obliterated all the coconut trees on the island. How might he find his buried treasure? (Dedicated to Howard Dachslager.)

I. *Solution by C. F. Pinzka, University of Cincinnati.* Since triangles AB_1C_1 and A_1BC_1 are congruent, angles C_1AB_1 and C_1A_1B are equal and then $\measuredangle A_1P_1A = \measuredangle A_1C_1A = 90°$. Thus P_1, and likewise P_2 and P_3, lie on the circle with diameter AB, and the treasure is buried midway between A and B.

II. *Solution by R. R. Seeber, Jr., IBM Corp., Poughkeepsie, N. Y.* While wondering how to proceed, the pirate watched three sea gulls engaged in a violent aerial battle. Presently all three birds fluttered to the ground, dead. Taking this as a favorable omen, the pirate proceeded with his original construction, using the locations of the three birds for replacements of the coconut trees. He found his treasure and was interested to note that it was buried at the midpoint of AB, a fact which had previously been concealed by a clump of shore-side coconut trees.

Also solved by A. G. Anderson, Robert Bart, J. L. Botsford, M. G. Boyce, Julian Braun, E. W. Brown, J. W. Clawson, C. W. Dodge, Brother Louis Francis, J. W. Gammill, J. W. Haake, A. R. Hyde, N. S. Kandalgaonkar, Geoffrey Kandall, Sister M. Kenneth Kolmer, J. D. E. Konhauser, M. A. Laframboise, Joe Lipman, Wallace Manheimer, D. C. B. Marsh, Beckham Martin, E. L. Nath, C. S. Ogilvy, D. S. Passman, J. L. Pietenpol, J. R. Sykes, Sister M. Stephanie, Ismael Torrecilla, and the proposer. Late solutions by C. D. Anderson, D. F. Atkins, Roger Bachmann, and J. B. Oenning.

A General Congruence

E 1293 [1957, 742]. *Proposed by J. B. Roberts, Reed College*

Prove that for every nonnegative integer t

$$3(1 + 6^t + 8^t) \equiv 1^t + 2^t + 3^t + \cdots + 9^t \pmod{18}.$$

Solution by Norman Miller, Queen's University. The congruence is readily verified for $t=0$ and $t=1$. Take $t>1$. The difference of the two expressions is

$$2^t + 3^t + 4^t + 5^t + 7^t + 9^t - 2 - 2\cdot 6^t - 2\cdot 8^t.$$

Since five terms are even and four terms are odd, this is divisible by 2. Three of the terms are divisible by 9. It remains to show that

$$2^t + 4^t + 5^t + 7^t - 2 - 2\cdot 8^t \equiv 0 \pmod 9). \tag{1}$$

The grouping $(7^t+2^t)+(5^t+4^t)-2(8^t+1)$ shows that this is true when t is odd. Take t even and write (1) in the form

$$7(7^{t-1} + 2^{t-1}) - 5\cdot 2^{t-1} + 5(5^{t-1} + 4^{t-1}) - 4^{t-1} - 16(8^{t-1} + 1) + 14.$$

Since $t-1$ is odd, each expression in parentheses is divisible by 9. Also

$$\begin{aligned} 5\cdot 2^{t-1} + 4^{t-1} - 14 &= 2^{2t-2} + 5\cdot 2^{t-1} - 14 \\ &= (2^{t-1} + 7)(2^{t-1} - 2) = 2(2^{t-1} + 1 + 6)(2^{t-2} - 1), \end{aligned}$$

where each factor in parentheses is divisible by 3. This completes the proof.

Also solved by Winifred Asprey, D. A. Breault, D. R. Brillinger, P. L. Chessin, T. W. Daniel, F. J. Duarte, Underwood Dudley, Louise Grinstein, J. W. Haake, R. Holt, A. R. Hyde, M. S. Klamkin, Sister M. Kenneth Kolmer, Joe Lipman, D. C. B. Marsh, C. T. Molloy, Jr., J. B. Muskat, Margaret Olmsted, C. A. Oster, F. D. Parker, W. G. Preble, Paul Schillo, R. R. Seeber, Jr., F. L. Wolf, David Zeitlin, and the proposer. Late solution by C. W. Trigg.

Limits of Two Recursive Sequences

E 1294 [1957, 742]. *Proposed by G. A. Harris, Jr., Yale University*

Having chosen two numbers a_1 and b_1 from the open interval (0, 1), define the sequences $\{a_n\}$ and $\{b_n\}$ recursively as follows:

$$a_{n+1} = a_1(1 - a_n - b_n) + a_n, \qquad b_{n+1} = b_1(1 - a_n - b_n) + b_n.$$

Prove that both sequences approach limits as $n\to\infty$, and find these limits.

Solution by B. H. Bissinger, Lebanon Valley College. By mathematical induction we can prove $a_n=a_1(1-d^n)/(1-d)$, where $d=1-a_1-b_1$. Since $|d|<1$, it follows that $\lim a_n=a_1/(a_1+b_1)$. By symmetrical considerations we have $\lim b_n=b_1/(a_1+b_1)$.

Also solved by D. S. Adorno, Philip Bacon, Edward Barbeau, Robert Bart, A. P. Boblétt, Julian Braun, D. R. Brillinger, P. L. Chessin, A. E. Danese, W. R. Derrick, Peter Duren, E. S. Eby, D. A. Freedman, Harry Goheen, Michael Goldberg, D. S. Greenstein, A. J. Gross and R. F. Potthoff (jointly), Emil Grosswald, J. W. Haake, Virginia S. Hanly and M. L. Slater (jointly), A. R. Hyde, N. S. Kandalgaonkar, A. F. Kaupe, Jr., D. A. Kearns, P. G. Kirmser, M. S. Klamkin, A. I. Lieberman and D. Solitar (jointly), Joe Lipman, D. C. B. Marsh, Norman Miller, C. T. Molloy, Jr., D. L. Muench, J. B. Muskat, E. L. Nath, C. S. Ogilvy, M. J. Pascual, D. S. Passman, Stanton Philipp, R. L. Pierce, J. L. Pietenpol, C. F. Pinzka, W. G. Preble, L. A. Ringenberg, Jeff Ritterman, Milton Rosenberg, Paul Schillo, D. L. Shell, D. L. Smith, G. S. Stoller, D. F. Templeton Jr., G. H. M. Thomas, A. L. Tritter, Chih-yi Wang, David Zeitlin, and the proposer. Late solutions by C. D. Anderson, D. F. Atkins, D. A. Breault, R. H. Hou, J. D. E. Konhauser, Morton Kupperman, Yoshio Matsuoka, Leo Moser, and O. E. Stanaitis.

A Transcendental Complex Equation with Only Real Roots

E 1295 [1957, 742]. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Show that all the roots of $\tan z = z/(1+m^2z^2)$, where m is real, are real.

I. *Solution by J. W. Haake, Armour Research Foundation, Tucson, Arizona.* We note that if z is a root of the given equation, then so are $\bar{z}$ and $-z$. Hence, in discussing imaginary roots $z=a+ib$, $b\neq 0$, we may, for convenience, assume $a\geqq 0$, $b>0$. Now in the given equation, set $z=a+ib(b\neq 0)$, $\tan z=(\sin z)/(\cos z)$, $\sin ib = i \sinh b$, $\cos ib = \cosh b$, and then multiply each side of the resulting equation by the conjugate of the denominator of that side. We obtain

$$\frac{(\sin 2a)/2 + (i \sinh 2b)/2}{\cos^2 a \cosh^2 b + \sin^2 a \sinh^2 b} = \frac{a[1+m^2(a^2+b^2)] + ib[1-m^2(a^2+b^2)]}{[1+m^2(a^2+b^2)]^2 + 4m^4a^2b^2}.$$

Since real and imaginary parts must be equal, we obtain, by taking the ratio of real part to imaginary part on each side,

$$\frac{\sin 2a}{\sinh 2b} = \frac{a}{b}\left[\frac{1+m^2(a^2+b^2)}{1-m^2(a^2+b^2)}\right].$$

For real m, this implies the impossible inequality

$$(\sin r)/(\sinh s) \geqq r/s, \qquad r \geqq 0,\ s > 0.$$

It follows that we cannot have $b\neq 0$, and z must be real.

II. *Solution by A. E. Danese, Eastman Kodak Co., Rochester, N. Y.* Let $F(z) = \tan z - z/(1+m^2z^2)$. Then the zeros of $F(z)$ are those of

$$g(z) = (1+m^2z^2)\tan z - z.$$

It is readily verified that

$$(2/z^3)\cos z\, g(z) = \int_0^1 (2m^2+1-t^2)\cos zt\, dt.$$

Application of the following theorem [Pólya and Szegö, *Aufgaben und Lehrsätz aus der Analysis*, Zweiter Band, New York, 1945, p. 69, 173] yields the result: Let $f(t)$ be twice continuously differentiable, $f(t)>0$, $f'(t)\leqq 0$, $f''(t)<0$ for $0\leqq t\leqq 1$. Then the function $\int_0^1 f(t)$ coz $zt\,dt$ has infinitely many, and only real, zeros.

Also solved by Robert Bart, D. C. B. Marsh, Nathan Schwid, and the proposers. Late solutions by R. H. Hou and J. D. E. Konhauser.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4776 [1958, 125]. Correction. *Proposed by D. J. Newman, AVCO Research and Development*

If $|\alpha| < 1/e$, then $\sum_{n=0}^{\infty} (z+\alpha n)^n/n!$ represents an entire function. Prove in fact that it is a simple exponential, $Ae^{\lambda z}$.

4793. *Proposed by M. H. Lietzke and C. W. Nestor, Jr., Oak Ridge National Laboratory*

In *Mathematics and the Imagination*, Kasner and Newman present the following problem. An equilateral triangle is inscribed in a circle of unit radius, another circle is inscribed in the triangle, a square in this circle, *etc.* Continue the procedure, increasing the number of sides of the regular polygon each time by one. As the number of sides of the inscribed polygon increases, the radii of the shrinking circles converge to a definite limiting value. Find this value. The proposed answer (approximately 1/12) seems to be considerably in error. What should it be?

4794. *Proposed by S. W. Golomb, California Institute of Technology*

Let V_n be binary n-space (the collection of n-vectors over the field of two elements). Consider two vectors of V_n to be in the same *class* if they differ only by a cyclic permutation of their components. Show that the number of classes is even, except when $n=2$.

4795. *Proposed by Y. L. Luke, Midwest Research Institute, Kansas City, Missouri*

Find the zeros of

$$H_n(x) = \sum_{m=0}^{n} \frac{(-2)^m(2n-m)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m+1)}.$$

4796. *Proposed by Chandler Davis, Institute for Advanced Study*

Say t_1 is *preferred* to t_2 provided (i) $0 \leqq t_1 < t_2 \leqq 1$, or (ii) $-1 \leqq t_2 < t_1 < 0$, or (iii) $-1 \leqq t_2 < 0 \leqq t_1 \leqq 1$. Determine a sequence of polynomials P_n such that, whenever t_1 is preferred to t_2, $P_n(t_2) = o(P_n(t_1))$.

4797. *Proposed by D. J. Newman, Massachusetts Institute of Technology*

Prove that all expressions like $7\sqrt{19}/4-3\sqrt{7}+8\sqrt{6}/5$ are irrational. More specifically, prove that the square roots of the square-free integers are linearly independent over the rationals.

SOLUTIONS

Fibonacci Numbers

4747 [1957, 437]. *Proposed by B. J. Boyer, Lafayette, Ind.*

Write $S_n=\sum_{k=0}^{n} B_k^2$,* where the B_k are Fibonacci numbers, $B_0=1$, $B_1=1$, $B_{n+2}=B_{n+1}+B_n$. Find the value of $\sum_{n=1}^{\infty}(-1)^n/S_n$.

Solution by D. C. B. Marsh, Colorado School of Mines. We have

$$S_n = B_n\cdot B_{n+1}, \tag{1}$$

$$\sum_{n=0}^{m}(-1)^n/S_n = B_m/B_{m+1}, \tag{2}$$

$$\lim_{m\to\infty} B_m/B_{m+1} = \tfrac{1}{2}(\sqrt{5}-1), \tag{3}$$

which is the desired result. (1) and (3) are well known (cf. this MONTHLY, vol. 60, 1953, p. 680 ff.) and (2) is easily demonstrated by induction.

Also solved by W. J. Blundon, Robert Breusch, P. G. Engstrom, Calvin Forman, J. Ginsburg, L. D. Goldberg, S. H. Greene, Emil Grosswald, A. R. Hyde, M. S. Klamkin, T. F. Mulcrone, J. K. Muskat, J. L. Pietenpol, R. C. Read, D. A. Robinson, D. L. Scheffler and R. A. Sebastian, Chih-yi Wang, E. J. Williams, L. K. Williams, and the proposer. Late solution by Yoshio Matsuoka.

Editorial Note. Robinson calls attention to the fact that the problem was proposed by Lucas (1) and solved by Mangon (2). This result contained an error which was noted in the same journal (3). Dickson (4) states Mangon's incorrect result but fails to take note of the later correction.

1. *Nouvelle Correspondance Mathematique*, vol. 6, 1880, p. 418.
2. *Ibid.*, p. 420.
3. *Ibid.*, p. 480.
4. Dickson, L. E., *History of the Theory of Numbers*, vol. 1, 1919, p. 402.

Rational and Irrational Numbers

4748 [1957, 509]. *Proposed by Alexander Oppenheim, University of Malaya*

Suppose that x and y are defined by the two series

$$x=\frac{1}{a_1}+\frac{1}{a_1^2}\,\frac{1}{a_2}+\frac{1}{a_1^2a_2^2}\,\frac{1}{a_3}+\cdots,$$

$$y=\frac{1}{a_1}+\frac{1}{a_1^3}\,\frac{1}{a_2}+\frac{1}{a_1^3a_2^3}\,\frac{1}{a_3}+\cdots,$$

* The upper limit of the summation was originally printed ∞ in error.

where the a_i are integers such that $a_i \geqq 2 (i=1, 2, \cdots)$. Prove that x and y are both rational or irrational.

Solution by Rimhak Ree, University of British Columbia. First we prove a lemma: *Let z be a positive rational number in* $(0, 1)$, *and let* $e \geqq 2$. *If the inequality* $0<(za-1)a^{e-1}<1$ *has a positive integral solution* a, *then the solution is unique.* If b is another integral solution, (we may assume $a<b$) and $z=m/n$ with integers m, $n>0$, then we will have

$$\frac{n}{a} < m \leqq m(b-a) < \frac{n}{a^{e-1}}$$

which is a contradiction. Thus the lemma is proved.

Let e be an integer $\geqq 2$, and set

$$z_i = a_i^{-1} + (a_i^e a_{i+1})^{-1} + (a_i^e a_{i+1}^e a_{i+2})^{-1} + \cdots.$$

Then $0<z_i<1$, and $(z_i a_i - 1)a_i^{e-1} = z_{i+1}$. Suppose z_1 is a rational number of the form m_1/n with integers m_1, $n>0$. Then z_i is a rational number of the form m_i/n with $0<m_i<n$. Since there are at most $n-1$ distinct values for m_i, there are positive integers j, k such that $z_j = z_{j+k}$. Then by the lemma, $a_j = a_{j+k}$ and hence $z_t = z_{t+k}$, $a_t = a_{t+k}$ for all $t \geqq j$. It is easily seen that if there are positive integers j, k such that $a_t = a_{t+k}$ for all $t \geqq j$ then z_1 is rational. Thus z_1 is rational if and only if there exist integers j, $k>0$ such that $a_t = a_{t+k}$ for all $t \geqq j$. Now, $x = z_1$ if $e=2$, and $y=z_1$ if $e=3$. Hence x and y are both rational or both irrational.

Also solved by Andrew Kraus, Bladovest Sendov, and the proposer.

Editorial Note. Suppose that

$$w = \frac{1}{a_1} + \frac{1}{a_1 b_1}\,\frac{1}{a_2} + \frac{1}{a_1 b_1 a_2 b_2}\,\frac{1}{a_3} + \cdots,$$

where a_i, b_i are infinite sequences of integers which satisfy the conditions $b_i \geqq a_i - 1 \geqq 1$, $(i=1, 2, \cdots)$ and $b_i = b_j$ whenever $a_i = a_j$. Then, as shown by the proposer, an argument similar to the above proves that w is rational if and only if the sequence (a_i) (and therefore also the sequence (b_i)) is ultimately periodic.

Unbounded Fourier Coefficients

4749 [1957, 509]. *Proposed by Joseph Lehner, Michigan State University*

Is it true that for every set E of positive measure contained in the interval $(0, 1)$, the numbers C_n have the property $|nC_n| < M$, $n=1, 2, \cdots$, where

$$C_n = \int_E e^{2\pi i n x} dx,$$

and M is a constant depending only on E?

Solution by Fritz Herzog, Michigan State University. We shall construct an open set E in $(0, 1)$ for which $\{nC_n\}$ is unbounded.

For positive integral p and for $q=1, 2, \cdots, 2p$, let E_{pq} denote the interval $(q-1/2)/(2p)!<x<q/(2p)!$. It is easily verified that no two of these intervals are overlapping. Let

$$E_p = \bigcup_{q=1}^{2p} E_{pq}, \qquad E = \bigcup_{p=1}^{\infty} E_p.$$

We shall show that, for the set E defined above, $nC_n \to \infty$ as $n \to \infty$ through the values $(2m)!$

For fixed m, let the integral

$$\int_X \exp\left[2\pi i(2m)!x\right]dx$$

be denoted by $I(X)$. Then

$$(1) \qquad I(E_{pq}) = \frac{\exp\left[2\pi iq(2m)!/(2p)!\right] - \exp\left[2\pi i(q-1/2)(2m)!/(2p)!\right]}{2\pi i(2m)!}.$$

The numerator on the right side of (1) vanishes when $p<m$, and it equals 2 when $p=m$. Hence

$$(2) \qquad C_{(2m)!} = I(E) = \frac{2m}{\pi i(2m)!} + I(E_{m+1} + E_{m+2} + \cdots).$$

To estimate $I(E_{m+1}+E_{m+2}+\cdots)$, we note that $E_{m+1}+E_{m+2}+\cdots$ is contained in the interval $0<x<1/(2m+1)!$, so that

$$(3) \qquad \left| I(E_{m+1} + E_{m+2} + \cdots) \right| < m(E_{m+1} + E_{m+2} + \cdots) < \frac{1}{(2m+1)!}.$$

It follows from (2) and (3) that

$$(2m)!\left| C_{(2m)!} \right| > \frac{2m}{\pi} - \frac{1}{2m+1},$$

which proves that the left side of the last inequality approaches ∞ as $m \to \infty$.

Therefore, a negative answer is indicated for the proposed question.

Also solved by S. P. Lloyd, Karl Zeller, and the proposer.

Gaussian Divisibility

4750 [1957, 509]. *Proposed by Leonard Carlitz, Duke University*

Find the denominator of

$$\binom{i}{m} = \frac{i(i-1)\cdots(i-m+1)}{m!}$$

where $i=\sqrt{-1}$ and the fraction is reduced to lowest terms in the field $R(i)$.

Solution by the proposer. Put $m!=2^e PQ$, where the primes dividing P are of the form $4k+1$, the primes dividing Q of the form $4k-1$, and

$$e=\left[\frac{m}{2}\right]+\left[\frac{m}{4}\right]+\cdots.$$

We shall show that the denominator of $\binom{i}{m}$ is

$$(1+i)^{2e-[m/2]}Q. \tag{1}$$

1. Let q be a rational prime $\equiv 3 \pmod 4$, so that q remains a prime in the Gaussian field $R(i)$ and the product $i(i-1)\cdots(i-m+1)$ is not divisible by q. Consequently the denominator of $\binom{i}{m}$ contains Q.

2. We have $2=-i(1+i)^2=i(1-i)^2$. A number $i+r$, where r is a rational integer, is divisible by $1+i$ if and only if $N(i+r)=1+r^2$, the norm of $i+r$, is even; but for r odd, $1+r^2\equiv 2 \pmod 4$, so that $i+r$ is not divisible by $(1+i)^2$. Thus $i(i-1)\cdots(i-m+1)$ is divisible by exactly $(1+i)^{[m/2]}$ and therefore the denominator of $\binom{i}{m}$ is divisible by $(1+i)^{2e-[m/2]}$ in accordance with (1).

3. It remains to show that

$$i(i-1)\cdots(i-m+1)\equiv 0 \pmod P. \tag{2}$$

Let p denote a rational prime $\equiv 1 \pmod 4$. Then $p=\pi\pi'$, where π, π' are distinct primes of $R(i)$. Choose a so that $a^2+1\equiv 0 \pmod p$; then $\pi \mid (i-a)$; we may suppose $1<a<p$. We recall that the numbers $0, 1, \cdots, p^n-1$ constitute a complete residue system (mod π^n) for every $n\geqq 1$. Thus in the sequence i, $i-1, \cdots, i-m+1$ there are $[m/p]$ multiples of π, $[m/p^2]$ multiples of π^2, and so on. Consequently $i(i-1)\cdots(i-m+1)$ is divisible by p^f, where

$$f=\left[\frac{m}{p}\right]+\left[\frac{m}{p^2}\right]+\cdots.$$

Similarly, starting with $\pi' \mid i-p+a$, we infer divisibility by π'^f. This completes the proof of (2).

Geometric Means

4752 [1957, 510]. *Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.*

Determine a set of n distinct, nonzero terms such that their geometric mean is the geometric mean of their arithmetic and harmonic means.

Solution by Emil Grosswald, University of Pennsylvania. Let S_j be the jth fundamental symmetric function of the n terms $a_1, \cdots, a_n$. Then their harmonic, geometric and arithmetic means are

$$H=nS_n/S_{n-1}, \qquad G=S_n^{1/n}, \qquad A=S_1/n,$$

respectively, and the condition $G^2=HA$ of the problem becomes

(1) $$S_{n-1} = S_1 S_n^{1-2/n},$$

It is therefore sufficient to take as $a_1, \cdots, a_n$ the roots of

(2) $$x^n - S_1 x^{n-1} + \cdots + (-1)^{n-1} S_{n-1} x + (-1)^n S_n = 0,$$

with arbitrary $S_2, S_3, \cdots, S_{n-2}$ and any S_1, S_{n-1}, S_n which satisfy (1). In order to have all terms different from zero it is sufficient to take $S_n \neq 0$; and $a_i \neq a_j$ for $i \neq j$ is assured if the coefficients of (2) are so chosen that its discriminant does not vanish.

Also solved by A. P. Boblétt, V. E. Hoggatt, W. S. Lawton, J. B. Muskat, Marlow Sholander, Michael Skalsky, and the proposer.

Editorial Note. Several explicit sets were proposed. The simplest are: (i) n successive terms of any geometric progression, (ii) $[n/2]$ distinct pairs of reciprocals, with the addition of the element 1 in case n is odd.

Integers of Special Form

4753 [1957, 596]. *Proposed by J. K. Senior, University of Chicago*

Are there any integers larger than 31 which can be represented in more than one way by the form $(a^m-1)/(a-1)$, a and m integers and $m>2$?

Comment by A. Makowski, Warsaw, Poland. We have

$$8191 = \frac{2^{13}-1}{2-1} = \frac{90^3-1}{90-1}.$$

This example was proposed by R. Goormaghtigh (*L'Intermédiaire des Math.*, 1917, pp. 88, 1530) who stated that for $A < 1{,}000{,}000$ the only solutions are 31 and 8191. A reference is given in Dickson, *History of the Theory of Numbers*, vol. 2, p. 703. See also *Mathesis*, 1957, p. 327.

Also solved by W. H. Benson and M. S. Klamkin.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Mathematics and Plausible Reasoning. By G. Pólya. Princeton Univ. Press, 1954. Vol. I, $5.50. Vol. II, $4.50. Both $9.00.

Vol. I. *Induction and Analogy in Mathematics.* xvi+280 pp.

The author has gathered an interesting list of problems which substantiate his thesis that mathematics requires imaginative guessing as well as deductive

argument. "You have to guess a mathematical theorem before you can prove it; you have to guess the idea of the proof before you carry through the details." His examples are chosen chiefly from arithmetic, algebra, and elementary geometry, but there are some sections which use the notions of differential and integral calculus.

Particularly interesting are the problems in maxima and minima which are solved without calculus. The author resists the temptation to give rules of thumb for handling various types of problems. Accordingly, the experienced teacher may be impatient with the author's leisurely style. Yet it is precisely because blind alleys have been deleted from published mathematics that makes it seem unmotivated to the student reader.

Vol. II. *Patterns of Plausible Inference.* x+190 pp.

Apparently the second volume is intended as a guide to any student who wishes to learn how to use induction and analogy in reasoning. This reviewer confesses disappointment in the result. The author belabors the point that creative reasoning is not always deductive. He spells out the precise pattern of guessing which can lead to the solution of a variety of additional problems. The moral seems to be "Go thou and do likewise," but, as hindsight is often clearer than foresight, it is doubtful whether creativity in mathematics would be encouraged by following this (or any other) set of rules.

The treatment of probability is superficial. Thus a linguistic example treats all letters as equally probable, and ignores borrowing. No attempt is made to use information theory to measure credibility, and the author supposes that no quantitative theory of plausible reasoning may be possible. The analogy with many-valued logics is also ignored. In conclusion the author says the exposition is "principally for students, but perhaps also useful to some teachers," but not to one who tries to make students understand things he does not understand himself.

ARTHUR BERNHART
University of Oklahoma

Elementary Differential Equations. By William Ted Martin and Eric Reissner. Addison-Wesley, Reading, 1956. xii+260 pp. $5.50.

Before evaluating this text, let us set up standards with which it may be compared. A textbook for a first course in differential equations which is intended primarily for students of science and engineering might be selected on the basis of the following requirements.

I. *Matters of principle:*
  1. The text should give a treatment of practical value.
  2. It ought to be mathematically sound.
  3. It has to be understandable for a student who has just squeezed through calculus.

The present, and some other recent books, come as close to passing this first test as one can reasonably expect. Hence let us formulate a second set of requirements, and really look at what's in the book.

II. *Contents*. The text should contain:

1. A chapter on the setting up of differential equations.
2. No chapter on first-order equations of special types for which there happen to be age-old tricks leading to solutions in closed form.
3. Several chapters scattered throughout the book which consist of the following commercial, printed in bold-faced capitals across two adjoining pages:

"LEARN A LITTLE GERMAN AND LOOK FOR THE SOLUTION OF ANY GIVEN DIFFERENTIAL EQUATION IN THE SOURCE BOOK BY

E. KAMKE

DIFFERENTIALGLEICHUNGEN. LÖSUNGSMETHODEN UND LÖSUNGEN. I GEWÖHNLICHE DIFFERENTIALGLEICHUNGEN. LEIPZIG 1944."

4. Several chapters on approximate solution of the equation $y'=f(x, y)$. The treatment should include graphical methods (isoclines), the methods of power series, successive approximation and difference equations, and an existence and uniqueness theorem. At this stage exact solutions may be given for the types "variables separable" and "linear."
5. A chapter on linear equations of second and higher order with constant coefficients.
6. A chapter on second-order linear equations with simple analytic coefficients, treating series solutions and the behavior of solutions near (regular) singular points, including infinity.
7. A chapter dealing with linear systems. It is desirable to introduce the student to vector-matrix notation here. This chapter is the proper place for the introduction of "variation of parameters."
8. No chapter on applications where the main difficulties are in the applications.
9. No last-chapter attempt to treat the whole theory of partial differential equations, including Fourier series and other orthogonal expansions.

The present book scores much higher on test II than many other recent texts. It gives an outstanding treatment of power series methods, both for first-order nonlinear and for higher-order linear equations. Nevertheless there are bad letdowns when it comes to points 3 and 9 above. The authors meet points 2 and 7 only part way.

There are some finer points on which the reviewer likes to comment.

The authors define a solution as a relation $g(x, y)=0$. The reviewer feels that this is too much of a concession to the special case of an exact equation of the first order. He prefers to use the word function and to tie a solution to an interval.

The reviewer likes to solve the equation $y' = f(x)/g(y)$ by writing $g(y)y' = f(x)$ rather than $g(y)dy = f(x)dx$. A similar remark applies to the exact equation: avoidance of differentials tends to improve the understanding.

The power series method is used to investigate the behavior of solutions of linear equations near a finite singularity, but nothing is done about behavior near infinity.

In dealing with linear differential equations with constant coefficients the authors first give the method of guessing exponential solutions. They follow it by the Laplace transform method which reduces the differential equation to an algebraic equation. They do not mention the operator method of D's which is more elegant than guessing and has the advantage over the Laplace transform method of being independent of special assumptions about the growth of right hand member and solutions.

J. Korevaar
The University of Wisconsin

Quality Control and Applied Statistics Abstracts (Including *Operations Research*). Editors: Robert S. Titchen, Arnold J. Rosenthal, Bruce Bollerman, Frank Nistico. Interscience, New York. One volume of approximately 1000 pages in 12 issues annually, starting June 1956. $60.00 per year.

Articles considered by the editors to contain new information of lasting interest to the technical worker in Quality Control, Applied Statistics or Operations Research are abstracted each month from approximately 400 domestic and foreign journals. Each abstract shows fully the important contributions of the article abstracted including such items as formulas, illustrations, tables and references. The abstracts are published in loose-leaf form on strong $6\frac{1}{2} \times 9\frac{1}{2}$ inch sheets of white paper. For filing convenience each abstract is given a code numeral indicating the type of treatment and a letter indicating the field of application. (Under type of treatment the main divisions are Statistical Process Control, Sampling Principles and Plans, Management of Quality Control, Mathematical Statistics and Probability Theory, Experimentation and Correlation, Managerial Applications, Measurement and Control Instrumentation.) Issue 1 of vol. 1 lists the journals reviewed and the classification of abstracts.

Issues 1 and 2 of vol. 1 and issue 2 of vol. 2 were found to contain xvi+80 pp. including 37 abstracts from 28 journals, 86 pp. including 43 abstracts from 30 journals and 96 pp. including 39 abstracts from 21 journals, respectively.

This is an essential journal for all workers in the fields that it surveys.

John C. Brixey
University of Oklahoma

Grundprobleme der Mathematischen Theorie Elektromagnetischer Schwingungen. By Claus Müller. Springer, Berlin, 1957. ix+344 pp. DM 49.60.

Experience indicates that the classical electromagnetic radiation field is essentially uniquely determined by the nature and distribution of the matter

present. This poses a formidable mathematical problem: given the mathematical equations governing the electromagnetic field, to specify adequate mathematical boundary conditions that will correspond to reasonable physical situations, and to show that under these various conditions the corresponding solutions exist and are unique.

In the admirable book under review, Professor Müller, who has made important contributions towards the solution of this problem, shows how it is solved for the case of electromagnetic oscillations when the boundary conditions do not involve the time; and in so doing he presents this aspect of the mathematical theory of electromagnetic oscillations in a state of completion and elegance comparable to that of potential theory. He assumes, of course, that the reader is mathematically sophisticated; for instance, in his preparatory chapter "The Fundamental Concepts of Vector Analysis" he plunges immediately into the problem of redefining differential vector operators as limits of multiple integrals so that they will be applicable to nondifferentiable operands.

After some preliminary orientation, Professor Müller treats the classical case of scalar oscillations in detail, presenting it as a complete entity while using it as an occasion to develop ideas needed for the more difficult vector case. The full presentation of the solution of the latter case for both homogeneous and nonhomogeneous media occupies the major portion of the book and is the outstanding achievement that makes the book unique.

Professor Müller is a gifted expositor. His book is notable not only for its mathematical importance and its high standard of rigor but also for its exceptional clarity.

BANESH HOFFMANN
Queens College

An Introduction to Genetic Statistics. By Oscar Kempthorne. Wiley, New York, 1957. xvii+545 pp. $12.75.

This is a large and exceptionally complete volume, presenting in detail the various statistical tools which the research geneticist needs in his work. Practical genetic examples are provided under each topic.

The mathematics involved in the presentation do not go beyond algebra and the differentiation of quadratic forms, yet the topics considered are abstruse enough that a good fundamental understanding of statistical methods and a thorough working knowledge of genetics are both essential prerequisites for the use of the book. The opening chapter is a highly condensed presentation of elementary probability. Then comes a chapter devoted to an equally strong distillation of the principles of genetics.

Individual topics follow, each treated in highly concentrated form. They include among others selection, inbreeding, tests of genetic hypotheses, estimation of genetic parameters, planning of genetic experiments, use of matrices, analysis of variance, multiple regression, path coefficients, correlations and equilibria. Each topic is discussed in relation to various genetic complications

and chromosomal aberrations.

The author has included not only expositions of the works of others, but also considerable original work on some of the more difficult phases of genetics. As concentrated as the presentation is, it covers more than 500 pages. The research worker in genetics will find the book indispensable for the statistical treatment of very specific problems, even the most recondite.

LAURENCE H. SNYDER
The University of Oklahoma

Report on A Survey of Training and Research in Applied Mathematics. By F. J. Weyl, Investigator. Monograph No. 1, Society for Industrial and Applied Mathematics, Philadelphia, 1956. $2.00.

The objectives of this Survey were:

(a) to determine the nature and extent of research in applied mathematics currently being carried forward in universities, government agencies, and industrial establishments;
(b) to estimate the nature and extent of the unrealized potentialities of mathematics in applied contexts;
(c) to suggest the kind and character of further research in this field which is needed in the national interest;
(d) to consider the kind of training in applied mathematics which is now available in the United States; and
(e) to relate this information to the need for trained mathematicians in future years.

These objectives were achieved through (i) an investigating committee, (ii) a questionnaire, (iii) conferences. Reports of the conferences (iii) were published by the National Science Foundation and also in this MONTHLY, vol. 61, No. 7, Part II, 1954.

The report under review, which deals with (i) and (ii) above, is in two parts: I, Summary and Recommendations; and II, Review of Current Developments. In I, among other things, it is recommended that mathematicians in industry and government be invited to give courses in universities, and that university faculty members who are able to do so should give courses in applied mathematics. Part II discusses the evolution of the "Scientific Substance," and gives a review of applied mathematics activities.

C. O. OAKLEY
Haverford College

Introductory College Mathematics. By Robert W. Wagner. McGraw-Hill, New York, 1957. 430 pp. $5.50.

There is nothing very unusual about the table of contents. We find Numbers, Equations and Inequalities, four chapters on elementary functions, one each on

Trigonometry, Conics, and Calculus (of polynomial functions), and finally some material on Interest, Annuities, Statistics, and Probability.

The text is long-winded, sometimes "folksy," and often merely absurd. We may classify some of the absurdities as (a) pointless, (b) confusing, (c) misleading, and (d) false.

(a) Pointless. Addition and multiplication of "counting numbers" are interpreted as taking unions and cartesian products of sets; the laws of arithmetic are "proved" from this interpretation. The student is then expected to apply this new insight to problems like (p. 12): "A line segment is 7 feet long. Given that there are 12 in. in a foot, how many inches long is the segment? Use the set of inches to justify your answer."

(b) Confusing. On page 13 we read: "Theorem. The number of primes is not a counting number." The first use of "number" here is not covered in the author's Chapter 1 catalogue: Counting Numbers, Negative Numbers, Rational Numbers, Real Numbers, Complex Numbers, and Approximate Numbers.

(c) Misleading. On page 45, "relation" means "equation or inequality." On page 81, a relation is a set-valued function; this definition is fortunately not pushed very far, and is violated on page 83, where the graph of an inequality is implied to be a union of lines rather than a collection of them.

(d) False. On page 5, "most practical sets can be put into one-to-one correspondence with a subset of the counting numbers," On page 140, concerning the functions $f(x)=kx^m$, "Since the uses of these functions with negative values of k are very uncommon. . . . "

As for general absurdity, a good example, again concerning $f(x)=kx^m$, is "A second property of these functions and the one which accounts for their wide use is that if the value of x is doubled or tripled the value of the image is multiplied by 2^m or 3^m." Another example is the naming of twelve, rather than the usual six, inverse trigonometric functions (p. 222).

The reviewer has the impression that the author is trapped by his desire to write an introduction to mathematical thinking rather than to mathematical methods. He does neither. Even the problems illustrate only the most routine calculations. Nonroutine problems are advertised by an asterisk (see Preface); there are precisely two such problems (pp. 215 and 364).

WALTER RUDIN
University of Rochester

Calculus. By Jack R. Britton. Rinehart, New York, 1957. xiv+584 pp. $6.50.

This is one of many calculus texts, and one of the better. The author's stated purpose—"to present an introductory course in the calculus as simply as possible but with due regard for the modern requirements of rigor"—has been attained at a level of exposition suitable for both college students and advanced-standing high-school students.

The use of tables of numerical values for velocity and tangent problems in the first chapter provides an excellent basis for the later development of limits and derivatives. The scope of the book is traditional, with derivatives introduced on page 22, differentials on page 93, integrals on page 108, infinite series on page 316, approximate integrals on page 388, partial differentiation on page 396, multiple integrals on page 434, and differential equations on page 467. There is a table of 105 integrals. The appendix includes 22 infinite series for reference and about 70 curves with their equations. The book is self-contained with respect to tables. There are ample numerical exercises and problems; answers for most odd-numbered problems are included at the back of the book.

Some instructors will want to emphasize the concept of "range" when variables are first introduced, to consider some of the applications of the integral earlier, and to consider approximate integration earlier. Such modifications should not be difficult. Even if considered necessary, their inconvenience would be more than compensated for by the "Review and Discussion Questions" at the end of most chapters and the excellent exposition throughout the book.

BRUCE E. MESERVE
State Teachers College
Montclair, New Jersey

Mathematical Analysis. By Tom M. Apostol. Addison-Wesley, Reading, 1957. xii+553 pp. $9.00.

This book treats the topics which usually fall under the heading of "Advanced Calculus." It includes rigorous proofs of many of the theorems which are usually considered too difficult for an advanced calculus text, but too elementary for a course in function theory. In the author's words, "the book helps to fill the gap between elementary calculus and advanced courses in analysis. More important than this, it introduces the reader to some of the abstract thinking that pervades modern mathematics."

The first chapter, "The Real and Complex Number System," forms an excellent introduction. Elements of set theory are presented in the next two chapters. The remaining chapters include a discussion of limits and continuity, differentiation, applications of partial differentiation, functions of bounded variation, connectedness, the Riemann-Stieltjes integral, multiple and line integrals, vector analysis, infinite series and products, Fourier series and integrals, and an introduction to the theory of functions of a complex variable.

The appearance of the book is pleasing. Theorems and definitions are stated in italics. In addition to the usual index, the book contains an index of special symbols.

The exercises which appear at the end of each chapter are unusually well chosen; many of them will present a real challenge to the better students. Lists of references for further study conclude many of the chapters.

Because of the difficulties involved when many omissions are made from a

text, this book would seem most suitable for students who have had some introductory work in advanced calculus. For such classes it should prove to be excellent both as a text and as a reference book.

F. M. MEARS
The George Washington University

Theorie der Beugung Elektromagnetischer Wellen. By W. Franz. Springer, Berlin, 1957. iv+123 pp. DM 21.60.

If the title of this book led one to seek in it a comprehensive exposition of the theory of the diffraction of electromagnetic waves, one would be disappointed by its lack of completeness. But the book could hardly aim at completeness in view of its mere hundred-odd pages, and is more aptly described as a monograph than as a treatise. It takes for granted existence and uniqueness theorems, and is principally, though by no means wholly, concerned with presenting a detailed, self-contained account of the method of solution by means of the Watson transformation, especially as applied to the cylindrical case for which Professor Franz himself first gave the solution. The Watson transformation is a powerful device whereby certain series can be transformed into other series that converge far more rapidly. The author treats the subject with authority and insight, and in ample detail, giving careful discussions, for example, of the paths of integration of the complex integrals that yield the Watson transformations for the cylindrical and spherical cases, and explaining clearly the significance of the residue waves as the so-called creeping waves.

These items are contained in the central section of the book. They are preceded by a preliminary section dealing with such necessary matters as Green's functions and dyadics, and boundary conditions; and they are followed by a section dealing with diffraction by objects having edges. This final section treats an assortment of special topics, from the Sommerfeld wedge theory to the recent Braunbek method, and covers, with a few notable exceptions, the principal problems whose solutions are known.

The book is up-to-date both in method and outlook. Despite an evident pressure of space, Professor Franz has illuminated it with frequent interpretative insights. It will be warmly welcomed by people working in the field, and should be of more than passing interest to other mathematicians.

BANESH HOFFMANN
Queens College

Numerical Analysis. By Kaiser S. Kunz. McGraw-Hill, New York, 1957. xv+381 pp. $8.00.

This book, based on graduate lecture courses at the Harvard University Computation Laboratory, is designed to "acquaint the student with the best procedures available for obtaining numerical solutions to problems arising in applied mathematics," with special attention to differential and integral equa-

tions. Eleven chapters discuss interpolation, in one and two variables, numerical differentiation and integration, the start and continuation of step-by-step methods for ordinary differential equations, and some methods for solving elliptic, parabolic and hyperbolic partial differential equations and Volterra and Fredholm integral equations. Four other chapters deal with the roots of equations and particularly polynomials, the summation of series, and the solution of linear algebraic equations and the inversion of matrices. A short appendix discusses simple error analysis.

There are also some surprising omissions: for example, the "cross-mean" iterative interpolation method of Aitken, the work of Comrie and others on subtabulation and the use of modified differences, the valuable implicit method for parabolic equations, for hyperbolic equations the method of integrating along characteristics, and for algebraic equations the concept of ill-conditioned matrices. Most important, there is no discussion of the eigenvalue problem, for either matrices or differential operators. This is surely an important problem in applied mathematics, not to be dismissed as a peripheral subject along with "smoothing, least-square approximation and harmonic analysis."

The material, however, is well presented, has little "advanced" mathematics, and is interesting and instructive to both student and expert. The chapter on integral equations is particularly good, that on bivariate interpolation more readable than most such expositions, and there is an interesting extension of the lozenge diagram for differentiation and integration. Remainder terms are derived for every formula, there is a useful discussion of error accumulation in step-by-step integration, and both worked examples and exercises for students are included in every chapter. The student, while undoubtedly benefiting from a knowledge of this book, will also have many questions to put to his supervisor.

L. Fox
Oxford

An Introduction to Diophantine Approximation. By J. W. S. Cassels. Cambridge Tracts in Mathematics and Mathematical Physics No. 45, Cambridge University Press, New York, 1957. x+166 pp. $4.00.

This tract presents basic techniques and outstanding results of Diophantine approximation in a way such that an undergraduate senior or first-year graduate student who has a knowledge of the elements of number theory should be able to read it. However, one chapter requires the elements of Lebesgue theory and another chapter requires the elements of algebraic number theory. Material needed from the geometry of numbers appears in the appendixes.

The first chapter on homogeneous approximation introduces the continued fraction process, equivalence, application to approximations and simultaneous approximation. This is followed by a study of indefinite binary quadratic forms, Markoff forms, the Markoff chain of forms and the Markoff chain for approximations. Chapter III introduces inhomogeneous approximation with a study of

the 1-dimensional case (two Minkowski theorems) and simultaneous approximation (Kronecker's theorem). The chapter on uniform distribution considers uniform distribution of linear forms and Weyl's criteria and consequences. The chapter on transference theorems studies transference between two homogeneous problems and application to simultaneous approximations, transference between homogeneous and inhomogeneous problems with application to inhomogeneous approximation, a quantitative Kronecker's theorem and successive minima. The sixth chapter is concerned with Roth's theorem and rational approximation to algebraic numbers. Chapter VII on metrical theory assumes a knowledge of Lebesgue theory. The concluding chapter studies theorems of Pisot, Vijayaraghavan and Salem on Pisot-Vijayaraghavan numbers. Each chapter has a careful introduction and is concluded with an illuminating section of notes. The three appendixes discuss bases in certain modules, tools from the geometry of numbers and Gauss's lemma. An excellent five-page bibliography is included.

JOHN C. BRIXEY
University of Oklahoma

International Series of Monographs in Pure and Applied Mathematics. Vol. 1. *An Introduction to Algebraic Topology.* By A. H. Wallace. Pergamon, New York, London, Paris, 1957. 198 pp. $6.50.

Homology theory has been given a beautiful axiomatization by Eilenberg and Steenrod in their book "Foundations of Algebraic Topology." In the preface to this volume the authors state "Homology theory is a transition (or function) from topology to algebra. It is this transition which is axiomatized."

The present volume serves as a motivation for this transition. The first third of the book gives the basic topology needed before the appearance of any algebra. A topological space is defined by means of the collection of all neighborhoods of each of its points, and these neighborhoods need not be open sets. It turns out that a neighborhood of a point p is any set containing any open set which contains p. The reviewer feels that "vicinity" is a better choice of name for such a set than "neighborhood." The other novelty about this topological discussion is the successful definition of arcwise connected spaces without defining arcs.

Chapters IV to IX, the remainder of the text, discuss the fundamental group and develop the theory of homology sequences. Chapters VI, VII, VIII prove the Homotopy, Excision, and Exactness Axioms of Eilenberg and Steenrod for the spaces discussed by the author. It is unfortunate that the Exactness Axiom comes so late in the book (p. 153) that no effective use is made of the powerful tool of commutativity diagrams.

It is a disappointment in an introductory text such as this that the Jordan Curve theorem, while assumed, is never explicitly mentioned. Practically all of the text consists of building up machinery which can be used in later courses, *e.g.* in a course based on Eilenberg and Steenrod. Important application will be

discovered by the student only if he digs into the numerous problems. One such application (Problem 2, p. 167) is the fixed-point property for solid r-dimensional Euclidean spheres. Problem 7, page 54, and Problem 1, page 61, contain errors.

DICK WICK HALL
Harpur College

The Teaching of Mathematics. By the Incorporated Association of Assistant Masters in Secondary Schools, Cambridge University Press, New York, 1957. ix+231 pp. $3.00.

This volume gives a complete and clear picture of the mathematics program in the grammar schools of England. The subject matter is outlined in detail, and the spirit and philosophy of instruction emerge in every chapter. The program reveals a true English conservatism that doggedly plods ahead to necessary reform. There is a great change from nineteenth-century sequential Euclidean geometry. The algebra program remains typically that of the English schools. Perhaps unique in this book is a summary of expectations from pre-secondary school study of mathematics and the large segment of space devoted to pedagogical matter. Among the latter are procedures for inculcating a love for mathematics, the organization of the class lesson, the role of questioning, tests and examinations; the use of audio, visual and other aids; the role of a mathematics homeroom or laboratory, a mathematics library and its use and the role of history in mathematics.

The teaching of statistics as a new subject (since 1948) reflects an influence of Yule and Kendall in determining the topics taught. Perhaps one novel characteristic is that of a sixth form which can occupy one, two or three years of study before the student continues to the university. The primary change, in point of view of organization, is the breakdown of barriers between arithmetic, algebra, geometry and analysis in order to achieve unity. It is doubtful that the proposed syllabus will achieve this unity. To a reader who is concerned with the teaching of mathematics in the U. S., there cannot help but come the feeling that the problems posed in this book are those with which we have been struggling in our country during the last 20 years and from which we have now moved forward.

HOWARD F. FEHR
Columbia University

BRIEF MENTION

Offerings and Enrollments in Science and Mathematics in Public High Schools, 1956. By Kenneth E. Brown and Ellsworth S. Obourn. U. S. Department of Health, Pamphlet No. 120, 1957. 44 pp. $0.25.

It is encouraging to note that the percentages of students taking science and mathematics courses have increased between 1954 and 1956, as a comparison of the study based on the 1954 data shows (see review this MONTHLY,

vol. 64, 1957, p. 130). The data are well presented. Not only national averages, which are, after all, somewhat meaningless, but also regional data are given, with the United States broken into nine regions. The national average shows that only eight per cent of the pupils in the twelfth grade are now in high schools which do not offer trigonometry, intermediate algebra, solid geometry or other comparable mathematics. However, the regional percentages range from 0.4 per cent to 19.1 per cent. This inexpensive little booklet should prove helpful to colleges and universities planning to revise their mathematics requirements in the realm of remedial and freshman mathematics.

Puzzle Math. By George Gamow and Marvin Stern. Viking, New York, 1958. 119 pp. $2.50.

A collection of 32 puzzles or brain twisters, some old, some new, but all charmingly presented in the authors' inimitable style. A pedant might quibble over the phrase "three times smaller" (p. 61), but this is picayune.

Mathematics Charts from the Dawn of History to 1950 A.D. By L. E. Christman and L. S. Overeem. Christman, Yorkville, Illinois. 13 pp. $1.50.

A set of charts which could serve as a framework for a talk on the history of mathematics. One may be surprised to see an ordinary desk calculator under the heading, "High Speed Computing Machines."

An Introduction to Scale Coordinate Physics. By William Bender. Burgess, Minneapolis, 1958. ix+340 pp. $7.50.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NATIONAL SCIENCE FOUNDATION SPONSORS CONFERENCE ON NATIONAL PROBLEMS IN MATHEMATICS

A special conference to consider national problems of research and training in mathematics was held at the University of Chicago, February 20 and 21, under the auspices of the National Science Foundation. Mathematicians representing all sections of the country held sessions on the following topics: (1) the need for centers of advanced study, (2) federal support for mathematical research in the universities, (3) texts for high school mathematics.

NOTES ON ANNUAL MEETING OF THE ASSOCIATION

Notes on the three program sessions of the forty-first annual meeting of the Mathematical Association of America have been prepared by Mr. Frank Kocher. Copies of these notes are available to members of the Association on request to Mr. Kocher at the address: Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania.

INTENSIVE COURSE IN OPERATIONS RESEARCH

An intensive course in Operations Research and Systems Engineering for business, industrial and government personnel was offered by the School of Engineering of the Johns Hopkins University from June 9 through June 20, 1958, at the Homewood Campus of the University. Introductory lectures were given by the Dean of the School of Engineering and the Directors of the University's Applied Physics Laboratory and Operations Research Office. These were followed by numerous case studies and expository lectures on Operations Research, Systems Engineering, Cost Data, Models, Human Engineering, Data Processing, Simulation, Information Theory, Quality Control, Design of Experiments, Game Theory, Flow Graphs, System Dynamics, Inventory Systems, Waiting Lines, Symbolic Logic, Stability and Linear Programming.

PRINCETON PROGRAM TO MODERNIZE COLLEGE PREPARATORY MATHEMATICS

A long-range approach to the objective of evolving a truly modern curriculum in college preparatory mathematics is being undertaken in Princeton, New Jersey, by the school boards of five different municipalities working in close cooperation with mathematicians associated with Princeton University and the Princeton-headquartered Educational Testing Service. Based upon nearly a year of discussions on the part of a Curriculum Revision Committee, brought into being by the Princeton Borough and Township Schools and more recently expanded to include three bordering municipalities, the newly-established program calls for a long, hard look at mathematics—from beginners' arithmetic to college calculus—as an integral part of the education of any future citizen.

The first step consists of a fifteen-week series of one-hour classes which are designed to give mathematics teachers in public elementary and secondary schools "added insight into the nature of mathematics as a creative endeavour in twentieth-century civilization." This orientation program, open on a voluntary basis to all mathematics teachers in the five cooperating school systems, has been divided into two sections, "advanced," meaning "algebra and beyond," and "elementary" for teachers in Grades IV through VII.

The orientation classes, continuing through June, will be supplemented next fall by a year-long "curriculum for teachers," which will be largely concerned with course-material in special areas, such as algebra, plane and solid geometry and trigonometry, and the relationships between the teachers' individual courses and new conceptions in mathematics.

In the near future, beyond the 1958–59 plan of study for teachers, who will also be urged to participate in summer institutes in mathematics, lies a revamped curriculum that will give students a view of contemporary mathematics as a whole and will enable qualified students to take the equivalent of an added year of advanced mathematics at the high-school level.

The present organization of the "Princeton Mathematics Program" has been carried forward by two expert parent-advisers to the schools' Curriculum Revision Committee, Dr. Marion G. Epstein, a member of the Mathematics Section in the Educational Testing Service's Test Development Division, and Professor Albert W. Tucker, chairman of the Princeton University's Department of Mathematics.

PERSONAL ITEMS

Professor G. B. Price, of the University of Kansas and Professor B. W. Jones of the University of Colorado have been appointed official delegates of the Association to the International Congress of Mathematicians, to be held in Edinburgh in August, 1958.

Professor G. B. Price represented the Association at the inauguration of Richard Aubrey McLemore as President of Mississippi College on March 19, 1958.

Professor M. F. Rosskopf of Teachers College, Columbia University, has received a Fulbright Award to lecture at the Pedagogisk Seminar, University of Oslo, for the year 1958–59.

Mr. M. T. Austin, Rutgers University, has accepted a position as mathematics analyst for the Chrysler Missile Division, Detroit, Michigan.

Mr. W. E. Beeman, North Texas State College, has been appointed Associate Professor at Arlington State College.

Assistant Professor W. S. Bishop, Howard College, has been promoted to Associate Professor.

Mr. C. K. Bradshaw, University of Nevada, is now a graduate assistant at Stanford University.

Dr. Louis DeBranges, Cornell University, has been appointed Assistant Professor at Lafayette College.

Mr. R. S. Dick, Columbia University, has been appointed Instructor at Queens College of the City of New York.

Mr. G. E. Duncan, Georgia Institute of Technology, has accepted a position as associate mathematical engineer for Lockheed Aircraft Corporation, Marietta, Georgia.

Miss Angeline W. Evans, Agnes Scott College, has accepted a position with the First National Bank of Atlanta, Georgia.

Pfc. P. J. Finn, St. John's College, Brooklyn, is now serving at the Combat Development Experimentation Center, Fort Ord, California.

Professor Michael Golomb, Purdue University, is on leave during 1957–58 and is a member of the Mathematics Research Center, University of Wisconsin.

Mr. R. G. Green, CONVAIR, San Diego, California, has accepted a position as dynamics engineer.

Dr. B. R. Levy, Institute of Mathematical Sciences of New York University, has been appointed Research Associate.

Mr. W. J. Lyche, North American Aviation, Los Angeles, California, has been appointed Assistant Professor at Long Beach State College.

Mr. W. J. Mays, Imperial Life Insurance Company, Asheville, North Carolina, has accepted a position as actuary with Western & Southern Life Insurance Company, Asheville.

Assistant Professor Ralph Playfoot, Lafayette College, has been promoted to Associate Professor.

Associate Professor T. E. Rine, Illinois State Normal University, has been promoted to Professor.

Mr. R. E. Schlea, Ohio Northern University, has been appointed Assistant Professor at Baldwin-Wallace College.

Mr. W. L. Shepherd, Texas Western College, has been appointed Assistant Professor.

Assistant Professor Peter Terwey, Jr., Davidson College, has been appointed Associate Professor at Lamar State College of Technology.

Professor L. F. Tolle, St. John's University, New York, has been appointed Chairman, Department of Mathematics.

Mr. V. D. Turner, Mankato State Teachers College, has been appointed Assistant Professor.

Dr. Jack Warga, Electro-Data Corporation, Pasadena, California, has accepted a

position as senior staff scientist with AVCO Manufacturing Corporation, Lawrence, Mass.

Associate Professor L. A. Warwick, Wesleyan College, has been appointed Head, Department of Mathematics at William Jennings Bryan University.

Mr. A. M. White, University of Santa Clara, has been appointed Assistant Professor.

Mr. R. S. Wolfe, Northwestern University, has been appointed Assistant Professor at Rollins College.

Miss Elaine G. Yodice, University of Wisconsin, has been appointed Instructor at the University of Massachusetts.

Mrs. Grace A. deForest died on April 5, 1958 after a brief illness. For the past nine years she has been in charge of the Buffalo office of the Association. She has served the Association well and faithfully and will be greatly missed.

Professor Emeritus Floyd Field, Georgia Institute of Technology, died February 14, 1958. He was a charter member of the Association.

Assistant Professor A. S. Hendler, Rensselaer Polytechnic Institute, died June 11, 1957.

Mr. E. H. Koch, Jr., retired, died January 26, 1958. He was a charter member of the Association.

Sister Ann Elizabeth, Professor and Registrar, St. Mary College, Kansas, died May 14, 1957. She had been a member of the Association for 23 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 96 persons have been elected to membership by the Board of Governors on applications duly certified.

HERBERT L. ARNOLD, Freezer-attendant, Campbell Soup Co., Chicago, Illinois.

DENVER BAILEY, M.S. (Kentucky) Teacher, St. Petersburg Junior College.

NICHOLAS S. BARKAS, M.A. (Boston) Professor, Varuakion Graduate School, Athens, Greece.

LOUIS J. BARRON, M.A. (Michigan) Instructor, St. Petersburg Junior College.

WILLARD E. BAXTER, Ph.D. (Pennsylvania) Asst. Professor, Ohio University.

HERBERT B. BEBB, Student, University of Oklahoma.

RAYMOND E. BLUME, A.A. (Riverside) Student, University of California, Riverside.

CHARLES E. BOYNTON, IV, B.A. (Emory) Grad. Asst., Emory University.

ALLAN U. BRENDER, Student, McGill University.

ALFRED J. BRUEY, Student, Fenn College.

KENNETH C. BULLOCK, B.S. (Oklahoma) Grad. Asst., Oklahoma State University.

JAMES W. CAFKY, Student, University of Oklahoma.

MRS. MARY L. CANTWELL, M.A. (Columbia) Teacher, St. Petersburg Junior College.

FRANCIS W. CARROLL, M.S. (Purdue) Grad. Fellow, Purdue University.

MISS YI CHANG, M.S. (Illinois) Grad. Asst., University of Illinois.

MRS. VIVIAN S. COHEN, M.A. (California) Instr., Sacramento Junior College.

LLOYD M. COOK, Ed.D. (Univ. of California, Berkeley) Professor, Chico State College.

ALEXANDER S. DAVIS, Ph.D. (North Carolina) Mathematician, National Security Agency, Fort Meade, Maryland.

FRANCES E. DAVIS, M.A. (Michigan) Asst. Professor, University of Hawaii.

LLOYD D. DAVIS, A.B. (Ohio Northern) Grad. Asst., University of Miami.

RAYMOND A. DICKEY, M.A. (Colorado State) Teacher, Central High School, Pueblo, Colorado.

ROBERT D. EAGLETON, Student, Abilene Christian College.

EDWARD M. EDWARDS, Student, University of British Columbia.

SHELDON J. EINHORN, M.A. (Pennsylvania) Asst. Instr., University of Pennsylvania.

PAUL R. ELBERT, Student, Butler University.

ROBERT ELLIS, Ph.D. (Pennsylvania) Asst. Professor, University of Pennsylvania.

FRANCIS J. FELIX, A.B. (Lycoming) Instr., Pennsylvania State University.

MRS. VIVIAN J. FIELDER, M.S. (Illinois) Grad. Asst., University of Illinois.

ROBERT M. FOOTE, M.A. (Texas) Mathematician, American Oil Co., Texas City, Texas.

CLYDE G. FORD, B.A. (Arkansas State) Teaching Asst., University of Oklahoma.

RICHARD K. FOSU, B.S. (McGill) Grad. Student, University of Toronto.

WALTER GAUTSCHI, Dr. (Basle) Mathematician, National Bureau of Standards, Washington, D. C.

FLORENCE E. GILLHAM, M.A. (Columbia) Teacher, St. Petersburg Junior College.

GERALD K. GOFF, M.Ed. (Phillips) Asst. Professor, Southwestern State College.

ROBERT V. GOORDMAN, Electronics Circuit Designer, Bell Telephone Laboratories, South Orange, New Jersey.

PETE A. GRACIA, Student, Lamar State College of Technology.

WAYNE W. GUTZMAN, Ph.D. (Iowa) Professor, University of South Dakota.

JACK L. HARRIS, B.S. (Georgetown) Grad. Asst., University of Kentucky.

EDITH M. HESS, M.L. (Houston) Data Analyst Specialist, Telecomputing Corp., Holloman AFB, New Mexico.

BILLY F. HOBBS, M.A. (Ball S.T.C.) Asst. Professor, Olivet Nazarene College.

GEOFFREY HORROCKS, M.A. (Oxford) Mathematician, Northern Life Assurance Co. of Canada, London, Ontario.

JOHN M. HORVATH, Ph.D. (Budapest) Asst. Professor, University of Maryland.

LAWRENCE C. HOUSE, A.B. (Boston) Asst. Instr., University of Connecticut.

DONALD F. JORDAN, Student, Texas Technological College.

ALVERN W. KAUFMANN, M.A. (Ohio S.U.) Professor, Roberts Wesleyan College.

FERDINAND KERTES, D.Sc. (Rutgers) Hd., Dept. of Math., Perth Amboy High School, New Jersey.

MIDDLETON LAMBERT, B.S. (London) Principal, Texada Elementary-Senior High School, Vananda, B. C.

ANTHONY T. LAURIA, M.S. (Purdue) Research Asst., Purdue University.

CLIFFORD A. LONG, M.S. (Illinois) Grad. Asst., University of Illinois.

RAYMOND A. LUTHER, Engineer, Western Electric, Kearny, New Jersey.

ROGER M. LUTZ, B.S. (Mt. Union) Teaching Fellow, University of Cincinnati.

JOHN E. McDOWELL, Switchman, Bell Telephone Co., Newark, New Jersey.

MRS. FLORENCE D. MESSINGER, B.A. (Wells) Instr., Texas Christian University.

CAREY G. MUMFORD, Ph.D. (Duke) Professor, North Carolina State College.

WILLIAM C. NEMITZ, M.S. (Ohio S.U.) Grad. Asst., Ohio State University.

PAUL T. NUGENT, A.B. (Franklin) Grad. Asst., Miami University.

MRS. INGRID OWREN, M.S. (Oslo) Instr., University of Alaska.

KYU S. PARK, B.S. (Seoul National) Instr., Dong-A University, Pusan, Korea.

JOSEPH A. PETERS, B.S. (St. Joseph's) Grad. Asst., University of Illinois.

WADE A. PETERSON, B.A. (Washington) Instr., University of Alaska.

GIDEON PEYSER, Ph.D. (New York) Instr., Newark College of Engineering.

ROBERT P. PIKUL, M.S. (Minnesota) Operations Research Analyst, United Aircraft Corp., East Hartford, Conn.

RAMAVARAPU R. RAO, M.S. (Banaral Hindu) Grad. Asst., University of Oklahoma.

JACK REYNOLDS, B.S. (Oklahoma) Instr., University of Oklahoma.

JAMES J. RHYNE, Student, University of Oklahoma.

CAROLYN S. RICE, Student, Texas Christian University.

C. DAVID ROBBINS, M.A. (Arkansas) Instr., University of Oklahoma.

RUTH M. ROBERTS, M.S. (Chicago) Asst. Instr., University of Pennsylvania.

MALCOLM S. ROBERTSON, Ph.D. (Princeton) Professor, Rutgers University.

WILFRED J. ROESLER, A.B. (Loyola) Research Asst., University of Minnesota.

FRANK D. RUSSELL, B.A. (DePauw) Data Analysis Engineer, Telecomputing Corp., Holloman AFB, New Mexico.

ROBERT F. RUTSCHOW, B.S. (V.M.I.) Grad. Student, Pennsylvania State University.

NORMAN SCHAUMBERGER, M.A. (C.C.N.Y.) Instr., Cooper Union.

DONALD C. SCOUTEN, Student, Oklahoma State University.

ANTHONY SEPAN, Student, Temple University.

RONALD M. SHELTON, M.S. (Illinois) Grad. Asst., University of Illinois.

WILLIAM A. SIBLEY, M.S. (Oklahoma) Research Asst., University of Oklahoma.

REBECCA E. SLOVER, Student, Georgetown College.

CLIFFORD W. SLOYER, JR., B.A. (Lehigh) Grad. Asst., Lehigh University.

WAYNE E. SMITH, M.A. (U.C.L.A.) Associate, University of California, Los Angeles.

DONALD N. SPOTH, Data Analyst Engineer, Telecomputing Corp., Holloman AFB, N. M.

KARL STROMBERT, M.A. (Oregon) Acting Instr., University of Washington.

J. ROGER TELLER, B.S. (Cincinnati) Grad. Student, University of Cincinnati.

WILLIAM J. THOMAS, Ph.D. (Michigan State) Asso. Professor, Baylor University.

RALPH N. TOWNSEND, M.S. (Illinois) Asst., University of Illinois.

SHIRLEY R. TREMBLEY, M.A. (Fresno S.C.) Instr., Bakersfield College.

JACK P. TULL, Ph.D. (Illinois) Instr., Ohio State University.

HARRIS W. VAYO, B.A. (Culver-Stockton) Asst., University of Illinois.

GEORGE E. WALLACE, M.A. (Stanford) Teacher, College of San Mateo.

LEROY J. WARREN, Ph.D. (Oregon) Asst. Professor, San Diego State College.

MARTHA F. WATSON, A.B. (Murray S.C.) Grad. Asst., University of Kentucky.

MARTIN T. WECHSLER, Ph.D. (Michigan) Asst. Professor, Wayne State University.

HELEN S. WEIHE, A.B. (Ursuline) Grad. Student, University of Kentucky.

KENNETH J. WHITCOMB, M.A. (Nebraska) Instr., Colorado State University.

BILL WILSON, Student, Baker University.

DAVID G. WILSON, Student, University of Oklahoma.

NEW DEPARTMENT

It is proposed to establish a department of the MONTHLY that would be concerned with mathematics education. This department would include news about current efforts to revise the high school and college curriculum in mathematics, about problems connected with teacher training and certification, and other similar activities. Until an associate editor is appointed to conduct the department, articles may be sent to the editor, Professor R. D. James.

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section, Mathematical Association of America was held at Haverford College on Saturday, November 30, 1957 with Professor Albert Wilansky, Lehigh University, Chairman of the Section, presiding. There were 67 present, including 59 members of the Association.

Dr. I. E. Block, Burroughs Corporation, was elected Chairman of the Section for 1957–58. Professor D. W. Western, Franklin and Marshall College, was elected to the Executive Committee.

The matter of participation by this Section in the National Mathematics Contest for high schools was discussed and the following action taken:

(1) The Philadelphia Section approves in principal that we participate in the Mathematics Contest for high schools in our region.

(2) The chairman shall appoint a committee to work out details with power to act for the Association. This committee shall plan to hold these contests beginning in March of 1959, and shall decide on methods of administration, publication, and awards. It shall, by March 30, 1958, submit its recommendations by mail to the entire membership for comments.

The committee appointed consists of Professors W. S. Lawton (Chairman), T. L. Koehler, E. R. Mullins, Jr., G. C. Webber and D. W. Western.

The following papers were presented at this meeting:

1. *Experimental statistics—some of the concepts and mathematical requirements*, by Dr. Stuart Hunter, American Cyanamid Company and Princeton University, introduced by the Secretary.

2. *Mass distributions on the circle and convex conformal maps*, by Professor I. J. Schoenberg, University of Pennsylvania.

An elementary method is presented which yields the solution in a number of special problems such as the following: Let $1<n_1<n_2< \cdots$ be a given infinite sequence of integers. (1) To find among all positive definite sequences $\{\mu_n\}$ $(n=0, \pm 1, \pm 2, \cdots)$ with the properties $\mu_0=1$, $\mu_{n_1}=\mu_{n_2}= \cdots =0$, the one which maximizes $|\mu_i|$. (2) For the class of power series $w=F(z)=z+c_{n_1}z^{n_1}+c_{n_2}z^{n_2}+c_{n_3}z^{n_3}+ \cdots$ which map $|z|<1$ onto a convex and univalent domain D, to find the largest circle $|w|<\rho$ which is covered by all these convex images D. The paper will appear shortly in the *Proceedings of the Netherlands Academy of Sciences*.

3. *A report on the recommendations of the Commission on Mathematics of the College Entrance Examination Board*, by Professor A. W. Tucker, Princeton University.

4. *Mathematics at a National Science Foundation Summer Institute*, by Professor David Rosen, Swarthmore College.

Professor Rosen's course on "Axiomatics of Number Systems" was given at the University of Pennsylvania in the summer of 1957. It was designed specifically for high school teachers.

G. C. WEBBER, *Secretary*

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-fifth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Loyola University, New Orleans, Louisiana on February 21–22, 1958. The Friday afternoon meeting was held in two concurrent sessions. Professor N. A. Childress, Mississippi Vice-Chairman, and Professor S. M. Spencer, Louisiana Vice-Chairman, presided. Professor A. C. Grimes of Mississippi State College, Chairman of the Section, presided at the Friday evening and Saturday morning sessions. There were 157 persons registered, including 65 members of the Association.

The following officers were elected for the coming year: Chairman, Professor C. G. Killen, Northwestern State College; Vice-Chairman for Mississippi, Professor S. R. Knox, Millsaps College; Vice-Chairman for Louisiana, Professor T. F. Mulcrone, S. J., Loyola University; Secretary-treasurer, Professor T. L. Reynolds, Millsaps College.

At the business meeting Professor H. T. Karnes of Louisiana State University gave a report of the committee on the High School Mathematics Contest, recommending methods for administering the contest within the section. The report was accepted.

The invited speaker for the Friday night meeting was Professor A. H. Clifford of Newcomb College, Tulane University who spoke on "Semigroups." The Saturday morning session heard a panel discussion on "The Undergraduate Mathematics Program at Tulane" by Professor A. D. Wallace, Mathematics Department, Moderator, Dean P. V. Grambsch, Business Administration, Dean L. H. Johnson, Engineering and Dr. C. W. Wing, Director of Admissions.

The following papers were presented:

1. *The High School Mathematics Contest*, by Professor W. H. Fagerstrom, Executive Director, The National Contest, Pan American College.

The speaker gave a brief history of the contest since its beginning in 1949 and discussed its national and international scope and pointed out that 73,000 high school students are scheduled to participate in the contest this year. The plan for giving awards was discussed and it was pointed out that a medal would be given to the leading student in each high school participating.

2. *Old curves, new definitions*, by Professor V. B. Temple, Mississippi Southern College.

A pedal curve is defined as the path traced by the vertex of a right angle as it moves according to specified geometric conditions. The following were defined as pedal curves: (1) The straight line, continuous and closed; (2) The parabola, continuous and closed; (3) The hyperbola as orthic (equilateral), and oblique (nonequilateral); (4) The hypo- and epi-cycloids; (5) The so-called rose curves (roses and clovers); (5) The modified Longchamps curve; (7) The cissoid of Diocles and the strophoid as subpedal curves.

3. *The graduate program in mathematics at Tulane University*, by Professor Fred B. Wright, Tulane University.

The current graduate program at Tulane has been developed since 1947. It is designed to lead a student to the areas of current research as quickly as possible. At the same time, it is so organized as to permit students of varying backgrounds to be inserted into the appropriate place in the curriculum. The success of the program can be measured by comparing the backgrounds of the persons who have been awarded the doctorate to the positions they have held since graduation.

A more recent program is embodied in the degree of Master of Arts in Teaching in Mathematics. This program is designed for teachers in secondary and primary school mathematics, and is administered with the advice of the Department of Education.

4. *A method of expansion of determinants*, by Professor S. R. Knox, Millsaps College.

A compact method of evaluating determinants is exhibited, noting advantages and disadvantages relative to similar methods.

5. *A note on the teaching of infinite series*, by Professor S. B. Murray, Mississippi State College.

In the usual textbook in Calculus the conditions for convergence of an alternating infinite series are given as (1) the Nth term must approach zero as N approaches infinity, and (2) the absolute value of the $N+1$ term must be less than the absolute value of the Nth term, after a certain N. Students often raise a question about the second condition, thinking it must follow from the

first. Two examples are presented to show that the first condition may be satisfied while the second is not satisfied.

6. *An experiment in large-section teaching at Louisiana State University*, by Professor F. A. Rickey, Louisiana State University.

A report on the teaching of Trigonometry in large sections of 180 each with two hour-lectures a week and small group discussions once a week. The results were favorable when compared with those of regular-sized trigonometry classes.

7. *Finding the sums of certain series*, by Professor W. B. Temple, Louisiana Polytechnic Institute.

The fact that $1^2+\cdots+n^2=n(n+1)(2n+1)/6$ can be easily proved by mathematical induction. In this paper this familiar formula is derived. The method used may be applied to deriving formulas for sum of cubes, fourth powers, *etc.*, of the positive integers up to n as well as $\sum_{i=1}^{n} 1/(i(i+1))$ and a number of other finite sums.

8. *Continuous solutions of two functional equations*, by Professor R. D. Boswell, Jr., Mississippi State College.

Let a, A, and k be real numbers where $A>0$. The author shows that the only continuous solutions of the equation $f(x+y)=f(x)+f(y)+a(1-A^x)(1-A^y)$ are those of the form $f(x)=kx-a(1-A^x)$ and the only continuous solutions of the equation $f(x+y)=A^xf(y)+A^yf(x)$ are those of the form $f(x)=kxA^x$.

9. *Mathematics and high-speed computers*, by Professor B. B. Townsend, Louisiana State University.

This paper was an expository presentation of more or less pertinent remarks concerning the mathematics used in digital-computer work, the author's views on what we should teach in this regard, and perhaps some very elementary explanations of what a digital computer does.

10. *The target-missile problem on the electronic-analog computer*, by Professor Margaret M. LaSalle, Southwestern Louisiana Institute.

The object is to plot a trajectory as a missile approaches a moving target. The differential equations of motion are set up, two-dimensional case. A servo-mechanism is used to convert from polar to rectangular coordinates so that an xy-plot is recorded for different initial conditions and angles of attack. Ascertained from this data is the curve requiring the shortest time for a hit.

11. *An analogue of the Chinese remainder theorem in group theory*, by Professor Eugene Schenkman, Louisiana State University, introduced by the Secretary.

For $i=1,\cdots,n$ let C_i be a collection of normal subgroups of G; and let B_i be the intersection of all the $C_j, j=1,\cdots,n, j\neq i$. Then the following two statements are equivalent: (1) If $a_1,\cdots a_n$ are arbitrary elements of G then the intersection of the cosets a_iC_i for $i=1,\cdots,n$ is not empty, and (2) $G=B_1B_2\cdots, B_n$.

T. L. REYNOLDS, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 14-15, 1958, at the University of Florida, Gainesville, Florida. Professors Trevor Evans, Chairman of the Section; D. E. South, Vice-Chairman; F. W. Kokomoor, Sectional Governor; and E. H. Hadlock presided over the general and divisional sessions. There were 219 in attendance, including 139 members of the Association.

The following officers were elected for the coming year: Chairman, Professor D. E.

South, University of Florida; Vice-Chairman, Professor T. C. Carson, East Tennessee State College; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College. The Section accepted the High School Contests Committee's report that at this time; since several states have a contest of their own, the Section should not attempt to administer the national contest. It was felt that both the decision to participate in and the administration of the contest should be on the state level. A new committee was named, each member being a state director except the chairman: Professor J. H. Banks (chairman), Professor Mariano Garcia, Jr. (Puerto Rico), Miss E. M. Lynch (Georgia), Professor J. D. Mancill (Alabama), Professor E. D. Nichols (Florida), Professor J. W. Sawyer (North Carolina), Professor C. E. Shuler (South Carolina) and Professor F. L. Wren (Tennessee).

As was done some twenty years ago, the Secretary arranged a display of 63 mathematics books published by members of the Southeastern Section.

The following program was presented:

1. *An algebraic solution of an extreme value problem in Euclidean n-space*, by Professor G. B. Huff, University of Georgia.

A fundamental problem in factor analysis may be reduced to the following problem in Euclidean n-space. If $z_1, \cdots, z_n$ is a set of unit vectors and $r<n$, what is the space S_r of dimension r which most nearly contains $z_1, \cdots, z_n$? It is shown by algebraic methods that the S_r for which the sum of the squares of the distances to $z_1, \cdots, z_n$ is a minimum is spanned by the r largest characteristic vectors of a matrix R, where r_{ij} is the cosine of the angle between z_i and z_j.

2. *A construction for some finite geometries*, by Professor J. R. Wesson, Vanderbilt University.

The finite plane with $n+1$ points on each line has points which may be named $1, 2, 3, \cdots, n^2+n+1$. In case n is a prime, the points on each line may be listed in a direct and straightforward manner. This construction is accomplished without "trial and error" and without using coordinates other than the labels $1, 2, \cdots, n^2+n+1$.

3. *A new proof of the theorem of mean value*, by Mr. Chung-Lie Wang, University of South Carolina.

A proof of the mean value theorem is given based on translation and rotation of axes, without any appeal to geometric intuition.

4. *Keeping track of Explorer*, by Dr. C. L. Bradshaw, Computation Laboratory, ABMA, Huntsville, Alabama.

The problem of tracking an artificial earth satellite demands that one possess computational procedures which are extremely fast and highly accurate. One must first determine (from launching conditions, first observations, *etc.*) a good set of initial conditions, and then provide flexible means of continually correcting the orbit as further observations are made. The ABMA approach used various methods of numerical integration to define the ephemeris and methods of partial derivatives and least squares for correction.

5. *On the commutative and associative laws*, by Professor Trevor Evans, Emory University.

This paper is a study of the types of identities which can be satisfied by algebraic systems with a binary operation. It is shown that no nontrivial group exists satisfying identities incompatible with commutativity or with finiteness. However, loops exist satisfying identities incompatible with associativity and with finiteness. Another question treated is the existence of an infinite irredundant set of identities compatible with associativity.

6. *Some recent developments in the undergraduate mathematics curriculum and their implications for the future*, by Professor H. E. Taylor, Florida State University.

Widespread calls for changes seem to be based on: (1) the fact that the profound mathematical advances of this century have had little effect on the curriculum; (2) the use by biological and social sciences of an increasing amount of new mathematics; (3) increasing concern for mathematics as part of liberal education. Developments which do more than rearrange traditional material are meeting varying degrees of success. Introduction of the ideas of sets and relations early makes it possible to introduce some "modern" developments and bring clarity, precision, insight, and efficiency to the study of solutions of equations, inequalities, analytic geometry and the function concept.

7. *An important responsibility of every university mathematics department*, by Professor W. A. Gager, University of Florida.

In most universities the education college depends upon the mathematics department of the liberal arts college to provide adequate mathematical training for those who plan to teach in high school. This paper raises questions concerning the mathematical content of the courses offered and the effectiveness with which the materials are presented to the prospective teachers. Modernization of the curricular offerings and extra-special attention to those who plan to be our future teachers of mathematics are urged.

8. *Innovations in mathematics teaching at North Carolina State College*, by Professor C. G. Mumford, North Carolina State College.

Activities recently initiated include the following: (a) a superior student program beginning at the freshman level, with some acceleration, but with emphasis on depth; (b) an honors program for engineering seniors conducted jointly by an engineering department and the mathematics department; (c) course work and research in computer mathematics with availability of analog and digital computers; (d) use of lecture-section teaching with about 90 students per section in calculus and introductory differential equations.

9. *Whither secondary mathematics?* by Professor E. D. Nichols, Florida State University.

The author discusses implications of the recommendations of the Commission on Mathematics for the future secondary-mathematics curriculum. The program in Algebra now in operation with selected eighth-graders at the University School of the Florida State University is considered as an example of an elementary-algebra course in which emphasis is placed on the structure of algebra rather than on manipulative skills.

10. *The stress distribution in a rotating limaçon*, by Professor C. B. Smith, University of Florida.

A thin plate bounded by the curve $r=b-2a\sin\theta$ $(b>2a)$ is assumed to be rotating about the y-axis at a constant angular velocity. The stresses arising in the plate are due to a centrifugal force or body force. The body force is removed from the problem by obtaining a particular solution, and the solution of the reduced problem is then found by using complex variables. The superposition of the two solutions gives the complete solution for the problem. When the constant a is set equal to zero, the solution readily reduces to the case of a rotating circular plate.

11. *A note on the synchronization problem*, by Professor Stephen Kulik, University of South Carolina.

Machines which are in operation are supervised. The number of operators is less than the number of machines. If a machine stops it is attended by the first available operator. The ratio of the time during which a machine is running productively to the total period of operation is its effi-

ciency. The solution for efficiency is given in the form of an algebraic equation of degree equal to the number of machines. It is assumed that (a) number of stops is proportional to productive running time; (b) the total time for repairs is small; (c) the distribution of the number of machines running productively at any moment follows approximately the binomial law.

12. *Optimal filtering and prediction*, by Professor Andrew Sobczyk, University of Florida.

A signal $\{z_k\}$ (discrete time series) consists of a desired message $\{x_k\}$ plus a noise $\{c_k\}$. It is required to determine a system of weights (filter) $\{w_n\}$ such that if $\{z_k\}$ is applied as input, the output $y_k=\sum_{n=0}^{N} w_n z_{k-n}$ will represent as closely as possible future values $\{x_{k+h}\}$ of the message. For the minimum mean-square difference criterion of optimality, the problem has a solution in terms of correlations which depend on infinite past and future of the time series. This paper contributes to the similar problem where, as in any practical case, only a finite amount of data is available.

13. *Some numerical results concerning the asymptotic distribution of Spearman's rank correlation coefficient*, by Professor N. C. Perry, Alabama Polytechnic Institute.

The average error per class interval (E) of an asymptotic probability function of Spearman's ρ coefficient was computed by comparison with the exact distribution for small sample sizes. The use of log log paper indicated a relation between E and sample size n approximately of the form $E=k/n^a$. Least square estimates of k and a show that to obtain probabilities of three decimal accuracy samples of size $n\geqq 11$ should be used; for five decimal accuracy $n\geqq 23$.

14. *Problem of Pellian equations*, by Professor J. W. Greiner, University of Florida.

Theorems and corollaries were given describing all integral solutions of the equation $\delta x^2-dy^2=1$, with $\delta\cdot d=D$, in terms of $t_n+u_n\sqrt{D}$, solutions of Pell's equation, $t^2-Du^2=1$, when t_1 is odd and D is a product of distinct odd primes.

15. *A note on Ky Fan's extension of the Bernstein theorem*, by Mr. B. L. Sanders, Florida State University.

F. P. Callahan and S. G. Kneale have successfully characterized the decompositions which will serve to prove the Bernstein theorem in the classical manner (*A note on the Schroeder-Bernstein theorem*, this MONTHLY, vol. 64, 1957, pp. 423–424). The Bernstein theorem was generalized by S. Banach and this theorem was then extended by Ky Fan (*Note on a theorem of Banach*, Mathematische Zeitschrift, vol. 55, 1952, pp. 308–309). The present paper discusses Ky Fan's extension in the light of the first paper mentioned above.

16. *Nonasymptotic semiorbits under expansive homeomorphisms*, by Professor B. F Bryant, Vanderbilt University.

Let X be a compact, self-dense metric space, and let f be an expansive self-homeomorphism of X (for terminology, see Gottschalk and Hedlund, *Topological Dynamics*, A.M.S. Coll. Pub., Vol. 36). It is shown that for each $x\in X$ and each $\epsilon>0$, there exists $y\in U(x,\epsilon)$ such that $0(x)$ and $0(y)$ are not doubly asymptotic. An example is given to show that the conclusion does not necessarily hold if X is not selfdense.

17. *On the Diophantine homogeneous quadratic equation* $f(x_1, x_2, x_3)=a$, by Professor E. H. Hadlock, University of Florida.

It is shown that the equation whose left member is a ternary quadratic form f with mild restrictions on its coefficients, and whose right member is a given integer, has solutions if any one of the three equations E, obtained from f by the elimination of its cross-products has a solution, provided the variables in E are subject to proper choice of signs and conversely.

18. *Some uses of linear spaces in analysis*, by Professor F. A. Ficken, University of Tennessee.

The speaker began this expository talk by an examination of the Picard theorem for the equation $y'=f(x, y)$, showing that the proof amounted to finding, by an iterative process, a fixed point of a mapping into itself of a suitable complete metric linear space. After some remarks on the algebraic and topological characteristics of infinite-dimensional linear spaces and their duals, he confined attention to Banach spaces. Illustrations included a boundary value problem, an integral operator, and the Euler expression for a variational problem. Many useful properties of a Hilbert space H were seen to be connected with the fact that H is self-dual. Examples included Fourier's series and integral and the spectral theorem for a symmetric (self-adjoint) transformation.

19. *A note on a development of logarithms using the function concept*, by Professor C. L. Seebeck, Jr. and Mr. H. C. Miller, Jr., University of Alabama, presented by Professor Seebeck.

The recent paper *A development of logarithms using the function concept*, this MONTHLY, vol. 64, 1957, pp. 667–668, is extended to include elementary proofs that the function log as defined is a unique, reversible function from R^+ onto R.

20. *A note on an involution of period thirteen*, by Professor W. R. Hutcherson, University of Florida.

A certain quintic surface invariant under a cyclic homography of period thirteen is investigated at the point (1, 0, 0, 0). Certain invariant curves on this surface and passing through this point were studied, both by the classical method and by the method of Godeaux.

21. *Circulants and their groups*, by Professor F. A. Lewis, University of Alabama.

The purpose of this paper is to determine the order of the largest permutation group leaving certain circulants invariant.

22. *The Fibonacci numbers and a dissection of a square*, by Cadet W. R. Alford, The Citadel.

A square of side u_n (the nth term of the Fibonnacci sequence) is divided into two congruent right triangles and two congruent trapezoids which can be reassembled in a rectangle form of dimensions u_{n-1} by u_{n+1} whose area differs from that of the square by unity.

23. *Some observations on the method of conjugate gradients*, by Dr. A. S. Householder, Oak Ridge National Laboratory.

A complete matricial formulation of the method is presented, which exhibits it in form somewhat more perspicuous than the usual scalar representation, and its relation to other biorthogonalization techniques is pointed out. The primary objective, however, is to exhibit effective bounds on the quantities arising in the computation, obtained in terms of a "condition number" of the matrix to be inverted. In cases where this condition number can be obtained, these bounds provide the scale factors needed for carrying out the computation in fixed point.

24. *Mathematics and biology*, by Mr. G. R. Flowers, Emory University.

This paper is a review of N. Rashevsky's *Topology and life: in search of general mathematical principles in biology and sociology*, Bulletin of Mathematical Biophysics, vol. 16, pp. 317–348, 1954. Emphasis is given to the idea of evolution as transformations of a directed graph.

25. *Moore-Smith limits in elementary calculus*, by Professor Herman Meyer, University of Miami.

The author describes a course in introductory calculus taught at the University of Miami.

Moore-Smith limits are introduced and basic limit theorems are proved in this general environment. They are then specialized to obtain the derivative and the integral.

26. *Taylor's series for complex variables*, by Mr. Chung-Lie Wang, University of South Carolina.

In the usual derivation of Taylor's series for a function of a complex variable Cauchy's integral formula is ordinarily employed. In this paper the theorem is established by using Taylor's theorem for a function of two real variables and without the use of Cauchy's integral formula.

27. *Solution of a class of quasilinear partial differential equations of first order*, by Professor Diran Sarafyan, University of Florida.

Let $A_i = a_i x^r + b_i y^r + c_i z^r + d_i$ where a, b, c, d and r are real constants, and $C(abc)$ and $D(abd)$ are third-order determinants made of indicated elements. Consider the quasilinear partial differential equations of first order $A_1 p + A_2 q = A_3$. The author shows that if C and D are zero then a class of particular solutions is given by $F_1 = C_1$ where F_1 is the determinant obtained from $C(abc)$ when c_1, c_2, and c_3 are replaced by x^r, y^r and z^r respectively and C_1 is an arbitrary constant. The consideration of $F_1 = C_1$ together with the system $dx/A_1 = dy/A_2 = dz/A_3$ makes possible the determination of another class of independent particular solutions $F_2 = C_2$. The class of all solutions are represented by $F_1 = g(F_2)$ where g is an arbitrary function.

28. *Criteria for polynomial solutions of a class of linear self-adjoint differential equations*, by Professor R. W. Cowan, University of Florida.

The linear self-adjoint differential equation is taken in such a form that the recurrence relation obtained by Frobenius' method will contain only two terms. All the singular points of the differential equation are regular. In order that one series solution reduces to a polynomial, conditions are imposed on the coefficients of the differential equation that at least one root of the indicial equation be a nonnegative integer and that the resulting series solution terminate.

29. *Differential equations whose coefficients involve q-periodic functions*, by Professor Tomlinson Fort, University of South Carolina.

Professor Fort discussed briefly q-periodic functions and almost-periodic functions. He then discussed the nature of the solution of certain differential equations whose coefficients involve q-periodic functions. He pointed out some of the problems involved in the study of differential equations with almost-periodic coefficients.

30. *Another proof of the fundamental theorem of algebra*, by Professor M. K. Fort, Jr., University of Georgia.

The notion of continuous square root of a mapping into the complex plane was introduced· Several basic theorems concerning this concept was stated, with outlines of proofs. A new proof of the fundamental theorem of algebra was then obtained by making use of these basic theorems about continuous square roots of mappings.

31. *On the characteristic roots of certain related matrices*, by Professor J. W. Ellis, Florida State University.

If $z = x + iy$ is any complex number, let $f(z)$ denote the real 2×2 matrix whose rows are $(x, -y)$ and (y, x). Now if M is any $n \times n$ matrix of complex numbers, let M' denote the $2n \times 2n$ real matrix obtained by replacing in M each m_{ij} by the elements of the matrix $f(m_{ij})$. Then the characteristic roots of M' are precisely those of M, plus their conjugates. A proof of this result is given using vector-matrix multiplication only. A by-product is a simple expression for characteristic vectors of M' in terms of the corresponding characteristic vectors of M.

32. *A Diophantine system*, by Mr. D. E. Thoro, University of Florida.

An elementary proof is given of the following theorem. A necessary and sufficient condition for the existence of integral solutions of two Diophantine equations in three unknowns is that the

g.c.d. of the second order determinants of the matrix of the coefficients is equal to the g.c.d. for the augmented matrix. Formulas for the general solution are obtained.

33. *A rational-valued metric on the space of rational pairs*, by Professor E. B. Shanks, Vanderbilt University.

A metric is defined over the space of rational pairs after a preliminary definition of a "pseudo square root." The pseudo square root is monotonic, strictly increasing and it is continuous except at the origin. A proof is given that the metric assumes only rational values on the space of rational pairs and assigns distances between pairs of points approximately the same as the Euclidean metric and that this approximation can be made as close as desired by choosing a parameter n involved in the definition large enough. The topology induced by the metric is discrete.

34. *Partly-ordered ideal-preserving groups*, by Mr. J. F. Andrus, University of Florida.

The postulate giving the relationship between the order relation and the group operation of a partly-ordered group (assumed to be additive) is replaced by a postulate which states that the sum of any element and any order ideal of the system must be either an order ideal or an order dual ideal. The principal theorem relates many properties of the resulting system to those of a certain normal subgroup which forms a partly-ordered group.

35. *A note on the conics*, by Professor Jose Gallego-Diaz, Vanderbilt University.

The function of a complex variable $w=z^2$ is used to map a system of conics which have the origin for focus into another system of conics which have the origin for center. More than twenty-five original examples are given of the use of this transformation in problems involving locus, envelopes, graphical constructions, maximum and minimum, orthogonal trajectories and so forth.

H. A. ROBINSON, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-eighth regular meeting of the Southern California Section of the Mathematical Association of America was held at Pasadena City College, Pasadena, California, on March 8, 1958. Professor P. B. Johnson, Chairman of the Section, presided. The registered attendance was 142, including 105 members of the Association.

At the business meeting the following officers were elected for the next academic year: Chairman, Professor P. J. Kelly, Santa Barbara College, University of California; Vice-Chairman, Professor D. H. Hyers, University of Southern California; Secretary-Treasurer, Mr. R. B. Herrera, Los Angeles City College. The following members were elected to the Program Committee for the coming year: Professor F. A. Valentine (Chairman), University of California, Los Angeles; Professor Lulu Bechtolsheim, University of Redlands; Professor S. E. Urner, Los Angeles State College; and Professor P. B. Johnson, Occidental College.

The following program was presented:

1. *Electromagnetic theory without Maxwell's Equations*, by Professor R. M. Redheffer, University of California, Los Angeles.

Interaction of transducers in a transmission line induces an algebra through which a "linear network" can be defined. Applied to a homogeneous dielectric sheet this algebra gives nonlinear functional equations for the transmission and reflection. The solution agrees with the classical results, when two complex constants that arise are related suitably to ϵ, μ, λ, θ. For an inhomogeneous sheet the process gives a Riccati equation for the right-hand reflection, and shows how the left-hand reflection and transmission follow therefrom by quadrature. This physical model suggests interesting properties of the general Riccati equation, which can be verified independently.

2. *Unimodular matrices with integral elements*, by Dr. Olga Taussky, California Institute of Technology, introduced by the Secretary.

This paper is a discussion of the following theorem and its generalizations: Every Abelian subgroup of the modular group is cyclic. The results are due to numerous authors.

This theorem can be interpreted not only as a result in number theory and matrix theory, but also as a result in abstract group theory or in non-Euclidean geometry. Analogous theorems on groups of matrices with elements in a fixed algebraic number field have also been investigated.

3. *Implications for college mathematics departments of the offering of sections in mathematical analysis in the senior high schools*, discussed by a panel of five persons.

The members of the panel discussing the announced topic were: Professor May M. Beenken, Immaculate Heart College, Chairman; Dr. Marian Cliffe, Los Angeles City Schools; Mr. Gerald Baughman, Pacific High School, San Bernardino; Professor C. G. Jaeger, Pomona College; Professor Clifford Bell, University of California, Los Angeles.

Senior courses in mathematical analysis in the Los Angeles city high schools are not intended to replace freshman college courses, but to prepare students better for these courses by teaching unifying concepts of mathematics. Such courses provide a simple introduction to concepts to be developed more fully in college; they help to develop skills in note-taking and study-reading of mathematical materials. Typical of a more rigorous course in elementary college mathematics is that successfully being offered to qualified seniors at Pacific High School, San Bernardino. Colleges welcome the introduction of such courses in the high school, and they will allow freshmen, who pass a placement test, to enroll in more advanced college mathematics courses.

4. *Measure theory by means of metric space methods*, by Professor H. G. Tucker, University of California, Riverside.

The development of measure and the Lebesgue integral is presented from a unified, metric space point of view. Both the unique extension of a sigma-finite measure on an additive class of sets to the completely additive class generated by this additive class and the definition of the Lebesgue integral are obtained by means of completion spaces of appropriate metric spaces. Various methods of integration (especially of functions taking values in a Banach space) can be compared by ordering according to the strengths of the corresponding metric topologies determining these integrals.

5. *Some mathematical problems in determining satellite orbits*, by Dr. S. E. Benesch, Jet Propulsion Laboratory, California Institute of Technology, introduced by the Secretary.

Work done under the author's direction was used to determine the orbit of the Explorer 1 satellite. Techniques used in preliminary orbit description (lower bound on perigee) before completion of the first pass were outlined. Global properties of the Doppler curve were developed to relate the change in altitude from horizon to horizon with asymetrical asymptotes, under an approximation that the product of velocity and distance from the earth's center is constant. Local properties of the Doppler curve relate the time of closest approach, $f(0)$, and the time for the point of inflection. Using Taylor's expansion in $f(0)$ for a slant range of 1000 miles and an acceleration of $\frac{1}{3}$ mile/sec.^{2}, it was found that the time difference was 5.5 sec., which was an important correction in refining the ephemeris by the use of Doppler data.

6. *Infinite-valued propositional logic*, by Professor C. C. Chang, University of Southern California, introduced by the Secretary.

In this paper algebraic systems called *MV*-algebras are introduced and studied. The study of such algebras is useful in establishing an algebraic proof of the completeness of the infinite-valued propositional logic of Łukasiewicz. Previously the completeness was proved only by methods from logic and metamathematics. Briefly, every *MV*-algebra is a subdirect product of linearly

ordered MV-algebras, and every linearly ordered MV-algebra is isomorphically embeddable into a segment of an ordered Abelian group. Since any two ordered Abelian groups are universally equivalent, every identity that holds in the segment [0, 1] of the reals must also hold for every MV-algebra. This paves the way to the algebraic proof of completeness.

7. *A new college of science*, by Professor R. C. James, Harvey Mudd College, Claremont.

Harvey Mudd College, the fifth of the Associated Colleges at Claremont, California, admitted its first freshman class in 1957. It was founded in the belief that there is a need for scientists and engineers with broad training in the humanities and social studies. Majors are offered in mathematics, physics, chemistry, and engineering science. The engineering science major will emphasize basic engineering principles, rather than specialized applications. The mathematics program has been planned to provide a calculus course carefully coordinated with the physics and chemistry courses. Upper division courses serve nonmajors as well as majors, giving a broad introduction to modern mathematics for majors, including topology, modern algebra, and operations analysis.

R. B. HERRERA, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 6, 1958.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958
NORTHEASTERN, College of the Holy Cross, Worcester, Massachusetts, November 29, 1958.
NORTHERN CALIFORNIA, University of California, Berkeley, June 17, 1958 (joint meeting with ASEE, mathematics division).
OHIO
OKLAHOMA, Oklahoma City University, October 24, 1958.
PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.
PHILADELPHIA, Lehigh University, Bethlehem, November 29, 1958.
ROCKY MOUNTAIN
SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.
SOUTHERN CALIFORNIA, University of Redlands, Redlands, March 14, 1959.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

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RECTIFIABLE CURVES AND THE WEIERSTRASS INTEGRAL

LAMBERTO CESARI, Purdue University

Recent research on surface area theory and the calculus of variations has drawn attention to the theory of rectifiable curves and the Weierstrass integral. Indeed, most basic theorems for curves have their two-dimensional counterpart, although the proofs and the underlying concepts for the two-dimensional case lie much deeper than for curves and show unexpected connections with topology, measure theory, and functional analysis.

The theory for rectifiable curves is presented sometimes in courses and books on real functions or on integration. The present brief exposition intends to emphasize those parts which are of greater interest because of their recent extensions to surfaces.

1. Parametric curves. We should first define the terms we need: parametric curves, continuous curves, rectifiable curves. For parametric curves we shall assume the following definition, which is adequate for the present exposition:

A system of equations

$$(1.1)\qquad C\colon\ x_1 = x_1(u),\ x_2 = x_2(u),\ \cdots,\ x_N = x_N(u),\qquad a \leqq u \leqq b,$$

where $x_i(u)$ are real functions of u is said to be a *parametric curve* C in the real Euclidean x-space E_N, $x=(x_1, \cdots, x_N)$, and $-\infty < a < b < \infty$.
Thus in vectorial notation we have

$$(1.1)\qquad C\colon\ x = x(u),\qquad a \leqq u \leqq b,$$

and C may be thought of as a *mapping* from $[a, b]$ into E_N. The curve C is said to be *continuous* if the functions $x_i(u)$, $i=1, \cdots, N$, are all continuous in $[a, b]$. We shall also define the *graph* $[C]$ of C as a concept apart from the curve C. By graph $[C]$ we mean, as usual, the set of all points $x \in E_N$ "covered," (or "occupied," or "travelled") by C in E_N, *i.e.*, $[C]=[x \in E_N;\ x=f(u),\ a \leqq u \leqq b]$. For every $u \in [a, b]$, $x(u)$ is said to be the *image* of u on C. It is often convenient to assume on $[a, b]$ the natural order from a to b. Then the curve C is said to be *oriented*. A point $x \in [C]$ may be covered once, finitely many times, infinitely many times by C; *i.e.*, a point $x \in [C]$ may be the image of one, finitely many, or infinitely many points $u \in [a, b]$. The examples (Fig. 1) for $N=2$ may illustrate the situation: $C_1\colon x_1=t,\ x_2=t^2,\ -1 \leqq t \leqq 1$; $C_2\colon x_1=\cos 2u,\ x_2=\cos 2u \tan u,\ -\pi/3 \leqq u \leqq \pi/3$; $C_3\colon x_1=x_1(u),\ x_2=x_2(u),\ 0 \leqq u \leqq 1$, where $x_1(u)=u \cos 2\pi u^{-1}$,

$x_2(u) = u \sin 2\pi u^{-1}$ if $0 < u \leqq 1$, $x_1 = x_2 = 0$ if $u = 0$; C_4: $x_1 = |u|$, $x_2 = x_2(u)$, $-1 \leqq u \leqq 1$, where $x_2(u) = |u| \sin 2\pi u^{-1}$ if $-1 \leq u \leqq 1$, $u \neq 0$, $x_2 = 0$ if $u = 0$.

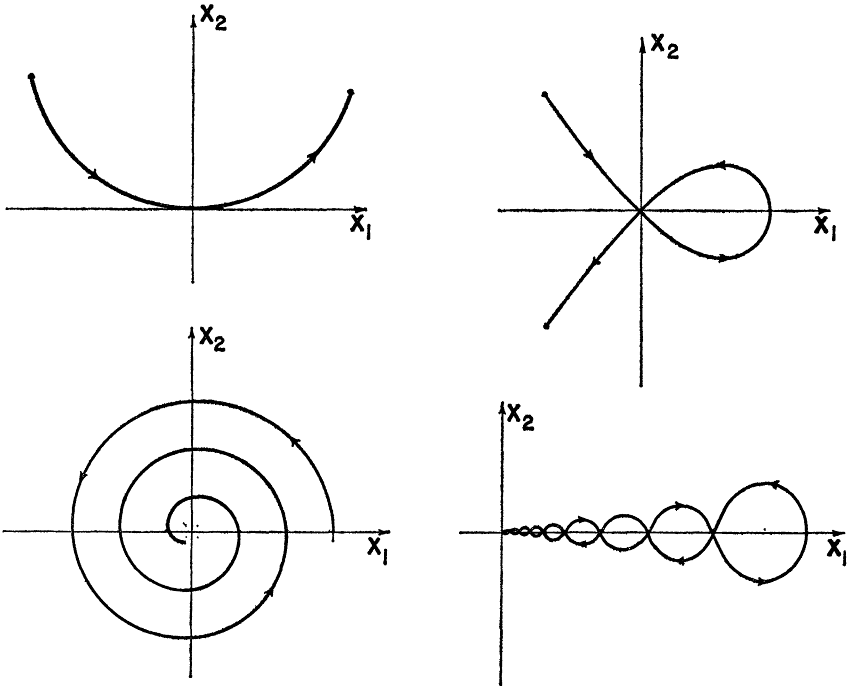

FIG. 1.

It may well occur that completely different curves C have the same graph. For instance, the curves D_k: $x_1 = \sin^2 ku$, $x_2 = 0$, $0 \leqq u \leqq \pi/2$, $k = 1, 2, \cdots$, all cover the segment $s = [D_k] = [0 \leqq x_1 \leqq 1, x_2 = 0]$ joining the points (0, 0), (1, 0) in E_2, but D_1 covers s just once, D_2 covers s twice (back and forth), D_3 covers s three times, *etc.* The curves $D_1, D_2, D_3, \cdots$ are certainly different curves (and indeed they will have different lengths $1, 2, \cdots$), but have the same graph. Analogously, the curves F_k: $x_1 = \cos ku$, $x_2 = \sin ku$, $0 \leqq u \leqq 2\pi$, all cover the circle $t = [x_1^2 + x_2^2 = 1]$, but F_1 covers t once ((1, 0) is covered twice), F_2 covers t twice, *etc.* The curves $F_1, F_2, \cdots$ are different curves but they have the same graph. The graph $[C]$ of a curve C may be a set of a very complicated nature (a characterization will be mentioned in Section 7). The graph of a curve C may well be a set of positive measure, and may have interior points; it may be a square, a cube, *etc.* (F. Osgood, G. Peano) (see *e.g.*, E. W. Hobson, *Functions of a real variable*, I, pp. 451–458).

There are situations where our intuition associates to two different sets of equations (1.1) the same entity. For instance, this occurs for the two curves in E_2

$$C: x_1 = u,\ x_2 = 0,\ 0 \leqq u \leqq 1, \qquad C': x_1 = \sin v,\ x_2 = 0,\ 0 \leqq v \leqq \pi/2.$$

More generally, suppose $u=\phi(v)$, $c\leqq v\leqq d$, to be a continuous, always-increasing function with $a=\phi(c)$, $b=\phi(d)$. Thus ϕ maps $[c, d]$ onto $[a, b]$, each value $a\leqq u\leqq b$ is taken by $\phi(v)$ only once, and the inverse function $v=\phi^{-1}(u)$, $a\leqq u\leqq b$, exists, is continuous, and maps $[a, b]$ onto $[c, d]$. Then any two curves as

$$C: x = x(u),\ a \leqq u \leqq b, \qquad C': x = x[\phi(v)],\ c \leqq v \leqq d,$$

are felt to "represent" the same entity. For instance, in the example above $\phi(v)=\sin v$, $0\leqq v\leqq\pi/2$. In Section 8 we shall introduce concepts of equivalence (Lebesgue, Fréchet). Thus C, C' will be denoted as Lebesgue equivalent. Classes of Lebesgue [Fréchet] equivalent curves will be said to define the same Lebesgue [Fréchet] curve.

Finally, it should be pointed out that "curves" as $x_1=t$, $x_2=t^2$, $-\infty<t<+\infty$ (a parabola), $x_1=(1-t)\cos t^{-1}$, $x_2=(1-t)\sin t^{-1}$, $0<t\leqq 1$ (a spiral), $x_1=t$, $x_2=(a^2+t^2)^{1/2}>0$, $0\leqq t<+\infty$ (half-hyperbola), are not parametric continuous curves according to the definition above since the parameter t does not range on an interval both closed and finite. A slightly more general concept of curve should be considered: $C: x=x(u)$, $u\in G$, where G is any set of real numbers. Some of the main theorems can be extended to this situation without effort (see, *e.g.*, S. Saks, *Theory of the integral*, Warsawa, 1937, pp. 121–125).

2. The Jordan length. By norm $|x|$ of $x\in E_N$ we mean, as usual, $|x| =(x_1^2+\cdots+x_N^2)^{1/2}\geqq 0$, and thus for any two points x, $y\in E_N$, $|x-y|$ is the Euclidean distance

$$(2.1)\qquad |x-y| = [(x_1-y_1)^2+\cdots+(x_N-y_N)^2]^{1/2} \geqq 0.$$

Thus, for $N=1$, $|x|$ is the absolute value of x, and $|x-y|$ the usual distance on the real axis.

For curves $C: x=x(u)$, $a\leqq u\leqq b$, for which the functions $x_1(u), \cdots, x_N(u)$, are continuous with their first derivatives $x_1'(u), \cdots, x_N'(u)$, the length of C is often assumed, by definition, to be the numerical value of the integral (length integral)

$$(2.2)\qquad \int_a^b |x'(u)|\, du = \int_a^b (x_1'^2+\cdots+x_N'^2)^{1/2} du.$$

This definition is certainly adequate under the hypotheses above, but it is not adequate under somewhat weaker assumptions, as the following examples show.

Let $\phi(u)$, $0\leqq u\leqq 1$, denote the well-known monotone nondecreasing continuous function with $\phi(0)=0$, $\phi(1)=1$, which is constant on each complementary interval $I\subset[0, 1]$ of the ternary Cantor set E in $[0, 1]$, where the countable sum of the lengths of the intervals I is exactly 1 [see, *e.g.*, E. W. Hobson, *Functions of a real variable*, I, p. 368]. Thus $\phi'(u)=0$ at all interior points of the intervals I and hence almost everywhere (a.e.) in $[0, 1]$. The curve (Fig. 2) C: $x_1=u$, $x_2=\phi(u)$, $0\leqq u\leqq 1$, in E_2, is a continuous curve of end points (0, 0), (1, 1). The chord through the end points of C has length $\sqrt{2}$, and we should

expect for the length of C a value $\geqq\sqrt{2}$. The formula (2.2) gives the value $\int_0^1(1+0)^{1/2}du=1$. (The Jordan length of C as defined below will be 2).

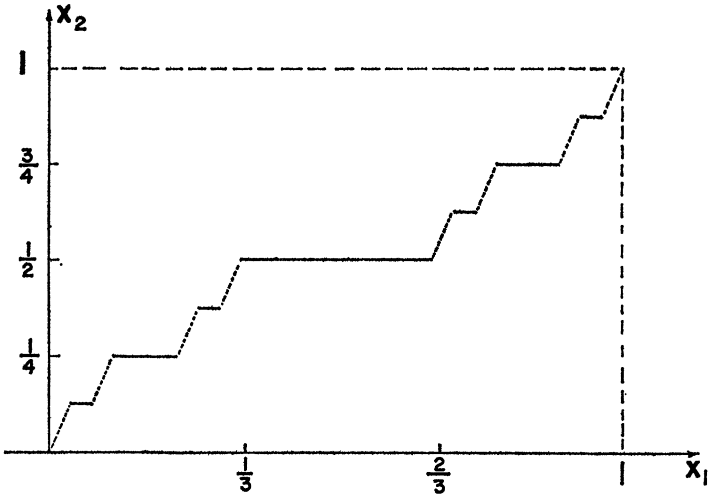

FIG. 2.

As another example consider the continuous curve $D: x_1=\phi(u)$, $x_2=0$, $0\leqq u\leqq 1$, in E_2. Essentially D is the segment s of end points (0, 0), (1, 0) (according to Section 8, D is actually Fréchet equivalent to the curve D_0: $x_1=u$, $x_2=0$, $0\leqq u\leqq 1$). Thus we should expect for the length of D the value 1 (or a value $\geqq 1$). The formula (2.2) gives the value $\int_0^1(0+0)^{1/2}du=0$.

These examples show that formula (2.2) is not adequate under hypotheses weaker than the one mentioned at the beginning of this section. C. Jordan, in 1884, proposed and studied a definition of length which is completely adequate for all parametric curves (continuous as well as discontinuous).

By *Jordan length* $l(C)$ of the curve C is meant the supremum of the elementary lengths of the inscribed polygonal lines. More precisely, if $D=[a=u_0<u_1<\cdots<u_n=b]$ denotes any finite subdivision of $[a, b]$ into consecutive subintervals, the definition of $l(C)$ is

$$l(C) = l_1(C) = \operatorname*{Sup}_D \sum_{i=1}^{n} \left| x(u_i) - x(u_{i-1}) \right| . \tag{2.3}$$

For $N=1$, C is any given real function of the real variable u in $[a, b]$, and $l(C)$ is its total variation. A curve C in E_N is said to be *rectifiable* if $l(C)<+\infty$.

If $I=[\alpha, \beta]$ is any subinterval of $[a, b]$, by $x(I)$ we denote the set of all points $x\in E_N$ with $x=x(u)$ for some $u\in[\alpha, \beta]$. By *oscillation* of $x(u)$ in $[\alpha, \beta]$ or osc $(x; \alpha, \beta)$ we denote, as usual, the diameter of the set $x(I)$, *i.e.*, osc $(x; \alpha, \beta)$ $=\operatorname{diam} x(I)=\operatorname{Sup}|x(u')-x(u)|$, where the supremum is taken over all $u, u'\in I$. Thus $|x(\alpha)-x(\beta)|\leqq$ osc $(x; \alpha, \beta)$ for every $I=[\alpha, \beta]$.

An alternative definition of Jordan length is the following one

$$l_2(C) = \operatorname*{Sup}_{D} \sum_{i=1}^{n} \text{osc}\,(x;\, u_{i-1},\, u_i). \tag{2.4}$$

(2.i) *For every continuous curve C we have $l_1(C) = l_2(C)$.*

The proof is the same as given in a previous article (this MONTHLY, vol. 65, 1958, pp. 317–332, Section 1, no. 1) for the total variation of a real function of one real variable. We shall refer to this article by I.

Given, as above, a curve C: $x = x(u)$, $a \leqq u \leqq b$, and a subdivision $D = [a = u_0 < u_1 < \cdots < u_n = b]$ of $[a, b]$, we shall denote by d and δ two possible *norms* of D with respect to C, namely $d = \max\,(u_i - u_{i-1})$, and $\delta = \max$ osc $(x;\, u_{i-1},\, u_i)$, where in both the maximum is taken for i ranging over $i = 1, 2, \cdots, n$. If C is a continuous curve, then $d \to 0$ implies $\delta \to 0$, but the converse is not true, as it is clear when $x(u)$ is constant on some subinterval of $[a, b]$. For continuous curves C the Jordan length is actually a limit of the sums (2.3), (2.4):

(2.ii) *For every continuous curve C we have*

$$l_1(C) = l_2(C) = \lim_{\delta \to 0} \sum_{i=1}^{n} \left| x(u_i) - x(u_{i-1}) \right| = \lim_{\delta \to 0} \sum_{i=1}^{n} \text{osc}\,(x;\, u_{i-1},\, u_i),$$

and the same holds as $d \to 0$.

The proof is the same as given in I for the total variation of a function of one real variable.

3. The Jordan theorem. We shall denote by V_r the total variation in $[a, b]$ of the function $x_r(u)$, $r = 1, \cdots, N$. Thus $x_r(u)$, $a \leqq u \leqq b$, is a function of bounded variation (BV) if and only if $V_r < +\infty$. Absolute continuous functions will be denoted as usual by AC functions (cf. I).

(3.i) (C. Jordan, 1884) *For every parametric curve C we have*

$$V_r \leqq l(C) \leqq V_1 + \cdots + V_N, \qquad r = 1, \cdots, N. \tag{3.1}$$

Hence C is rectifiable if and only if all functions $x_r(u)$, $r = 1, \cdots, N$, are BV in $[a, b]$.

Proof. For every two points $x, y \in E_N$ we have, from (2.1),

$$\left| x_r - y_r \right| \leqq \left| x - y \right| \leqq \left| x_1 - y_1 \right| + \cdots + \left| x_N - y_N \right|, \; r = 1, \cdots, N.$$

Hence, for every subdivision D of $[a, b]$ we have

$$\sum_{i=1}^{N} \left| x_r(u_i) - x_r(u_{i-1}) \right| \leqq \sum_{i=1}^{n} \left| x(u_i) - x(u_{i-1}) \right| \leqq \sum_{r=1}^{N} \sum_{i=1}^{n} \left| x_r(u_i) - x_r(u_{i-1}) \right|,$$

$r = 1, \cdots, N$. By taking the supremum of each member we deduce relation

(3.1). This relation implies that $l(C)<+\infty$ if and only if all numbers V_r, $r=1, \cdots, N$, are finite.

Theorem (3.i) was proved by Jordan in 1884 in just the same form, *i.e.*, for continuous as well as discontinuous curves. The Jordan length has a number of formal properties in common with the total variation of a real function $f(u)$ $[N=1]$, namely all those properties of the latter which involve only vectorial properties of the real system. One of these properties is the following one:

(3.ii) *If $I=[a, b]$ and $I_1, \cdots, I_m$, is a finite subdivision of I into subintervals, and we denote by $C, C_1, \cdots, C_m$, the curves defined by the vector function $x(u)$ on $I, I_1, \cdots, I_m$, respectively, then $l(C)=l(C_1)+\cdots+l(C_m)$.*

4. Lower semicontinuity of the Jordan length. This is the basic property of the Jordan length and holds for continuous as well as discontinuous parametric curves. This property is well expressed by the following theorem:

(4.i) *If $C: x=x(u)$, $a\leqq u\leqq b$, $C_n: x=x_n(u)$, $a\leqq u\leqq b$, $n=1, 2, \cdots$, are given curves and $x(u)=\lim x_n(u)$ as $n\rightarrow+\infty$ for every $u\in[a, b]$, then*

$$l(C) \leqq \liminf_{n\rightarrow\infty} l(C_n). \tag{4.1}$$

Proof. Given $\epsilon>0$ there is a subdivision D of $[a, b]$ such that

$$\sum_{i=1}^{N} \mid x(u_i) - x(u_{i-1}) \mid > l(C) - \epsilon, \text{ or } 1/\epsilon,$$

according as $l(C)<+\infty$, or $l(C)=+\infty$. Since $x_n(u_i)\rightarrow x(u_i)$ as $n\rightarrow\infty$, $i=0, 1, \cdots, N$, there is an n_i such that $|x_n(u_i)-x(u_i)|<\epsilon/2N$ for all $n\geqq n_i$, $i=0, 1, \cdots, N$, and hence, also, for $n\geqq n_0=\max n_i$. For $n\geqq n_0$ we have also $|x_n(u_i) - x_n(u_{i-1})| > |x(u_i) - x(u_{i-1})| - \epsilon/N$, and finally $l(C) - \epsilon$, or $1/\epsilon$ $< \sum_{i=1}^{N} [|x_n(u_i)-x_n(u_{i-1})|+\epsilon/N]\leqq l(C_n)+\epsilon$ for all $n\geqq n_0$. This implies (4.1). This proof is the same as in I, (7.i).

A variant of (4.i) for continuous curves is

(4.ii) *If $C: x=x(u)$, $a\leqq u\leqq b$, $C_n: x=c_n(u)$, $a\leqq u\leqq b$, $n=1, 2, \cdots$, are continuous curves and $x(u)=\lim x_n(u)$ as $n\rightarrow\infty$ for at least all u of a set which is everywhere dense in $[a, b]$, then* (4.1) *holds.*

Remark 1. In neither of the statements (4.i), (4.ii) uniform convergence is required, and, in (4.i), the continuity of the curves is not required.

Remark 2. Examples show that equality sign does not hold necessarily in (4.1). For instance, let $C: x_1=u, x_2=0, 0\leqq u\leqq 1$, and $C_n: x_1=u, x_2=n^{-1}\sin 2\pi n^s u$, $0\leqq u\leqq 1$, where n, n^s are all integers. Then $l(C)=1$, and, by (3.1), also $4n^s[(4n^s)^{-2}+n^{-2}]^{1/2}\leqq l(C_n)\leqq 4n^s[(4n^s)^{-1}+n^{-1}]$.

If $n=m^2$, $s=1/2$, $m=1, 2, \cdots$, we have $l(C)=1=\lim l(C_m)$ as $m\rightarrow+\infty$. If $s=1$, $n=1, 2, \cdots$, we have $l(C)=1<17^{1/2}\leqq\lim l(C_n)\leqq\limsup l(C_n)\leqq 5$. If $s=2$, $n=1, 2, \cdots$, we have $l(C)=1<\lim l(C_n)=+\infty$.

Remark 3. A continuous curve $C: x=x(u)$, $a \leqq u \leqq b$, is said to be a *polygonal line* (and $x(u)$ quasilinear in $[a, b]$), if there is some finite subdivision D of $[a, b]$ into parts on each of which $x(u)$ is linear or constant. Then each part is mapped by $x(u)$ onto a segment σ and the elementary length $l_e(C)$ is the sum of the lengths of these segments. Obviously the Jordan length $l(C)$ coincides with the elementary length for every polygonal line C.

Now let us consider any continuous curve $C: x=x(u)$, $a \leqq u \leqq b$, and the class γ of all sequences $\gamma_n=[x=x_n(u),\ a \leqq u \leqq b]$, $n=1, 2, \cdots$, of polygonal lines with $x_n(u) \rightarrow x(u)$ uniformly in $[a, b]$ as $n \rightarrow \infty$. By (4.i) we have

$$l(C) \leqq \liminf_{n\to\infty} l(\gamma_n) \equiv \liminf_{n\to\infty} l_e(\gamma_n),$$

while, by the definition of Jordan length (Section 2), we know that there is some sequence with $l_e(C_n) \rightarrow l(C)$ as $n \rightarrow \infty$. This remark suggests the following alternative definition of Jordan length

$$l(C) = \operatorname*{Inf}_{\gamma}\ [\liminf_{n\to\infty} l_e(\gamma_n)].$$

5. The function $s(u)$. For every $a \leqq u \leqq b$, let $s(u)$ denote the length of the curve defined by (1.1) on the interval $[a, u]$, and by $v_r(u)$ the total variation of $x_r(u)$ in $[a, u]$ .Then by (3.ii) and the corresponding theorem for total variations (cf. I, Section 1, Remark 1), we deduce that, for each interval $[\alpha, \beta]$ of $[a, b]$, the differences $s(\beta)-s(\alpha)$, $v_r(\beta)-v_r(\alpha)$ are the length of the curve defined by (1.1) on $[\alpha, \beta]$, and the total variation of $x_r(u)$ on $[\alpha, \beta]$, $r=1, \cdots, N$, respectively. The functions $s(u)$, $v_r(u)$ are monotone nondecreasing in $[a, b]$, and $s(a)=v_r(a)=0$, $s(b)=l(C)$, $v_r(b)=V_r$, $r=1, \cdots, N$.

(5.i) *For every curve C and any subinterval $[\alpha, \beta]$ of $[a, b]$ we have*

$$v_r(\beta) - v_r(\alpha) \leqq s(\beta) - s(\alpha) \leqq \sum_{r=1}^{N} [v_r(\beta) - v_r(\alpha)], \qquad r = 1, \cdots, N,$$

$$| x_r(\beta) - x_r(\alpha) | \leqq | x(\beta) - x(\alpha) | \leqq s(\beta) - s(\alpha), \qquad r = 1, \cdots, N.$$

This statement is a consequence of the definitions and of (3.i).

(5.ii) *For every continuous rectifiable curve C the functions $s(u)$, $v_r(u)$ $a \leqq u \leqq b$, are continuous in $[a, b]$, and $s(a)=v_r(a)=0$, $s(b)=l(C)<+\infty$, $v_r(b)=V_r<+\infty$, $r=1, \cdots, N$.*

This statement is a consequence of (5.i) and of the same property for the total variations $v_r(u)$ (cf. I, Section 6).

(5.iii) *For every continuous rectifiable curve C, the function $s(u)$ is AC if and only if all functions $v_r(u)$, $r=1, \cdots, N$, are AC, and hence if and only if all functions $x_r(u)$, $r=1, \cdots, N$, are AC.*

The first part of this statement is a consequence of (5.i), the second part of the analogous statement for the total variation (cf. I, Section 6).

6. The Tonelli theorems.

(6.i) (L. Tonelli, 1908–12) *For every continuous rectifiable curve* $C: x=x(u)$, $a \leqq u \leqq b$, $x(u)=[x_r(u), r=1, \cdots, N]$, *we have*

$$l(C) \geqq \int_a^b |x'(u)|\, du, \tag{6.1}$$

and the $=$ *sign holds if and only if all functions* $x_r(u)$, $r=1, \cdots, N$, *are AC.*

Proof. By (3.i) all functions $x_r(u)$ are BV in $[a, b]$ and, by a theorem of Lebesgue [cf. I, Section 11], the derivatives $x_r'(u)$ exist almost everywhere (a.e.) in $[a, b]$ and are L-integrable. Also, the nondecreasing function $s(u)$, $a \leqq u \leqq b$, $s(a)=0$, $s(b)=l(C)$ has derivative $s'(u)$ a.e. in $[a, b]$, $s'(u)$ is L-integrable in $[a, b]$, and, by (5.i), we may deduce that $|x'(u)| \leqq s'(u)$ a.e. in $[a, b]$. By the same Lebesgue theorem we have now

$$l(C) = s(b) - s(a) \geqq \int_a^b s'(u)du \geqq \int_a^b |x'(u)|\, du,$$

and (6.1) is proved. Suppose that the $=$ sign holds in (6.1). Then $s(b)-s(a) = \int_a^b s'(u)du$, while, for every $a \leqq u \leqq b$, by (6.1) we have

$$s(u) - s(a) \geqq \int_a^u s'(u)du, \qquad s(b) - s(u) \geqq \int_u^b s'(u)du. \tag{6.2}$$

By addition and comparison we have $+\infty > s(b)-s(a) \geqq \int_a^b s'(u)du = s(b)-s(a)$ and thus the $=$ sign must hold in both relations (6.2) for all $a \leqq u \leqq b$. This implies that $s(u)=s(a)+\int_a^u s'(u)du$ for all u, *i.e.*, $s(u)$ is an integral function, and thus $s(u)$ is AC. By (5.iii) all $x_r(u)$, $r=1, \cdots, N$, are AC in $[a, b]$. Suppose finally that all functions $x_r(u)$, $r=1, \cdots, N$, are AC in $[a, b]$. Then by (5.iii) and for every subdivision D of $[a, b]$ we have

$$\begin{aligned} \sum_{i=1}^n |x_r(u_i) - x_r(u_{i-1})| &= \sum_{i=1}^n \left[\sum_{r=1}^N \left(\int_{u_{i-1}}^{u_i} x_r'(u)du \right)^2 \right]^{1/2} \\ &\leqq \sum_{i=1}^n \int_{u_{i-1}}^{u_i} \left(\sum_{r=1}^n x_r'^2(u) \right)^{1/2} du = \int_a^b |x'(u)|\, du. \end{aligned} \tag{6.3}$$

By taking the supremum of the first member for all D we have $l(C) \leqq \int_a^b |x'(u)|\, du$. By comparison with (6.1) we conclude that the $=$ sign holds in (6.1).

Remark. The central inequality in (6.3) has been obtained by applying a known inequality for real numbers, often given in either form

$$[(\sum a_i)^2 + (\sum b_i)^2 + (\sum c_i)^2]^{1/2} \leqq \sum (a_i^2 + b_i^2 + c_i^2)^{1/2},$$

$$\left(\left[\int_I f du\right]^2 + \left[\int_I g du\right]^2 + \left[\int_I h du\right]^2\right)^{1/2} \leqq \int_I (f^2 + g^2 + h^2)^{1/2} du.$$

See, *e.g.*, S. Saks, p. 171. The first one is just a form of the so-called triangle inequality. The various parts of the previous proof can be traced, with some simplifications, in the original papers of L. Tonelli, [*Acc. Sci. Torino*, 43, 1908, 783–800, and 47, 1912, 1067–1075].

(6.ii) (L. Tonelli, 1912). *For every continuous rectifiable curve C we have $s'(u) = |x'(u)|$ a.e. in $[a, b]$.*

A simple proof of this theorem is given in L. M. Graves, *Functions of real variables*, 1946, p. 213.

7. The Noebeling theorem. Since a parametric continuous curve is, by definition, a continuous mapping from a finite closed interval $a \leqq u \leqq b$ into E_N, the set $[C]$ has a number of properties, namely $[C]$ is bounded, and closed (thus compact), and connected (thus a continuum). In addition it is locally connected, even "uniformly locally connected." A set $M \subset E_N$ is said to have this property if, given $\epsilon > 0$, there is a $\delta > 0$ such that for any two points $x, y \in M$, $|x-y| < \delta$ there is some subcontinuum $m \subset M$, with $x, y \in m$, diam $m < \epsilon$. By a theorem of H. Hahn and Mazurkiewicz, a set M is the graph $[C]$ of a (parametric continuous) curve C if and only if M is bounded, closed, connected and uniformly locally connected. Such a set is also said to be a Peano space. A square in E_2, a cube in E_3, certainly satisfy these conditions and thus are the graphs of some continuous curve. In topology such a set is said to be "a continuous curve"; it is unfortunate that the use of this term clashes with the one of the present article, of differential geometry, of calculus of variations, and other parts of mathematics.

For each point $x \in [C]$ let us denote by $M(x)$ the number (finite or $+\infty$) of the disjoint points $u \in [a, b]$ with $x = f(u)$; *i.e.* $M(x)$, is the number of points u of $[a, b]$ of which x is the image under C. For each $x \in E_N - [C]$, let $M(x) = 0$. Then the *multiplicity function* $M(x)$, $0 \leqq M(x) \leqq +\infty$, is defined for all $x \in E_n$, and is nonnegative and integral valued. We shall denote by H^1 the 1-dimensional Hausdorff measure in E_N.

(7.i) (G. Noebeling, 1940) *For every continuous parametric curve C we have*

$$l(C) = \int_{E_N} M(x) dH^1.$$

This theorem connects the Jordan length of a continuous curve C with the concept of 1-dimensional Hausdorff measure of a set. If $\phi(x)$ is the characteristic function of the set $[C]$, *i.e.*, $\phi = 1$ for $x \in [C]$, $\phi = 0$ for $x \in E_N - [C]$, then

$$H^1([C]) = \int_{E_N} \phi(x) dH^1 \leqq \int_{E_N} M(x) dH^1 = l(C),$$

and it may well occur that $H^1([C]) < l(C)$, as, *e.g.*, for the curves D_k, F_k, $k \geqq 2$, of Section 1.

For every set $A \subset E_N$, Lebesgue measure, say $|A|$, and N-dimensional Hausdorff measure coincide, say $|A| = H^N(A)$. For $N > 1$, the measure $|[C]|$ of the graph $[C]$ of a continuous curve C in E_N is not necessarily zero [indeed, $[C]$ may be a cube in E_N, *etc.*]. Nevertheless, if C is rectifiable and $N > 1$, then $|[C]| = 0$ [see, *e.g.* L. Cesari, *Surface area*, 1956, p. 93]. (For the concept of measure H^1, which will not be used in the following, see also, *e.g.*, S. Saks, *loc. cit.*, p. 53).

8. Lebesgue and Fréchet equivalences. If $C: x = x(u)$, $a \leqq u \leqq b$, we may identify a and b, and then we must require $x(a) = x(b)$. Then C is said to be a *closed* curve, and $[a, b]$ becomes topologically equivalent to a circle, *i.e.*, a 1-sphere. Otherwise we say that C is *open*, and in this case a and b are not identified, and $[a, b]$ is a 1-cell. Since the concept of equivalence for closed curves requires a few more words than for open curves, we will suppose that all curves in this section are (parametric, continuous, and) open. Also, we shall suppose that they are all oriented.

A mapping $u = h(v)$, $v \in J$, $u \in I$, $h(c) = a$, $h(d) = b$, from the interval $J = [c \leqq v \leqq d]$ onto the interval $I = [a \leqq u \leqq b]$, is said to be a homeomorphism from J onto I if it is one-one and continuous together with its inverse, $v = h^{-1}(u)$. Given any two (oriented, open, continuous, parametric) curves

$$C: x = f(u), \ u \in I = [a, b], \qquad C_1: x = g(v), \ v \in J = [c, d],$$

in E_N, we say that C is Lebesgue-equivalent to C_1 (or L-equivalent to C_1) if there is a homeomorphism from J onto I, $u = h(v)$, $v \in J$, $u \in I$, such that $f[h(v)] = g(v)$ for all $v \in J$. It is easy to prove that L-equivalence has the standard symmetric, reflexive, transitive properties for an equivalence. Thus we may say, as usual, that two curves C, C_1 are L-equivalent. We have already given in Section 1 an example of two L-equivalent curves.

A more general concept of equivalence is the following one. Given any two curves C, C_1 as above, we say that C is Fréchet equivalent to C_1 (F-equivalent, $C \sim C_1$) provided, given $\epsilon > 0$, there exists a homeomorphism $u = h_\epsilon(v)$, $v \in J$, $u \in I$ (which may depend on ϵ), such that $|f[h_\epsilon(v)] - g(v)| < \epsilon$ for all $v \in J$. Again it is easy to prove that the F-equivalence has the same properties for an equivalence mentioned above, *i.e.*, 1. $C \sim C$; 2. $C \sim C_1$ implies $C_1 \sim C$; $C \sim C_1$, $C_1 \sim C_2$ implies $C \sim C_2$.

Obviously F-equivalent curves are L-equivalent, but the converse is not true, as the following example shows. Suppose (Fig. 3) $C: x_1 = u$, $x_2 = \cdots = x_N = 0$, $u \in I = [0, 1]$; $C_1: x_1 = g(u)$, $x_2 = \cdots = x_N = 0$, $u \in J = [0, 1]$, where $g(u) = 3u/2$ if $0 \leqq u \leqq 1/3$, $g(u) = 1/2$ if $1/3 \leqq u \leqq 2/3$, $g(u) = 1 + (3/2)(u - 1)$, if $2/3 \leqq u \leqq 1$. Let us prove that $C \sim C_1$. For every $n \geqq 4$ let $u = h_n(v)$, where $h_n(v) = 3v/2$ if $0 \leqq v \leqq (1/3) - (1/n)$; $h_n(v) = 1 + (3/2)(v - 1)$ if $(2/3) + (1/n) \leqq v \leqq 1$; $h_n(v) = (1/2) + 9(n+6)^{-1}(v - 1/2)$ if $(1/3) - (1/n) \leqq v \leqq (2/3) + (1/n)$. Let $h(v) = g(v)$,

$f(u)=u$. Then, by elementary computations, we have $|h_n(v)-h(v)|<3/n$ for all $0\leqq v\leqq 1$. Thus $|f[h_n(v)]-f[h(v)]|\to 0$ uniformly in $[0, 1]$, and this proves, since $f[h(v)]=g(v)$, that $C\sim C_1$.

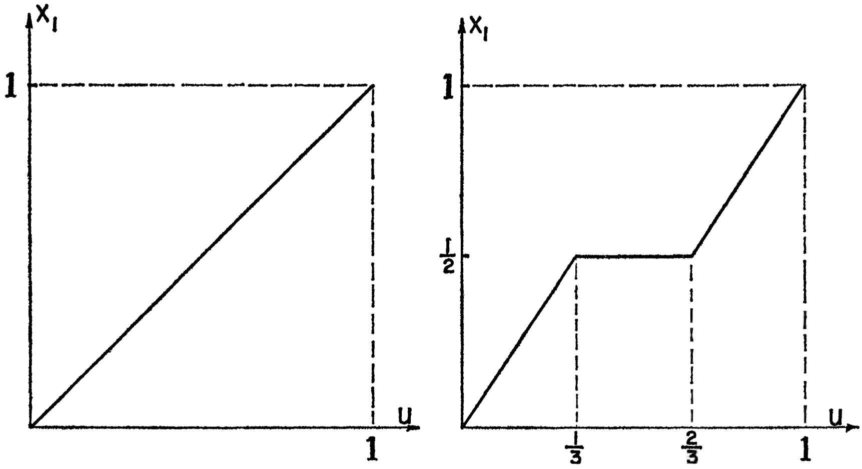

FIG. 3.

(8.i) *If C, C' are L-equivalent, or F-equivalent, then $l(C)=l(C')$, i.e., the Jordan length is invariant with respect to both L- and F-equivalences.*

Proof. For L-equivalence, observe that, if C: $x=f(u)$, $a\leqq u\leqq b$, C': $x=g(v)$, $c\leqq v\leqq d$, then $f[h(v)]=g(v)$, $f(u)=g[h^{-1}(u)]$ for some homeomorphism h, and any subdivision D of $[a, b]$ is mapped by h into a subdivision D' of $[c, d]$ (and conversely). Since the corresponding sums (2.3) are equal, we deduce $l(C)\leqq l(C')$ as well as $l(C')\leqq l(C)$. Thus $l(C)=l(C')$. For F-equivalence observe that for each $n=1, 2, \cdots$, there exists a homeomorphism h_n such that, if C_n: $x=f_n(v)$ $\equiv f[h_n(v)]$, $c\leqq v\leqq d$, then $|f_n(v)-g(v)|=|f[h_n(v)]-g(v)|<1/n$ for all v, *i.e.*, $f_n(v)\to g(v)$ as $n\to\infty$. By the lines above and (4.i) we have $l(C_n)=l(C)$, $l(C')$ $\leqq \lim l(C_n)$ as $n\to\infty$, and thus $l(C')\leqq l(C)$. Analogously we can prove that $l(C)\leqq l(C')$, and thus $l(C)=l(C')$.

9. Fréchet distances of two curves. Given any two continuous curves in E_N, say, C: $x=f(u)$, $u\in I=[a, b]$, C_1: $x=g(v)$, $v\in J=[c, d]$, we shall consider the class $\{h\}$ of all homeomorphisms $u=h(v)$ from J onto I. Then for every $h\in\{h\}$ the expression $|f[h(v)]-g(v)|$, $v\in J$, has an absolute maximum $m[h]\geqq 0$ and we shall consider the infimum of this maximum $m[h]$ for all $h\in\{h\}$, say

$$\|C, C_1\| = \operatorname*{Inf}_{h} \max_{v\in J} |f[h(v)] - g(v)|.$$

We shall denote $\|C, C_1\|$ as the Fréchet distance of C to C_1. In other words,

$\|C, C_1\|$ is the infimum of all numbers $\epsilon \geqq 0$ having the following property: there exists a homeomorphism $u = h_\epsilon(v)$ from J onto I such that $|f[h_\epsilon(v)] - g(v)| \leqq \epsilon$ for all $v \in J$.

(9.i) *If C, C_1, C_2 denote continuous curves in E_N, we have* 1. $\|C, C_1\| \geqq 0$; 2. $\|C, C_1\| = \|C_1, C\|$; 3. $\|C, C_1\| \leqq \|C, C_2\| + \|C_2, C_1\|$; 4. $\|C, C_1\| = 0$ *if and only if* $C \sim C_1$.

The proof does not present difficulties. It should be pointed out here the rather trivial fact that the Fréchet distance $\|C, C_1\|$ of two curves C, C_1 is a quite different concept than the distance $\{[C], [C_1]\}$ of the two sets (graphs) $[C]$, $[C_1]$, the latter being defined, as usual, as the infimum of all distances $|x-y|$ between all pairs of points $x \in [C]$, $y \in [C_1]$. The inequality $\{[C], [C']\} \leqq \|C, C_1\|$ holds as an immediate consequence of the definitions. Obviously the sign $<$ may well hold. For instance, if $N=2$, C: $x_1 = u$, $x_2 = 0$, $0 \leqq u \leqq 1$, C_1: $x_1 = 0$, $x_2 = 1+u$, $0 \leqq u \leqq 1$, we have $\{[C], [C_1]\} = 1$, $\|C, C_1\| = \sqrt{5}$. For the two curves F_1, F_2 of Section 1 we have $[F_1] = [F_2]$, $\{[F_1], [F_2]\} = 0$, $\|F_1, F_2\| = 2$.

We may state now a remarkable extension of the property of lower semicontinuity of the Jordan length:

(9.ii) *If C, C_n, $n = 1, 2, \cdots$, are continuous curves in E_N, if $\|C_n, C\| \to 0$ as $n \to \infty$, then $l(C) \leqq \liminf l(C_n)$ as $n \to \infty$.*

Proof. Let C: $x = x(u)$, $u \in I$, C_n: $x = x_n(v)$, $v \in J_n$, $n = 1, 2, \cdots$, and $\delta_n = \|C, C_n\|$. Hence $\delta_n \to 0$ as $n \to \infty$. For every n there is a homeomorphism $v = h_n(u)$ such that $|x_n[h_n(u)] - x(u)| < \delta_n + n^{-1}$ for all $u \in I$, $n = 1, 2, \cdots$. Hence, if C_n' : $x = X_n(u) = x_n[h_n(u)]$, $u \in I$, $n = 1, 2, \cdots$, we have $|X_n(u) - x(u)| < \delta_n + n^{-1}$ for all $u \in I$ and hence $X_n(u) \to x(u)$ as $n \to \infty$ for all $u \in I$. By (4.i) and (8.i) we have $l(C_n') = l(C_n)$, $l(C) \leqq \liminf l(C_n')$, and hence $l(C) \leqq \liminf l(C_n)$ as $n \to \infty$. Thereby (9.ii) is proved.

Curves C, C_1 (both oriented and open), which are F-equivalent, are often considered as "representations" of the same entity. To obtain this we shall denote by a *Fréchet curve* $\tilde{C}$ (or F-curve) any family $\tilde{C} = \{C\}$ of all curves C which are F-equivalent to one another. The equality of two F-curves, say $\overline{C} = \{C\}$, $\overline{C}_1 = \{C_1\}$, is denoted by $\overline{C} = \overline{C}_1$, and is simply defined by the identity of the two families $\{C\}$ and $\{C_1\}$. By F-distance $\|\overline{C}, C_1\|$ of two F-curves $\overline{C} = \{C\}$, $\overline{C}_1 = \{C_1\}$, we shall denote the *F-distance* $\|C, C_1\|$ of any two curves $C \in \{C\}$, $C_1 \in \{C_1\}$. Such a number does not depend upon the particular choice of C and C_1 in the two families (because of (8.i)). Now theorem (9.i) has the following formulation:

If $\overline{C}$, $\overline{C}_1$, $\overline{C}_2$ denote F-curves we have (i) $\|\overline{C}, C_1\| \geqq 0$ *and* $\|\overline{C}, \overline{C}_1\| = 0$ *if and only if* $\overline{C} = \overline{C}_1$; (ii) $\|\overline{C}, \overline{C}_1\| = \|\overline{C}_1, \overline{C}\|$; (iii) $\|\overline{C}, \overline{C}_1\| \leqq \|\overline{C}, \overline{C}_2\| + \|\overline{C}_2, \overline{C}_1\|$.

Thus all F-curves $\overline{C}$ form a metric space whose metric is the F-distance.

Also, if we denote by Jordan length $l(\overline{C})$ of an F-curve $\overline{C} = \{C\}$ the Jordan

length $l(C)$ of any one of the elements $C \in \{C\}$, then $l(\overline{C})$ does not depend upon the choice of C in $\{C\}$ (by (8.i)), and (9.ii) has the following formulation:

(9.iii) *If $\overline{C}$, $\overline{C}_n$, $n=1, 2, \cdots$, are F-curves, and $\|\overline{C}_n, \overline{C}\| \to 0$ as $n \to \infty$, then $l(\overline{C}) \leqq \liminf l(\overline{C}_n)$ as $n \to \infty$.*

10. The representation theorem.

(10.i) *For every continuous curve C: $x = x(u)$, $a \leqq u \leqq b$, with $L = l(\mathcal{C}) < +\infty$, there is another curve C': $x = X(v)$, $a_1 \leqq v \leqq b_1$, for which $C' \sim C$, $X(v)$ is an AC vector function, even Lipschitzian with constant* 1, *and*

$$l(C) = l(C') = \int_{a_1}^{b_1} | X'(v) | \, dv.$$

In other words: For every F-curve $\tilde{C} = \{C\}$, with $L = l(\tilde{C}) < +\infty$, there is an element for which the Jordan length is given by the integral (2.2). In somewhat imprecise terms, we may say that each rectifiable curve has at least one representation for which the Jordan length is given by the integral (2.2).

Proof of (10.i). If $L=0$, then $x(u) = x_0$ is a constant, and we may assume $X(v) = x_0$ for all $a_1 \leq v \leq b_1$. Suppose $L > 0$, and observe that $s(a) = 0$, $s(b) = L$, and $s(u)$, $a \leqq u \leqq b$, is monotone nondecreasing and continuous in $[a, b]$. Also,

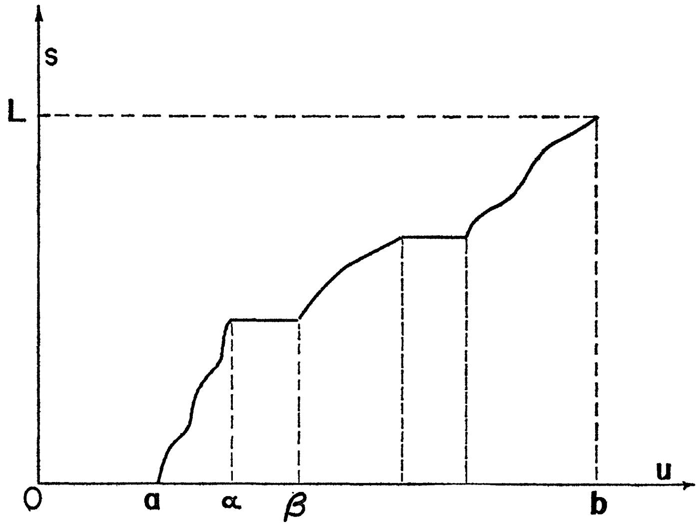

FIG. 4.

$s(u)$ is constant on an interval $[\alpha, \beta]$ if and only if $x(u)$ is constant there (Fig. 4). For each s_0, $0 \leqq s_0 \leqq L$, there is either a single point $u_0 = \alpha = \beta$, or a maximal interval $\alpha \leqq u \leqq \beta$, $a \leqq \alpha \leqq \beta \leqq b$, with $s(u) = s_0$, and, in the latter case, $x(u)$ is constant on $[\alpha, \beta]$. Let $X(s_0) = x(u)$, $\alpha \leqq u \leqq \beta$. Then the vector function $X(s)$ is defined for

all $0 \leqq s \leqq L$. Let us prove that $X(s_0-0)=X(s_0)$ for every $0<s_0 \leqq L$. Indeed we have then $\alpha>a$ and, given $\epsilon>0$ there exists σ such that $0<\sigma \leqq \alpha-a$, and $|x(u)-x(\alpha)|<\epsilon$ for all $\alpha-\sigma \leqq u \leqq \alpha$. Since $[\alpha, \beta]$ is a maximal interval of constancy for $s(u)$, we have $s(\alpha-\sigma)<s_0$, *i.e.*, $\delta=s_0-s(\alpha-\sigma)>0$. For every s with $s_0-\delta \leqq s<s_0$, there is now some u with $\alpha-\sigma \leqq u \leqq \alpha$ and $X(s)=x(u)$. Thus $|X(s)-X(s_0)|=|x(u)-x(\alpha)|<\epsilon$, and hence $X(s_0-0)=X(s_0)$. Analogously we can prove that $X(s_0+0)=X(s_0)$ for every $0 \leqq s_0<L$. Thus $X(s)$ is continuous and $C': x=X(s)$, $0 \leqq s \leqq L$, is a continuous parametric curve. Let us prove that $C' \sim C$. Indeed, we may well approximate $s=s(u)$ by means of quasilinear, strictly increasing functions $s=\phi_n(u)$, $a \leqq u \leqq b$, $\phi_n(a)=0$, $\phi_n(b)=L$, with $|\phi_n(u)-s(u)|<1/n$ for all $a \leqq u \leqq b$, $n=1, 2, \cdots$. Then $X[\phi_n(u)] \rightarrow X[s(u)]$ uniformly in $[a, b]$ as $n \rightarrow \infty$, *i.e.*, for every given $\epsilon>0$ there is some n with $|X[\phi_n(u)]-X[s(u)]|<\epsilon$, or $|X[\phi_n(u)]-x(u)|<\epsilon$ for all $a \leqq u \leqq b$. Thus $C' \sim C$. We have now $|X(s')-X(s)|=s'-s$, hence, if $X=(X_1, X_1 \cdots, X_N)$, also $|X_r(s')-X_r(s)| \leqq s'-s$, for all $0 \leqq s<s' \leqq L$. Thus $X(s)$ is an AC vector function and also Lipschitzian with constant 1. By (8.i) and (6.i) we have $L=l(C)=l(C')=\int_0^L|X'(s)|\,ds$, where $|X'(s)| \leqq 1$. Since $0=\int_0^L[1-|X'(s)|]\,ds$, we conclude that $|X'(s)|=1$ almost everywhere in $[0, L]$.

Remark. The last line of the proof above shows that for every rectifiable curve C and a.e. in $[0, L]$, $L=l(C)$, we have $(dX_1/ds)^2+\cdots+(dX_N/ds)^2=1$. Since $ds/ds=1$, this relation is also a consequence of (6.ii).

A continuous mapping $C: x=p(u)$, $a \leqq u \leqq b$, is said to be *light* if for every $x \in [C]$, the set $p^{-1}(x)$ is totally disconnected, *i.e.*, its components are single points. In other words, $p(u)$ is light if and only if $p(u)$ is constant on no proper subinterval of $[a, b]$. Obviously $x=X(s)$ is a light mapping. Thus (10.i) yields that every rectifiable F-curve has a light representation. This statement is actually general:

(10.ii) *Every F-curve has a light representation.*

(10.iii) *Any two F-equivalent continuous light mappings $x=f(u)$, $a \leqq u \leqq b$, $x=g(v)$, $c \leqq v \leqq d$, are L-equivalent.*

For any continuous mapping $C: x=f(u)$, $a \leqq u \leqq b$, let $\Gamma=\{\gamma\}$ denote the collection of all maximal intervals of constancy for $f(u)$ in $[a, b]$ (*i.e.*, proper closed intervals, and single points). Then Γ is a decomposition of $[a, b]$ into disjoint parts, proper closed intervals and single points. For a light mapping, Γ is the collection of the single points of $[a, b]$. We shall think of Γ as ordered on $[a, b]$ in the natural order.

(10.iv) *If $C_1: x=f(u)$, $a \leqq u \leqq b$, $C_2: x=g(v)$, $c \leqq v \leqq d$, are continuous mappings, and Γ_1, Γ_2 are the corresponding collections, then $C_1 \sim C_2$ if and only if there exists a one-one ordered correspondence between Γ_1 and Γ_2 such that $f(\gamma_1)=g(\gamma_2)$ for every pair $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ of corresponding elements.*

11. The line integral as a Weierstrass integral. For curves $C: x=x(u)$, $a \leqq u \leqq b$, $x=(x_1, \cdots, x_N)$, for which the real functions $x_r(u)$, $r=1, \cdots, N$, are continuous, with their first derivatives $x_1', \cdots, x_N'$, and $x_1'^2+\cdots+x_N'^2>0$ everywhere on $[a, b]$, by line integral is often assumed, by definition, the integral

$$I(C,f) = \int_C f(x, t) = \int_a^b f[x(u), x'(u)]du, \tag{11.1}$$

where $f(x, t)$, $x=(x_1, \cdots, x_N)$, $t=(t_1, \cdots, t_N)$, is any given function of (x, t), continuous in (x, t) for all t and all $x \in [C]$, and thus I is said to be the line integral of f on C. In order to assure that the integral $I(C,f)$ has the same value on equivalent curves, *i.e.*, on different representations of the same Lebesgue, or Fréchet curve, the further condition is required (h): $f(x, Kt)=Kf(x, t)$ for all $K \geqq 0$, t, and $x \in [C]$ (this implies $f(x, 0)=0$ for all $x \in [C]$). Condition (h), *i.e.*, f positive homogeneous of degree one in t, is known to be sufficient and, essentially, necessary for the required invariance of I with respect to Lebesgue and Fréchet equivalence for all curves C. If $f(x, t)=|t|=(t_1^2+\cdots+t_N^2)^{1/2}$, then $I(C,f)$ is the length integral (Sec. 2, (2.2)); if $f(x, t)=f(x)t_1$, then $I(C,f)$ is the "work" of a force $f(x)$ in the direction of the x_1-axis; if $N=2$, C is closed, and $2f(x, t)=x_1t_2-x_2t_1$, then $I(C,f)$ is the "signed area" linked by the plane curve C.

It may be pointed out that $x(u)$ denotes the current point on C, that $x'(u)/|x'(u)|=[x_r'(u)/|x'(u)|, r=1, \cdots, N]$ denotes the vector of the direction cosines of the tangent t to C at $x(u)$, and that (11.1), by condition (h), can be written in the equivalent form $I(C, f)=\int_a^b f[x(u), x'(u)/|x'(u)|]|x'(u)|du$. Thus $I(C,f)$ depends on the chain of values taken by f in correspondence to each point $x(u)$ of C and the associate direction of the tangent t to C at $x(u)$.

Unfortunately, the definition of line integral by means of formula (11.1) is subject to the same criticism which was expressed in Section 2 for the length integral. Namely, definition (11.1), though adequate under the hypotheses of continuity and differentiability mentioned at the beginning of this section, is not adequate under weaker assumptions, for instance in the class of all (parametric, continuous) rectifiable curves C. In this general situation an integral (W-integral) can be defined which is quite adequate, and which coincides with the integral (11.1) whenever all functions $x_r(u)$, $r=1, \cdots, N$, are absolutely continuous (AC) in $[a, b]$ (thus certainly under the more restrictive assumptions mentioned at the beginning of the section). The W-integral, or Weierstrass integral, is defined by a process of limit similar to the one we have discussed for Jordan length (2.ii).

Let $C: x=x(u)$, $a \leqq u \leqq b$, $x(u)=[x_1(u), \cdots, x_N(u)]$, be any (parametric, continuous) rectifiable curve, let $D=[a=u_0<u_1<\cdots<u_n=b]$ be any subdivision of $[a, b]$ and δ_D the norm of D (Section 2). For each interval $[u_{i-1}, u_i]$ we may consider the two points $p_{i-1}=x(u_{i-1})=(x_{i-1,r}, r=1, \cdots, N)$ $p_i=x(u_i)=(x_{ir}, r=1, \cdots, N)$ on C, and, if $p_{i-1} \neq p_i$, the chord $s=p_{i-1}p_i$, whose

direction cosines are the N numbers $\alpha_{ir}=(x_{ir}-x_{i-1,r})$ solidus $|p_i-p_{i-1}|$. If α_i is the vector $\alpha_i=(\alpha_{i1}, \cdots, \alpha_{iN})$, then α_i may be thought of as the vector of the "average" direction cosines for the "arc $(p_{i-1}p_i)$ of C." If t_i denotes any point $u_{i-1}\leqq t_i\leqq u_i$, and $\tilde{p}_i=x(t_i)$ is its image on the "arc $(p_{i-1}p_i)$ of C," then the sum

$$S_D = \sum_{i=1}^{n} f(\tilde{p}_i, p_i - p_{i-1}) = \sum_{i=1}^{n} f(\tilde{p}_i, \alpha_i) \,|\, p_i - p_{i-1} |$$

can be thought of as an "approximate" value for a line integral of f on C. Note that the intervals $[u_{i-1}, u_i]$ with $p_{i-1}=p_i$, *i.e.*, $p_i-p_{i-1}=0, f(\tilde{p}_i, 0)=0$, have no bearing on the first of the sums above, and the same can be said for the second sum if we assume that the corresponding vector α_i is any arbitrary unit vector.

The following theorems hold

(11.i) *If* $C: x=x(u)$, $a\leqq u\leqq b$, $x(u)=[x_r(u),\ r=1, \cdots, N]$, *is any (parametric, continuous) rectifiable curve in* E_N, *if* $f(x, t)$ *is continuous in* (x, t) *for all* t *and* $x\in[C]$, *if* $f(x, Kt)=Kf(x, t)$ *for all* $K\geqq 0$, t, $x\in[C]$, *then the limit exists and is finite*

$$J(C,f) = \lim_{\delta\to 0} S_D = \lim_{\delta\to 0} \sum_{i=1}^{n} f(\tilde{p}_i, p_i - p_{i-1}).$$

The limit $J(C,f)$ *is said to be the Weierstrass integral of* f *on* C, *and it is proved that* $C\sim C'$ *implies* $J(C,f)=J(C',f)$.

(11.ii) *Under the conditions above, if all functions* $x_r(u)$, $r=1, \cdots, N$, *are* *AC on* $[a, b]$, *then*

$$J(C,f) = \int_a^b f[x(u), x'(u)]du.$$

The last integral is said to be a Lebesgue-Tonelli integral. Because of (6.i) we can say that the Weierstrass integral $J(C,f)$ is given by the formula (11.1) (as an L-integral) whenever the Jordan length $l(C)$ of C is given by the length integral. By (10.i) every rectifiable Fréchet curve has a representation for which the Weierstrass integral is given by the Lebesgue-Tonelli integral.

Both theorems (11.i), (11.ii), and other properties of $J(C,f)$ are best proved as consequences of general theorems on interval functions, and the proofs are omitted. New proofs have been given by the author [L. Cesari, *Additive set functions and Weierstrass integral.*] The W-integral $J(C, f)$ was first proposed by Weierstrass, and then studied consistently by Tonelli as the main tool for the calculus of variations [L. Tonelli, *Rend. Acc. Lincei*, 21/1, 1912, 448–453, 554–559; 21/2, 1912, 132–137, *etc.*]. It was successively discussed by K. Menger, N. Aronszajn, G. Bouligand, and others. On the general topic of the present article the reader may consult L. Cesari, *Surface area*, quoted above, and T. Rado, *Length and area*, Amer. Math. Coll. Publ., vol. 30, 1948.

LOGIC AND MATHEMATICS COURSES

LOUIS O. KATTSOFF, Harpur College

The increasing tendency to introduce some work in logic into mathematics courses at the introductory level appears to me to need careful attention in terms of the aims and objectives of such introductory courses.

One of the first things that needs to be underlined is that for a great number of our undergraduates the introductory course is also the terminal course. This gives it a unique importance not shared by other courses. If the student is to learn anything about mathematics at all this is his only opportunity, for it is all too evident quite often that the mathematics he has had in the secondary school left him with no idea about mathematics except that of mechanical memorization and strange manipulations that somehow give the right answers. The introductory course for these students is the only place where they can begin to understand mathematics and its place in our civilization. Since I am writing this paper chiefly for mathematicians, I do not need to argue that no person can consider himself educated who does not to some degree comprehend mathematics and its role in creating, fashioning, and comprehending the kind of world we have.

Granted this need to give these students an understanding of what it is to do mathematics and what it is that mathematics does, the very utilitarian character of mathematics means that the student's education in other fields is impossible unless he acquires certain fundamental mathematical techniques. This too is so obvious that I shall not take the time to document it. What it means, however, is that these technical operations at least must be taught in the introductory course. For those who like myself prefer nonutilitarian reasons, I would suggest that an understanding of mathematics without some immersion in techniques is like trying to understand a youth in love by one who has never loved. It is, I know, the fashion today to decry what is called in derogatory tones "drill" or "technique," and to emphasize "thinking things out." This, as I shall show later, involves a basic misconception concerning the meanings of "thinking things out" and "technique." Be that as it may, one eliminates formulas and techniques only at the risk of doing students a disservice since it will make their education more difficult. Not even mathematics is so isolated and autonomous that it can be taught with neglect of its ramification. Even the "queen" must make her decisions in the light of the needs of her subjects.

It would be splendid if we could talk about our subject only in terms of this last group of students that I want to mention. These are those students for whom the introductory course lays the basis for a major and perhaps a career in mathematical research. For this group, quite obviously, the introductory course is not at all terminal but initiatory. Yet I venture to urge that the needs of this group are not exclusive but inclusive of those of the other two. These students need not only to understand the doing of mathematics and its applications, but they need to create in themselves the knack of technical skills and the fund of

knowledge without which they either become technical hacks or remain sterile. The basic difference between this group and the others is that this one needs more subject-matter and this perhaps cannot all be crowded into one course. Those who prepare for advanced work can, in any case, obtain the understanding and applications over a longer period of time. In a sense, however, I think the whole division I have made is quite artificial; but I will leave the problem of working out the details of a course for all three sets of students to others. The chief point I need to make is that there are these three aims of beginning courses in mathematics.

I have made this approach to my problem in order to have some basis for discussing the growing introduction of logic into introductory courses. Evidently, if one adds one or more chapters to an introduction to mathematics in which basic logical concepts are presented, there must be some justification other than enlarging the volume. It should be apparent, indeed, that to justify the inclusion of these notions one is best advised to endeavor to indicate that doing so will help achieve some desirable goal. Having stated a set of aims, we can now consider whether or not logical notions do make their attainment possible.

To help us in deciding our problem, let us look, rather sketchily to be sure, at the relation between logic and mathematics as it developed over the centuries. Logic, traditionally, was the science of correct reasoning. Mathematics was the science of quantity. The two, once upon a time, were separate studies. For many centuries the core of logic was the study of immediate inference and the syllogism. In later years Euler's and Venn diagrams were introduced and the interpretation of categorical propositions in terms of class-inclusion relations was added. So, throughout the passing centuries, logic was studied by those who wanted to learn how to derive valid conclusions from a given set of premises. Mathematics, on the other hand, was concerned to discover the properties of two kinds of quantities—the discrete and the continuous. In short, one who wanted to know the properties of space and time (which were quantities) learned mathematics. True, to derive the theorems of mathematics one did in fact use reasoning, still this was not so surprising. This attitude is quite prevalent. Let me quote one fairly-recent textbook in general college mathematics. "Mathematics is impossible without logic, for we must use correct arguments if we are to solve problems, prove theorems, *etc.*"

However, this quotation and the chapter on logic come at the end of the book. Indeed, if the student needs to be told that mathematics is impossible without logic in this sense, he must be quite stupid. Such a justification for the introduction of a chapter on logic which demonstrates no clear relation between logic and mathematics is as valueless as a chapter on logic in a textbook on surgery, for the justification would be exactly the same. In fact, this particular chapter to which I refer seems to me to be completely useless since it neither aids the student to understand mathematics nor gives any clear impression of the value of logic. It has no connection with the rest of the book, and I understand it is so considered by instructors who use it.

More significant reasons for including work in formal logic (or mathematical logic) can be given. In point of fact the relation between logic and mathematics is quite intricate, so much so that a true understanding of the nature of mathematics is virtually impossible without some ideas about logic. Even more significant is the fact that the distinction between logic and mathematics loses its sharpness, so that at times it is not possible to say whether one is working in logic or in mathematics. All of this is important even if we do not mention the application of symbolic logic to problems in algebras, statistics and computer-construction.

Credit is usually given to G. Frege for the first explicit and detailed attempt to derive the concept of natural number from logical notions. Frege actually was inspired by two objectives. He wished to construct logic in such a way that all the consequences of a set of statements could be derived purely mechanically. Deductive logic, he felt, needed to be just as much a calculus as arithmetic so that one could, by defining certain operations, perform all "reasoning" by manipulating the symbols according to these rules. Secondly, Frege sought to show that the number concept was at bottom constructible by definition from the elementary logical notions. He approached the problem by considering quantity, and in particular number, to be a property of groups (or classes of things) and a purely formal property at that. Since a class could be defined in terms of "propositional functions" by considering it to be composed of all those elements for which the function resulted in a true proposition, this rather obviously made it possible to start with logical notions and build up classes. Adding to this the notion of (1, 1) correspondence, Frege and later Russell came up with the definition of a number as "the class of all classes that can be put into (1, 1) correspondence with each other." It is plain to see at this point that this rather startling development, which was made better known by Russell in his *Philosophy of Mathematics* (Vol. I) and then with A. N. Whitehead in *Principia Mathematica* using a much better symbolism than had Frege, signifies that whoever wants to work at the roots of arithmetic needs to go back to logic. This means more than that. If the work of Frege and Russell was correct, then mathematics was but a branch of logic, and it is also possible to conceive of logic as mathematics. If logic could serve as a matrix for mathematical notions, could one not start with mathematics and define logical concepts in its terms? We will not try to answer this question but merely to point out that this paves the way for the expanded notions of "algebras," and the conception of an algebra as a logical system or a logical system as an algebra.

In a context such as this, I must mention George Boole. Boole too was inspired by the Leibnizean and Cartesian notion of a "universal science of order and measurement." He wanted to reduce the so-called "laws of reason" to mechanical operations. Accordingly, in his *Law of Thought*, Boole tried to do so and gave us what has been called "Boolean Algebra." This I mention simply to indicate again the close connection between algebra and logic and the stages that make it easy to identify the two.

The Frege-Russell derivation of finite arithmetic from the concept of class

(or function) was fortunately based upon a notion that led to the paradoxes. Russell formulated one of the earliest of these in the Russell paradox based on the definition of a particular class. Soon other paradoxes were published. The effect was to release a host of studies aimed at "saving" the foundations of mathematics. We need not describe these here except to indicate some of the things that resulted.

Attempts to correct the definition of a class to obtain one that would not lead to paradoxes brought about an intensive development of the theory of classes, both finite and transfinite. The notion of a class could be constructed or simply taken as an interpretation of a set of axioms. This in its turn complemented the development of axiomatic theory. The work of David Hilbert and his associates in this area is now part both of the history of logic and of mathematics. Out of it in large measure has come metamathematics and a deepening of our insight into what mathematics is. The result has been elaboration of such notions of "proof," "function," "rules of substitution," "rules of replacement," "rules of operation," analysis of recursion formulas, the distinction between variable and constant, and so on. Metamathematics is, in brief, the theory of mathematics and talks *about* mathematics. But above all it has tended to underline the notion of mathematics as the mechanical development of the deductive consequences of an initial set of statements. This explained and explains why the axiom-system is the ideal type of mathematical structure.

Hilbert's work on the axiomatization of geometry clarified the basic elements of such systems, too. It brought to the fore the importance of the problems of consistency, independence, completeness, and the nature of mathematical existence. The first of these are logical problems being concerned with the relation of implication. The last of these problems became of prime importance in the controversy with L. E. J. Brouwer, the Dutch mathematician, and centered around the principle of classical (or two-valued) logic known as the "law of excluded middle." It resulted in a clarification of such expressions in mathematics as "there exists a function such that . . . ." But it also led into the whole theory of logical matrices by means of which the multivalued logics were developed, and which play an important role in the construction of certain types of computers.

All of these developments brought out the fact that pure mathematics ultimately reduces to the mechanical (*i.e.*, deductive) manipulation of sets of symbolic arrangements according to pre-established rules. This is why I said earlier in this paper that there is a fundamental misconception involved in the attack on drill and technique. If pure mathematics teaches one to think, it instructs in the art of mechanical elaboration of initial premises. Facility in such work, as in all mechanical operations, comes through repetition. To get the "feel" of such manipulation, even when one is simply trying to understand what mathematics is rather than to become an expert mathematician, one does need to do this sort of work. But I do not want to involve myself in debating

this issue. It is, I believe, to the realm of applied mathematics that those refer who speak in terms of "thinking things through." In this area we enter upon the significance of mathematics for our civilization. This is, however, a different matter.

All of these remarks make it quite evident that the interrelations between symbolic logic and mathematics are many and intricate. It is, in my opinion, simply impossible to understand the nature of mathematics without some introduction to the various calculi of symbolic logic. It is more and more evident that both from a practical as well as a theoretical view, symbolic logic is of importance to mathematicians. This calls for a fundamental revision of introductory courses in the direction of an introduction of the basic concepts and procedures of symbolic logic. To make room for this, something clearly will need to be omitted from the standard general mathematics course. The two alternatives have been either to cut out so-called more difficult and useless material or to cut down on the drill. Both of these are mistaken directions. Rather, if one examines standard-type texts and sees the vast amount of work devoted to so-called "review of fundamentals," the possibility arises of doing something here. There are always three or four chapters that include material that students should have had in high school. This could easily be replaced by other material. I believe it is not too much to ask of college freshmen, especially in these days of mass influx and limited facilities, to be able to perform the fundamental operations, to work with linear equations and to graph elementary functions. But one must warn against simply replacing chapters devoted to these reviews with some devoted to logic with no organic relation between the logic and the mathematics indicated. An excellent example of the sort of thing that can be done is found in the recent book by John Kemeny, J. L. Snell and Gerald Thompson entitled *Introduction to Finite Mathematics.** As I read it, it meets the three needs I have indicated. It begins with logic and shows the relation of logical to mathematical notions, thereby making for a better understanding of the nature of mathematics. It devotes much time to probability, matrices, and sets, providing a basis for further work in mathematics, and it devotes attention to the application of mathematics in the behavioral sciences, giving insight into the place of mathematics in our civilization.

Knowledge of symbolic logic will inevitably profit both the professional mathematician, pure and applied, and the beginning student for it will help

* John G. Kemeny, J. Laurie Snell, Gerald L. Thompson, Introduction to Finite Mathematics, 1957.

Cf. also for the relation of logic to mathematics, E. R. Stabler, An Introduction to Mathematical Thought, 1953; and for an approach via logic, Patrick Suppes, Introduction to Logic, 1957.

For a general introduction to the foundations of mathematics cf. L. O. Kattsoff, A Philosophy of Mathematics, Iowa University Press, 1948.

For a rather technical study cf. S. C. Kleene, Introduction to Metamathematics, Princeton, 1952.

For a good introduction to set-theory, cf. A. Fraenkel, Abstract Set Theory, 1955.

display to them what McShane has called the "two legs" on which mathematics stands—its "innate beauty and austere elegance" as well as its "usefulness to scientists and technicians of all kinds."*

* E. J. McShane, Maintaining communication, this MONTHLY, vol. 64, 1957, pp. 309–317.

PSEUDO-INVERSES IN ASSOCIATIVE RINGS AND SEMIGROUPS

M. P. DRAZIN, Northwestern University

1. Introduction. E. H. Moore ([5], and *cf.* also Penrose [6], Rado [8]) has shown how, given any square matrix x with elements in the complex field, it is possible to define another matrix, coinciding with x^{-1} whenever x^{-1} happens to exist, and having (even for singular x) properties leading to simple proofs of several theorems which yield to other methods only with difficulty (if at all). More precisely, Moore's result (stated in Penrose's form) is that, given any square* complex matrix x, then there is always exactly one complex matrix c satisfying

$$cxc = c \quad \textit{and} \quad xcx = x,$$

$$cx \textit{ and } xc \textit{ are both hermitian.}$$

Following Penrose, we denote this unique matrix c by $x^{\dagger}$, and call it *the generalized inverse of* x; for evidence of the value of this concept, we refer the reader to the work of Moore and Penrose ([5], [6], [7]).

It is clear from Penrose's arguments that the function $x^{\dagger}$ thus defined for square complex matrices can be extended to the elements of any finite-dimensional algebra R over the complex field by the standard device of mapping R isomorphically into the algebra M_n of all $n \times n$ complex matrices (*e.g.* with $n = 1 + \dim R$), *provided* that the image of R in M_n admits the operation of forming transposed complex conjugate matrices. However, there seems to be no prospect of extending Moore's generalized inverse concept so as to apply to finite-dimensional algebras over a given division ring F unless F at any rate admits an involutory anti-automorphism $\lambda \to \bar{\lambda}$ such that $\lambda_1\bar{\lambda}_1 + \cdots + \lambda_m\bar{\lambda}_m = 0$ implies $\lambda_1 = \cdots = \lambda_m = 0$. (This, of course, would restrict F to having zero characteristic.)

In this note we show how to define a function, somewhat analogous to the generalized inverse function, over the elements of *arbitrary* finite-dimensional algebras, and even of an extensive class of associative rings. Indeed, all of our

* In fact, Moore and Penrose considered rectangular matrices, but this aspect of the result is not relevant here.

theorems in Section 2 below apply equally well to arbitrary semigroups; however, we shall for brevity, generally use the language of ring theory.

Given any associative ring R, and any element x of R, we shall call c *pseudo-invertible* (*in* R) if an element c of R exists satisfying

(i) $$cx = xc,$$

(ii) $$x^m = x^{m+1}c \text{ for some positive integer } m,$$

and

(iii) $$c = c^2x.$$

By Theorem 1 below, these three conditions determine c uniquely when it exists, so we may refer to c as *the* pseudo-inverse of x. We show also (*inter alia*) that the existence of such a c in fact follows from the apparently much weaker hypothesis that elements a, b of R exist such that

$$x^p = x^{p+1}a, \qquad x^q = bx^{q+1}$$

(for some positive integers p, q); in other words, pseudo-invertibility (as applied to a given element x of a given ring) coincides with Azumaya's property [**2**] of *strong π-regularity*. Since it is known that every element of any algebraic ring (as defined in [**3**]), or of any ring with minimal condition on left or right ideals, is strongly π-regular, it follows that our pseudo-inverse function is applicable to all such rings (and, in particular, to matrices and all finite-dimensional algebras). The specialization of pseudo-invertibility in which (ii) above holds with $m=1$ was discussed by Azumaya, and indeed some of our arguments below are, formally, only slight generalizations of his: the essence of our present contribution is that (i), (ii) and (iii) determine c uniquely even when m is unrestricted (and indeed variable). The case $m=1$ of pseudo-invertibility, but with (iii) omitted, was discussed by A. H. Clifford [**10**], who called the corresponding property "relative invertibility"; in view of our Theorem 4 below, this property of relative invertibility is, in fact, precisely equivalent to pseudo-invertibility with $m=1$.

In our third section, we use pseudo-inverses to obtain a simple proof of another result of Azumaya ([**2**] Theorem 4). At the suggestion of a referee, we describe, in a concluding section, another way of looking at pseudo-inverses in any semigroup S: it turns out that our definition can be equivalently expressed, in a natural and simple way, in terms of certain idempotents and maximal subgroups of S. After Moore's and Penrose's work, it seems not unreasonable to hope that the pseudo-inverse may prove to be a useful tool in dealing with those rings in which it is everywhere defined, and that the publication here of the basic properties of pseudo-inverses may perhaps serve as a starting-point for other applications.

2. General results on pseudo-inverses. In this section we establish a number of facts about pseudo-inverses in rings (or semigroups) not subjected to any global conditions. First and most fundamentally, we have

THEOREM 1. *Let R be any given associative ring (or semigroup), and x any element of R. Then x has at most one pseudo-inverse in R, and, if a pseudo-inverse for x does exist, it commutes with every element of R which commutes with x.*

Proof. Let c_1, c_2 satisfy conditions corresponding to those imposed on c by (i), (ii) and (iii) of our definition of pseudo-invertibility, say with integers m_1, m_2 in (ii). Then, on writing $m = \max(m_1, m_2)$, the conditions on c_1, c_2 arising from (i), (ii) certainly imply

$$\text{(A)} \qquad c_1 x^{m+1} = x^m = x^{m+1} c_2,$$

while those arising from (i), (iii) certainly imply

$$\text{(B)} \qquad c_1 = c_1^2 x, \qquad c_2 = x c_2^2.$$

And in fact (A), (B) together imply that $c_1 = c_2$. For, by induction from (B), we have $c_1 = c_1^{k+1} x^k$, $c_2 = x^k c_2^{k+1}$ $(k = 1, 2, \cdots)$, and in particular $c_1 = c_1^{m+1} x^m$, $c_2 = x^m c_2^{m+1}$, whence, by (A),

$$c_1 = c_1^{m+1} x^m = c_1^{m+1} x^{m+1} c_2 = c_1 x c_2 = \cdots = c_2.$$

Thus x has at most one pseudo-inverse c. Also, if y is any element of R satisfying $xy = yx$, then, by (ii) and (i), we find

$$c x^m y = c y x^m = c y x^{m+1} c = c x^{m+1} y c = x^m y c,$$

whence $c^{m+1} x^m y = x^m y c^{m+1}$; but, as above, (iii) gives $c = c^{m+1} x^m$, and so, using (i) again, we conclude that

$$cy = c^{m+1} x^m y = x^m y c^{m+1} = y c^{m+1} x^m = yc,$$

as required.

In view of Theorem 1, we may denote the unique pseudo-inverse of a given pseudo-invertible element x by x'; obviously, whenever x^{-1} exists in the ordinary sense, then x' exists and $x' = x^{-1}$. It should be noted that, when it is defined at all, x' is independent of what ring we think of x as lying in (in the sense that, if R_1, R_2 are subrings of a given ring R and x lies in $R_1 \cap R_2$, having pseudo-inverses c_1, c_2 in R_1, R_2 respectively, then c_1, c_2, being pseudo-inverses for x in R, must coincide).

COROLLARY 1. *If $x_1, \cdots, x_j$ are given pseudo-invertible elements (of some ring) with $x_s x_t = 0$ $(s, t = 1, \cdots, j;\ s \neq t)$, then $x_1 + \cdots + x_j$ is also pseudo-invertible, with $(x_1 + \cdots + x_j)' = x_1' + \cdots + x_j'$.*

Proof. There will clearly be no loss of generality in supposing from the outset that $j = 2$; and it will also be convenient to write u, v in place of x_1, x_2. By (i), the hypothesis $uv = vu$ $(= 0)$, and a double application of the last part of Theorem 1, we see that u, v, u', v' all commute with one another; further, by (iii),

any product of these which simultaneously involves u or u' and also v or v' can be expressed as a product involving both u and v, and must consequently vanish. Hence

$$(u' + v')(u + v) = (u + v)(u' + v'),$$

$$(u' + v')^2(u + v) = u'^2u + v'^2v = u + v,$$

and, choosing m so large that $u^m = u^{m+1}u'$, $v^m = v^{m+1}v'$, we have

$$(u + v)^{m+1}(u' + v') = u^{m+1}u' + v^{m+1}v' = u^m + v^m = (u + v)^m;$$

by the first part of Theorem 1, the result now follows.

Given any pseudo-invertible element x of a ring, then, by (ii), there will be a unique (positive) integer $i(x)$ such that $x^m = x^{m+1}x'$ for every $m \geqq i(x)$ but for no $m < i(x)$. We shall refer to $i(x)$ as *the index of* x; again, provided that it does indeed exist, this integer $i(x)$ does not depend on what ring we regard x as lying in. If x is not pseudo-invertible, then we take $i(x) = \infty$ conventionally.

The special case of Theorem 1 in which $m_1 = m_2 = 1$ is, apart from differences in terminology, just Azumaya's ([2] Lemma 1) with the existence clause omitted;* our next result, however, is of interest chiefly when $i(x) > 1$.

THEOREM 2. *Let x be any pseudo-invertible element (of some given ring) and k any positive integer. Then x^k is pseudo-invertible, with $(x^k)' = (x')^k$, and $i(x^k)$ is the unique (positive) integer q satisfying $0 \leqq kq - i(x) < k$.*

Proof. By (i), $x^k(x')^k = (x')^k x^k$. Also, by induction from (ii), (iii) respectively, we have $x^{i(x)} = x^{i(x)+j}(x')^j$, $x' = (x')^{j+1}x^j$ $(j = 1, 2, \cdots)$, so that, since $kq \geqq i(x)$,

$$(x^k)^q = x^{kq-i(x)}x^{i(x)} = x^{kq-i(x)}x^{i(x)+k}(x')^k = (x^k)^{q+1}(x')^k,$$

$$(x')^k = (x')^{k-1}(x')^{k+1}x^k = ((x')^k)^2x^k.$$

Thus $(x')^k$ satisfies the conditions for $(x^k)'$, and $i(x^k) \leqq q$.

Finally, $i(x^k) < q$ would mean that $(x^k)^{q-1} = (x^k)^q(x')^k$, and, since $x' = x^{k-1}(x')^k$, this in turn implies that $x^{k(q-1)} = x^{kq-(k-1)}x' = x^{k(q-1)+1}x'$, whence, by the definition of $i(x)$, we should have $k(q-1) \geqq i(x)$, contrary to our definition of q.

THEOREM 3. *Given any element x of a ring, then if x is pseudo-invertible so is x'; in fact x' has index 1 whenever it exists, and then $x'' = x^2x'$.*

To prove this, one has merely to verify that, if c satisfies (i), (ii) and (iii), then $d = x^2c$ satisfies $cd = dc$, $c = c^2d$, $d = d^2c$. We omit the details, and also leave to the reader the proofs of the following equally trivial joint corollaries of Theorems 1, 2 and 3:

* The last clause of Theorem 1 in this special case has also been set as a problem by T. Skolem [9].

COROLLARY 2. *Given any element x of a ring R, then $x''=x$ if and only if x is pseudo-invertible with index* 1; *and, when this is the case, for any given element y of R, x commutes with y if and only if x' does.*

COROLLARY 3. *Given any pseudo-invertible element x of a ring, then $(x^k)''=x^k$ for every integer $k \geqq i(x)$.*

COROLLARY 4. *For any pseudo-invertible element of a ring, $x'''=x'$.*

We have already noted that an element x of a ring R is called *strongly π-regular* in R if elements a, b of R and positive integers p, q exist such that

$$x^p = x^{p+1}a, \qquad x^q = bx^{q+1}. \tag{1}$$

We now prove

THEOREM 4. *Given any element x of a ring R, then x is pseudo-invertible in R if and only if it is strongly π-regular in R.*

Proof. That pseudo-invertibility implies strong π-regularity is obvious (even without using (iii)). Conversely, we shall show that (1) implies (i), (ii) and (iii) with $m = \max(p, q)$ and $c = x^m a^{m+1}$.

We note first that (1) gives

$$x^{m+1}a = x^m = bx^{m+1},$$

so that $x^m a = bx^{m+1}\cdot a = b\cdot x^{m+1}a = bx^m$, whence, by induction, $x^m a^k = b^k x^m$ $(k=1, 2, \cdots)$. Thus our choice $c = x^m a^{m+1}$ can equivalently be written $c = b^{m+1}x^m$, and we have:

(i) $xc = x\cdot x^m a^{m+1} = x^{m+1}a\cdot a^m = x^m a^m = b^m x^m = \cdots = cx$ by symmetry;

(ii) By another induction, $x^m = x^{m+k}a^k$ $(k=1, 2, \cdots)$, and so

$$x^{m+1}c = x^{m+1}\cdot x^m a^{m+1} = x^{m+(m+1)}a^{m+1} = x^m;$$

(iii) By (i) and (ii) which we have just proved,

$$c^2x = c\cdot xc = c\cdot x^{m+1}a^{m+1} = x^{m+1}c\cdot a^{m+1} = x^m a^{m+1} = c.$$

It is clear from this proof that (1) ensures that $i(x) \leqq \max(p, q)$, and (cf. [2] Lemma 3) indeed it is easy to see that in fact $i(x) \leqq \min(p, q)$: for if, say, $p<q$, then (1) gives $x^p = x^{p+(q-p)}a^{q-p} = x^q a^{q-p}$, so that $bx^{p+1} = bx\cdot x^q a^{q-p} = bx^{q+1}\cdot a^{q-p} = x^q a^{q-p} = x^p$. In view of this, our Theorem 4 includes Azumaya's Theorem 3, in which he proved that strong π-regularity implies the *existence* of c satisfying (i) and (ii); we can arrange for uniqueness only by introducing some additional restriction such as (iii). For a discussion of the implications subsisting between strong π-regularity (*i.e.*, pseudo-invertibility), right π-regularity, left π-regularity and π-regularity, we refer the reader to Azumaya's paper.

COROLLARY 5. *Let R be any finite-dimensional algebra. Then, for any given $x \in R$, x' exists and lies in the subalgebra generated by x.*

Proof. If R has dimension k, then $x, x^2, \cdots, x^{k+1}$ are linearly dependent, and so, for some $j \leqq k+1$, x^j is a linear combination of $x^{j+1}, x^{j+2}, \cdots$, hence even of x^{j+2} and higher powers. Thus x is strongly π-regular, with $a=b$ in (1) expressible as polynomials in x (without constant terms); the corollary now follows from the form of c in the proof of Theorem 4 above. More generally, the result clearly holds even for all "algebraic rings" (in which, by definition, there corresponds to each element x an integer $j(x)$ such that $x^{j(x)}$ is a linear combination of $x^{j(x)+1}$ and higher powers).*

We remark that, immediately from the definition and uniqueness property, the operation of taking the pseudo-inverse (when such exists) of a given element commutes with all homomorphisms and antihomomorphisms of the containing ring. In particular, for any matrix algebra over the complex field, the pseudo-inverse of the complex conjugate (or transpose) of a given matrix is the same as the complex conjugate (or transpose) of the pseudo-inverse. Thus the pseudo-inverse of a given complex matrix x is real (symmetric, hermitian, *etc.*) whenever x is; and, even for arbitrary square complex matrices, it can be shown (*e.g.*, by using (iii) and Corollary 5) that the property of having all eigenvalues real and nonnegative is also preserved.

3. Sufficient conditions for pseudo-invertibility in rings.

Following Azumaya, we call an element x of a ring R *right π-regular* in R if a positive integer p and an element a of R exist such that $x^p = x^{p+1}a$. In these circumstances we shall define the *right index of* x in R, denoted by $r(x)$ (or more precisely $r_R(x)$), as the smallest integer p occurring in any such representation; and we take $r(x) = \infty$ whenever x is not right π-regular. Thus $r(x)$ is defined for *all* elements of R; and we define the *left index* $l(x) = l_R(x)$ of x in R similarly. Then, for any given x, we know from Theorem 4 that $i(x)$ is finite (*i.e.*, x is pseudo-invertible) if and only if $r(x)$, $l(x)$ are both finite, while our remark following Theorem 4 shows that then in fact§ $r(x) = l(x) = i(x)$.

This "local" result (*i.e.*, concerning only the single element x) is new, but a still more striking "global" result has been known for some time ([4] p. 74). Given any subset T of a ring R, let us write

$$i(T) = \sup i(x), \qquad r(T) = \sup r(x), \qquad l(T) = \sup l(x),$$

where each upper bound is taken over all $x \in T$ (with the natural convention for infinite values‡). Obviously $r(T) \leqq i(T)$, and Kaplansky's result (or rather

* In the case of a square matrix x over an algebraically closed field F, we could alternatively prove the existence of x' by combining Corollary 1 (for $j=2$) with the well-known fact that x is similar over F to diagonal (u, v), where u, v are square matrices over F with u (if occurring) nilpotent and v (if occurring) nonsingular.

§ For a pseudo-invertible element x, the common value of $r(x)$, $l(x)$ and $i(x)$ coincides with the value of the "index of x" as defined by Azumaya (whose definition concerned only pseudo-invertible elements); but it is important in what follows that $r(x)$, $l(x)$ and $i(x)$ are meaningful for *arbitrary* x, and that $r(x)$, $l(x)$ (separately) can be finite even for nonpseudo-invertible x.

‡ More precisely, we take $i(T) = \infty$ whenever the $i(x)$ with $x \in T$ are finite but unbounded, and also whenever $i(x) = \infty$ for some $x \in T$; and similarly for $r(T)$, $l(T)$.

a slightly-strengthened form of it) is that, if $r(R)$ is finite, then so is $i(R)$ and in fact $i(R)=r(R)$; moreover, by our final remark in the previous paragraph, we even have $i(x)=r(x)=l(x)<\infty$ for each $x\in R$.

To motivate the next theorem (which was discovered by Azumaya, and does not seem readily extensible to semigroups), we note first that our definitions of $i(x)$, $r(x)$, $l(x)$ are of course consistent with the accepted meaning of the "index" of a given nilpotent element of a ring. More precisely, every nilpotent element x is clearly pseudo-invertible (with $x'=0$), so that $i(x)=r(x)=l(x)<\infty$, and satisfies $x^{i(x)}=0$, $x^{i(x)-1}\neq 0$ (since $x^{i(x)}=x^{i(x)+k}(x')^k$ for arbitrarily high k, while $x^{i(x)-1}=0$ would contradict the definition of $i(x)$). Thus, if $N=N(R)$ denotes the set of all nilpotent elements of the ring R, then we always have $i(N)=r(N)=l(N)$ (possibly all infinite), and $i(N)<\infty$ expresses the condition that the nilpotent elements of R have bounded index.

In ([2] Theorem 4), Azumaya generalized Kaplansky's result that the finiteness of $r(R)$ implies that of $i(R)$; replacing the finiteness of $r(R)$ by that of $i(N)$, he showed that this obviously weaker hypothesis in fact implies that $i(x)\leqq i(N)$ whenever $r(x)$ (or $l(x)$) is finite. We present now a new and very simple proof, bearing only a slight resemblance to Azumaya's (and none to Kaplansky's), of this result.

THEOREM 5. *Let R be any associative ring whose nilpotent elements have bounded index (i.e., with finite bound $i(N)$). Then every right π-regular element x of R is pseudo-invertible, with $i(x)=r(x)=l(x)\leqq i(N)$.*

Proof. Given any right π-regular element x of R, say with $x^p=x^{p+1}a$, we shall first show that x is necessarily pseudo-invertible; by Theorem 4, it will be enough to find $b\in R$ and a positive integer q such that $x^q=bx^{q+1}$.

Now, by induction, we have $x^p=x^{p+k}a^k$, and so

$$x^{p+k}(x^p - a^kx^{p+k}) = (x^p - x^{p+k}a^k)x^{p+k} = 0 \qquad (k = 1, 2, \cdots).$$

Also each of the 2^t monomials in the expansion of $(x^p-a^kx^{p+k})^t$ has x^{p+k} as a right-hand factor provided only that $pt\geqq p+k$, and so, for each k, we have $(x^p-a^kx^{p+k})^{t+1}=0$ for large enough t. Hence, by our hypothesis that $i(N)$ is finite, it follows that $(x^p-a^kx^{p+k})^{i(N)}=0$, so that $x^{i(N)p}\in Rx^{p+k}$ $(k=1, 2, \cdots)$. Choosing $k=(i(N)-1)p+1$, we deduce that an element b of R exists such that $x^q=bx^{q+1}$, where $q=i(N)p$.

Thus x is pseudo-invertible, so that (i) and (ii) clearly give $(x-x^2x')^m=0$ for some m, whence $(x-x^2x')^{i(N)}=0$; hence, by (i), $x^{i(N)}=x^{i(N)+1}y$ for a suitable polynomial y in x, x' (with integer coefficients and no constant term), and so, finally, by (i) and the proof of Theorem 4, we can conclude that $i(x)\leqq i(N)$. (Also, incidentally, we have the explicit representation $x'=x^pa^{p+1}$).

COROLLARY 6. *Let R be any associative ring whose nilpotent elements have bounded index. Then, if every element of R is right π-regular, R must in fact be boundedly right π-regular, i.e. $r(R)$ is finite, and indeed $r(R)=i(R)=i(N)<\infty$.*

To deduce this from the theorem, we have only to note that $r(R) \leqq i(R) \leqq i(N) = r(N) \leqq r(R)$ (the inequality $i(R) \leqq i(N)$ following from Theorem 5 and the hypothesis of the corollary, while the other relations are obviously true in any ring); and of course Kaplansky's result is included in Corollary 6. We remark also that, by Azumaya's ([2] Theorem 5), one may, in the hypothesis of Corollary 6, replace "right π-regular" by "π-regular."

Some familiar examples of rings satisfying the hypotheses of Corollary 6 are (a) rings with minimal condition on (say) right ideals, and (b) algebraic rings of bounded degree (or, more generally, of bounded index); but it is already known (see [1] Theorem 3.1 and [4] p. 74) that rings of type (a) have $r(R)$, $l(R)$ finite, and this is obvious in case (b). However, Arens and Kaplansky's proof that $r(R)$ is finite in case (a), though brief enough, depends on the Wedderburn-Artin structure theory, and it would be gratifying to find a direct proof. We mention in this connection that, since every element x of any ring of type (a) is (from consideration of the descending chain of principal right ideals generated by the powers of x) clearly right π-regular, it would, in view of Corollary 6, suffice to prove (directly) the finiteness of $i(N)$; and, for this, since the rings in question have nilpotent (Jacobson) radicals, only the semisimple case need be considered.

4. Concluding remarks. The author is grateful to a referee for pointing out that, even in the context of arbitrary semigroups, the definition of pseudo-invertibility can be rephrased in a way that may perhaps make it more immediately available as a tool for tackling certain problems. To do this, given any element e of a semigroup S, let G_e denote the set of all elements $a \in S$ such that

$$(\alpha)\ ae = ea = a, \quad \text{and} \quad (\beta)\ ab = ba = e \text{ for some } b \in S.$$

Clearly, if G_e is nonempty, then e is necessarily idempotent, and, conversely, we have (this being, essentially, Lemma 1.3 of Clifford's paper [10]).

THEOREM 6. *If e is idempotent, then G_e is a group with e as identity element, and moreover every subgroup G of S containing e is in fact a subgroup of G_e.*

Proof. To prove the first assertion, it clearly suffices to show that, given any $a \in G_e$, we can find an inverse for a in G_e; and (e being idempotent) this is immediate, since (α), (β) for a and any corresponding b imply that $ebe \in G_e$, so that ebe is the inverse of a in G_e.

For the second assertion, we have only to note that e must be the identity of any subgroup G in which it lies, so that (α), (β) are immediate for all $a \in G$, as required.

COROLLARY 7. *The maximal subgroups of S are precisely the subgroups G_e with e idempotent.*

We next have

THEOREM 7. *Given any element x of a semigroup S and any positive integer m, then the statement*

(1) *x is a pseudo-invertible element of S with index at most m*

is equivalent to

(2) *there is an idempotent* $e \in S$ *such that* $ex = xe \in G_e$ *and* $x^m \in G_e$.

Proof. To see that (1) implies (2), we mention first that (i) and (iii) of the definition of pseudo-invertibility imply that $e = xc = cx$ is idempotent. Also, by (i), $ex = xe$ and $(ex)e = e(ex) = ex$, while (iii) gives $(ex)c = c(ex) = e$; hence $ex = xe \in G_e$. Similarly, (ii) becomes $x^m = x^m e$, so that $x^m e = ex^m = x^m$, while (i) and (iii) give $x^m c^m = c^m x^m = e$, whence $x^m \in G_e$ and (2) follows.

Conversely, given (2), let c denote the inverse of $ex = xe$ in the group G_e. Then $ce = ec = c$ and $cex = xec = e$, whence $cx = xc = e$ and $c = ce = c^2 x$, which are (i) and (iii). Finally, if $x^m \in G_e$, then we have $x^m = x^m e = x^{m+1} c$, which is (ii).

COROLLARY 8. *An element x of S is pseudo-invertible with* $i(x) = 1$ *if and only if x belongs to some maximal subgroup of S.*

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ACKNOWLEDGMENT

At the time of publication, the author of *Solution of a ranking problem from binary comparisons*, this MONTHLY, vol. 64, No. 8, Part II, October, 1957, was unaware of the work of R. A. Bradley and M. E. Terry along similar lines. Had he known of this, he would have made reference to their paper, *The rank analysis of incomplete block designs* I; *The method of paired comparisons*, Biometrika, vol. 39, Parts 3 and 4, December, 1952, both in the body of the text and in the bibliography.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the eighteenth William Lowell Putnam Mathematical Competition held on February 8, 1958, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of the Polytechnic Institute of Brooklyn, Brooklyn, New York. The members of the team were Martin Isaacs, Donald Passman, and Lawrence A. Shepp; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Jonathan L. Alperin, Everett C. Dade, and Stephen Lichtenbaum; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were David R. Brillinger, (Mrs.) Marion Green, and Carl R. Riehm; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the University of Manitoba, Winnipeg, Manitoba. The members of the team were Ian Connell, Wilbur Jacob Jonsson, and Philip James Peebles; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are David R. Brillinger, University of Toronto; Donald J. C. Bures, Queen's University; Richard Dudley, Harvard University; Joseph Lipman, University of Toronto; and Lawrence A. Shepp, the Polytechnic Institute of Brooklyn. Each of these will receive a prize of fifty dollars.

The six succeeding persons (tenth and eleventh tied) ranking highest in the examination, named in alphabetical order, are Robert Hartshorne, Harvard University; John R. F. Hewett, University of Toronto; Stanley Kaplan, Cornell University; H. Jerome Keisler, California Institute of Technology; Stephen Lichtenbaum, Harvard University; and J. G. Petersen, Stanford University. Each of these will receive a prize of twenty dollars.

The following teams, named in alphabetical order, won honorable mention: California Institute of Technology, Pasadena, California, the members of the team being Luis Baez-Duarte, H. Jerome Keisler, and Charles J. Stone; College of the City of New York, New York, New York, the members of the team being Samuel Klein, Eugene Luks, and David Shelupsky; Rice Institute, Houston, Texas, the members of the team being David M. Dahm, Donald J. Deckard, and

Burton Randol; and Wesleyan University, Middletown, Connecticut, the members of the team being Benjamin Day, Justus Diller, and Robert T. Fisher.

Fourteen individuals were given honorable mention. The names, alphabetically arranged, are: Everett C. Dade, Harvard University; David M. Dahm, Rice Institute; Ronald Graham, University of Alaska; George Hannauer, Oberlin College; Paul B. Kantor, Columbia University; Samuel Klein, College of the City of New York; Alan Vernon Lemmon, Washington University; Dale Nelson, Knox College; Guillermo Owen, Fordham University; Donald Passman, Polytechnic Institute of Brooklyn; Philip James Peebles, University of Manitoba; Carl R. Riehm, University of Toronto; William Silvert, Brown University; Harold Nathaniel Ward, Swarthmore College.

A total of 539 individuals from 96 institutions entered the competition this year. Of this number 109 individuals and 3 institutions were unable to compete, due to various reasons. Therefore, a total of 430 undergraduates from 93 institutions actually took part in the competition.

The following is a list, in alphabetical order, of all colleges and universities which entered teams in the competition: Agricultural and Mechanical College of Texas, Alabama Polytechnic Institute, Arizona State College (Flagstaff), Arizona State College (Tucson), Blackburn College, Boston University, Brandeis University, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Case Institute of Technology, Central Michigan College, College of the Holy Cross, Columbia College, Cornell University, Dartmouth College, Fairleigh-Dickinson University, Fordham University, Geneva College, Georgia Institute of Technology, Harvard University, Incarnate Word College, Kenyon College, Knox College, Lebanon Valley College, Livingstone College, Massachusetts Institute of Technology, McGill University, Mississippi State College, New York University, Oberlin College, Polytechnic Institute of Brooklyn, Princeton University, Purdue University, Queen's University (Kingston, Ontario), Radcliffe College, Sacramento State College, Saint Francis Xavier University, Saint Martin's College, Saint Olaf College, San Jose State College, Southwestern Louisiana Institute, Stanford University, State College of Washington, Stevens Institute of Technology, Swarthmore College, The Cardinal Stritch College, The College of Saint Thomas, The College of the City of New York, The Rice Institute, Tufts University, University of Buffalo, University of British Columbia, University of California (Berkeley), University of California (Davis), University of California (Los Angeles), University of Detroit, University of Houston, University of Illinois, University of Kansas, University of Kentucky, University of Manitoba, University of Michigan, University of Minnesota, University of Notre Dame, University of Oregon, University of Rochester, University of Santa Clara, University of Toronto, Union College, United States Naval Academy, Wake Forest College, Wesleyan University, and Yale University.

The following colleges and universities, in alphabetical order, entered individual contestants only: Albertus Magnus College, Arizona State College (Tempe), Bowling Green State University, Central College, Duke University, Duquesne University, Florida Southern College, Grinnell College, Hope College, Kent State University, Marian College, Park College, Queens College (Flushing, N. Y.), Southeastern State College, The Ohio State University, University of Alaska, University of Ottawa, University of Washington, Ursinus College, Washington University, and West Virginia Wesleyan College.

The individual rankings of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

The problems given to those participating in the competition, together with a write-up of the solutions, will appear in a later issue of this MONTHLY.

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California

REMARK ON A RECENT NOTE ON LINEAR FORMS

P. T. BATEMAN, University of Illinois and Institute for Advanced Study

Suppose that a, d, and s are positive integers and that a and d are coprime. For given integral n consider the solvability in integers $x_0, x_1, \cdots, x_s$ of

$$(1) \qquad n = ax_0 + (a+d)x_1 + \cdots + (a+sd)x_s,\ x_0 \geqq 0, x_1 \geqq 0, \cdots, x_s \geqq 0.$$

Put $N = ga + (d-1)(a-1)$, where g is the smallest integer greater than $(a-2)/s$. Generalizing a classical result [2] for the case $s=1$ and a theorem of Alfred Brauer [1] for the case $d=1$, J. B. Roberts [3] has recently proved:

If $n > N$, *then* (1) *is solvable. If* $n = N-1$, *then* (1) *is not solvable.* (In short, $N-1$ is the largest integer for which (1) is not solvable.)

Although Roberts' proof is elementary, it is very involved. Here we present a much simpler proof.

Putting $y_i = \sum_{j=i}^{s} x_j$ for $i = 0, 1, \cdots, s$, we see that the solvability of (1) in integers $x_0, x_1, \cdots, x_s$ is equivalent to the solvability in integers $y_0, y_1, \cdots, y_s$ of

$$(2) \qquad n = ay_0 + d(y_1 + \cdots + y_s),\ y_0 \geqq y_1 \geqq \cdots \geqq y_s \geqq 0.$$

Now for a given y_0 the integers expressible in the form $y_1 + \cdots + y_s$ with $y_0 \geqq y_1 \geqq \cdots \geqq y_s$ are precisely the integers z such that $0 \leqq z \leqq sy_0$. Thus the solvability of (2) in integers $y_0, y_1, \cdots, y_s$ is equivalent to the solvability in integers y, z of

$$(3) \qquad n = ay + dz, \qquad 0 \leqq z \leqq sy.$$

Accordingly it suffices to discuss the solvability of (3).

First suppose $n \geqq N$. Since a and d are coprime there exists an integer z such that $dz \equiv n \pmod{a}$ and $0 \leqq z \leqq a-1$. Hence $n - dz = ay$, where y is an integer. Further

$$ay = n - dz \geqq n - d(a-1) \geqq N - d(a-1) = ga - (a-1) > (g-1)a.$$

Thus $y > g-1$; that is, $y \geqq g$. By assumption $sg > a-2$; that is, $sg \geqq a-1$. Consequently

$$sy \geqq sg \geqq a - 1 \geqq z.$$

Thus (3) is solvable and therefore (1), so that the first assertion of Roberts' theorem is proved.

Now let $n = N-1$. Suppose y and z are integers such that $n = ay + dz$ and

$z \geqq 0$. Since $N-1=a(g-1)+d(a-1)$, we have $z \equiv a-1 \pmod a$. Hence $z \geqq a-1$ and $y \leqq g-1$, so that

$$sy \leqq s(g-1) \leqq a-2 < a-1 \leqq z.$$

Thus (3) and (1) are not solvable in this case and Roberts' theorem is completely proved.

References

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THE MATRIX EQUATION $X^2-I=0$ OVER A FINITE FIELD*

JOHN H. HODGES, University of Buffalo

1. Introduction and notation. Let $GF(q)$ denote the finite field of order q. In this paper we consider the problem of determining the number $N=N\{x^2-1, m\}$ of $m\times m$ matrices X over $GF(q)$ such that $X^2-I=0$. The more general problem of determining the number $N\{E(x), m\}$ of $m\times m$ matrices X over $GF(q)$ which satisfy the equation $E(X)=0$, where $E(x)$ is a polynomial with scalar coefficients over $GF(q)$, will be treated in a later paper.

Except as indicated, Roman capitals $A, B, \cdots$ will denote square matrices of order m over $GF(q)$. I_t will denote the identity matrix of order t. We will need the following well-known formula for the number g_t of nonsingular matrices of order t over $GF(q)$:

$$g_t = q^{t^2}\prod_{i=1}^{t}(1-q^{-i}) = \prod_{i=0}^{t-1}(q^t-q^i). \tag{1.1}$$

We shall consider separately the two cases where q is odd and even in Section 2 and 3 respectively.

2. $X^2-I=0$ for q odd. Let $N=N_0$ for q odd. If X is an $m\times m$ matrix which satisfies the equation $X^2-I=0$, since the minimum polynomial of X must divide $x^2-1=(x-1)(x+1)$, the elementary divisors of $xI-X$ are of the form $x\pm 1$. If $xI-X$ has t elementary divisors of the form $x-1$ and $m-t$ of the form $x+1$, then it follows ([2] p. 241) that X is similar to $J_t=\operatorname{diag}(I_t, -I_{m-t})$ with $0\leqq t\leqq m$. Conversely, it is easily seen that if X is similar to some J_t, then $X^2-I=0$. Thus X is a solution of the equation if and only if it is similar to some J_t, $0\leqq t\leqq m$.

Let $S_0(m, t)$ be the number of distinct $m\times m$ matrices X which are similar to J_t. Since no matrix X is similar to more than one such J_t it follows that

* Invited address to the Upper New York State Section of the Mathematical Association of America at Skidmore College on May 4, 1957.

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$$N_0\{x^2 - 1, m\} = \sum_{t=0}^{m} S_0(m, t). \tag{2.1}$$

Now as P runs through all nonsingular matrices of order m, $P^{-1}J_tP$ runs through all $m\times m$ matrices X which are similar to J_t, each X appearing the same number of times, namely as many times as J_t itself appears. But if $P^{-1}J_tP=J_t$, then P is a nonsingular matrix which commutes with J_t. Therefore the number of times each matrix X which is similar to J_t appears as the value of the form $P^{-1}J_tP$ is the number $C_0(m, t)$ of nonsingular matrices P of order m which commute with J_t. Recalling that g_m is the number of nonsingular matrices of order m and that $S_0(m, t)$ is the number of $m\times m$ matrices X which are similar to J_t, it follows that we have

$$g_m = S_0(m, t)C_0(m, t) \qquad 0 \le t \le m. \tag{2.2}$$

It can easily be shown by a direct calculation that

$$C_0(m, t) = g_t g_{m-t}. \tag{2.3}$$

Then by substituting (2.3) into (2.2), solving for $S_0(m, t)$ and substituting into (2.1) we get

THEOREM 1. *The number of $m\times m$ matrices X over $GF(q)$, q odd, satisfying the equation $X^2-I=0$ is*

$$N_0\{x^2 - 1, m\} = g_m \sum_{t=0}^{m} \frac{1}{g_t g_{m-t}}, \tag{2.4}$$

where g_t is given by (1.1) *for $0<t\leqq m$ and $g_0=1$.*

Using the q-binomial coefficients defined by

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{(1 - q^m) \cdots (1 - q^{m-r+1})}{(1 - q) \cdots (1 - q^r)} \text{ for } 0 < r \leqq m, \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1, \tag{2.5}$$

we are able to give an interesting formula for $N_0\{x^2-1, m\}$. Substituting for g_m, g_t, g_{m-t} in (2.4) the explicit values given by (1.1) and simplifying gives

$$N_0\{x^2 - 1, m\} = \sum_{t=0}^{m} q^{2t(m-t)} \prod_{k=1}^{m-t} \frac{(1 - q^{-t-k})}{(1 - q^{-k})}, \tag{2.6}$$

where the value of the inner product is taken as 1 when $t=m$. For fixed t, $0\leqq t<m$, in view of the definition (2.5),

$$\prod_{k=1}^{m-t} \frac{(1 - q^{-t-k})}{(1 - q^{-k})} = q^{-t(m-t)} \prod_{k=1}^{m-t} \frac{(1 - q^{m+1-k})}{(1 - q^k)} = q^{-t(m-t)} \begin{bmatrix} m \\ m - t \end{bmatrix}. \tag{2.7}$$

Finally, substituting (2.7) into (2.6) we get

$$N_0\{x^2 - 1, m\} = \sum_{t=0}^{m} q^{t(m-t)} \begin{bmatrix} m \\ m - t \end{bmatrix}. \tag{2.8}$$

3. $X^2 - I = 0$ for q even. Let $N = N_e$ for q even. In this case, if X is an $m \times m$ matrix which satisfies the equation $X^2 - I = 0$, since the minimum polynomial of X must divide $x^2 - 1 = (x-1)^2$, the elementary divisors of $xI - X$ are of the form $x - 1$ or $(x-1)^2$. If $xI - X$ has t elementary divisors of the second form and $m - 2t$ of the first, then X is similar to $H_t = \operatorname{diag}(I_{m-2t}, E_1, \cdots, E_t)$ where $E_i = E$ for all i and E is the companion matrix of $(x-1)^2$. Thus we see that X is a solution of the equation if and only if it is similar to some H_t, $0 \leqq 2t \leqq m$.

Proceeding as in the first case, let $S_e(m, t)$ be the number of distinct $m \times m$ matrices X which are similar to H_t. Then we have

$$N_e\{x^2 - 1, m\} = \sum_{0 \leqq 2t \leqq m} S_e(m, t). \tag{3.1}$$

Also as before, if $C_e(m, t)$ is the number of nonsingular matrices of order m which commute with H_t, then

$$g_m = S_e(m, t) C_e(m, t). \tag{3.2}$$

Now, L. E. Dickson ([1] p. 235) has given a somewhat complicated formula for the number of nonsingular matrices of order m commuting with a given matrix A of order m, in terms of the degrees of the prime factors of the minimum polynomial of A and the number and exponents of the various powers of these prime factors which are elementary divisors of A. We will not take the space to state this formula here, but using it we find that the number of nonsingular matrices of order m which commute with H_t, for $0 \leqq 2t \leqq m$, is

$$C_e(m, t) = q^{t(2m-3t)} g_t g_{m-2t}, \tag{3.3}$$

where g_t is defined by (1.1) for $0 < t \leqq m$ and $g_0 = 1$. Therefore, using (3.3), (3.2) and (3.1), we get

THEOREM 2. *The number of $m \times m$ matrices X over $GF(q)$, q even, satisfying the equation $X^2 - I = 0$ is*

$$N_e\{x^2 - 1, m\} = g_m \sum_{0 \leqq 2t \leqq m} \frac{q^{-t(2m-3t)}}{g_t g_{m-2t}}, \tag{3.4}$$

where g_t is given by (1.1) *for $0 < t \leqq m$ and $g_0 = 1$.*

The author's investigations have not disclosed a representation for N_e corresponding to the representation of N_0 given by (2.8).

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AN ELEMENTARY PROOF OF THE ERDÖS-MORDELL THEOREM

LEON BANKOFF, Los Angeles, California

The Erdös-Mordell Theorem states that *if O is an arbitrary point within a triangle ABC and P, Q, R are the feet of the perpendiculars from O upon sides BC, CA, AB respectively, then* $OA+OB+OC \geqq 2(OP+OQ+OR)$.

A request by Paul Erdös for a proof of this theorem appeared as Problem 3739 of this MONTHLY (June–July 1935, p. 396), and almost two years later (April 1937, pp. 252–254) a brief trigonometric solution by L. J. Mordell was published, along with a considerably longer trigonometric solution by D. F. Barrow, who, in addition, showed that the equality sign holds only when triangle ABC is equilateral and O is the centroid of the triangle. A discussion of the theorem by L. F. Tóth in 1953 ([**1**] pp. 12–14, 28) induced the search for a simple and elementary proof of the theorem by the methods of synthetic geometry. Such a proof has recently been supplied by D. K. Kazarinoff [**2**], and this proof has been extended by N. D. Kazarinoff [**3**] to establish an analogous theorem about the tetrahedron. An appropriate analog for the theorem in Euclidean n-space, $n>3$, has not yet been formulated.*

The following synthetic proof of the Erdös-Mordell Theorem seems simpler and more elementary than that of D. K. Kazarinoff.

Let P_1P_2, Q_1Q_2, R_1R_2 denote the respective orthogonal projections of RQ, PR, QP on BC, CA, AB. Then

$$\begin{aligned}(1)\qquad OA + OB + OC \geqq\ & OA(P_1P + PP_2)/RQ + OB(Q_1Q + QQ_2)/RP \\ & + OC(R_1R + RR_2)/PQ.\end{aligned}$$

Because of the equality of angles ROB and RPP_1 in the cyclic quadrilateral $BPOR$, the right triangles PRP_1 and OBR are similar and $P_1P = RP\cdot OR/OB$. In a like manner we find $PP_2 = PQ\cdot OQ/OC$; $Q_1Q = PQ\cdot OP/OC$; $QQ_2 = RQ\cdot OR/OA$; $R_1R = RQ\cdot OQ/OA$ and $RR_2 = RP\cdot OP/OB$.

Substituting in (1) and grouping terms with respect to the factors OP, OQ, OR, we obtain

$$\begin{aligned}OA + OB + OC \geqq\ & OP\left(\frac{RP\cdot OC}{PQ\cdot OB} + \frac{PQ\cdot OB}{RP\cdot OC}\right) + OQ\left(\frac{PQ\cdot OA}{RQ\cdot OC} + \frac{RQ\cdot OC}{PQ\cdot OA}\right) \\ & + OR\left(\frac{RP\cdot OA}{RQ\cdot OB} + \frac{RQ\cdot OB}{RP\cdot OA}\right) \geqq 2(OP + OQ + OR).\end{aligned}$$

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* The references given in the second paragraph were furnished by the referee.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

TERSE TRIGONOMETRY

C. M. FULTON, University of California, Davis

This is an attempt at formulating a new approach to trigonometry. Its conciseness and rigor should make it suitable for good students. The basic idea is a truly general definition of the trigonometric functions.

Let $P(x, y)$, $x^2+y^2=1$, be on the terminal side of an angle θ and $P'(x', y')$, $x'^2+y'^2=1$, on its initial side. The angle with its vertex at the origin O is considered positive or negative according as the rotation involved is counterclockwise or clockwise. We define

$$\cos\theta = xx' + yy', \qquad \sin\theta = yx' - xy'. \tag{1}$$

Interchanging the sides of the angle it is seen that

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta. \tag{2}$$

The following identities are easily established:

$$(xx' + yy')^2 + (yx' - xy')^2 = 1, \tag{3}$$

$$(x - x')^2 + (y - y')^2 = 2 - 2(xx' + yy'). \tag{4}$$

The meaning of (3) is obvious. Identity (4), on the other hand, is crucial in that it shows the invariance of the definition of cosine: if the angle is rotated, the distance PP' remains unchanged and so does the cosine. Hence, because of (3) the same is true for the square of the sine. Moreover, a continuity argument will take care of the sine itself. It should be apparent to the reader that, in order to save space, we are using technical rather than pedagogical language in this discussion.

We now use the invariance demonstrated above and place the angle in standard position letting $x'=1$, $y'=0$. The definitions (1) reduce to

$$\cos\theta = x, \qquad \sin\theta = y. \tag{5}$$

Let us remind ourselves at this stage that the much-stressed trigonometric identities involving one angle only could all be written as algebraic identities by using (5). Returning to definitions (1) we denote by α and α' the angles in standard position whose terminal sides are OP and OP', respectively. Then θ and $\alpha-\alpha'$ differ by some integral multiple of 360° and with the aid of (5) we obtain the formulas for $\cos(\alpha-\alpha')$ and $\sin(\alpha-\alpha')$. On account of (2) the corresponding formulas for $\alpha+\alpha'$ are easily derived. Eventually, the so-called

reduction formulas follow as special cases. All these derivations are, of course, completely general.

Clearly, the definitions (5) can be changed to the usual form $\cos\theta = x/r$, $\sin\theta = y/r$ by taking a point P at a distance r from the origin. This leads directly to the solution of right triangles. Furthermore, identity (4) is generalized to

$$(x - x')^2 + (y - y')^2 = r^2 + r'^2 - 2rr' \cos\theta,$$

which is precisely the law of cosines. May we be allowed to suggest at this point that for practical purposes the latter be rewritten as a haversine formula, namely

$$c^2 = (a - b)^2 + 4ab \operatorname{hav} \gamma.$$

Finally, to find a trigonometric formula for the area of triangle $OP'P$, let $x' = r'$, $y' = 0$, $y > 0$. Then the area $K = \frac{1}{2}r'y = \frac{1}{2}r'r \sin\theta$. We can now prove the law of sines without difficulty, using the formula for K.

GENERAL SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

R. D. LARSSON, Clarkson College of Technology

Texts on ordinary differential equations define linearity and develop the properties of the linear differential equations, but then fail to use these properties to the maximum extent possible.

Consider the homogeneous differential equation with constant coefficients

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0.$$

Assuming a solution of the form $y = e^{mx}$ we obtain roots of the auxiliary equation $m = m_1, m_2$ and the general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ if $m_1 \neq m_2$.

Using the property that linear combinations of solutions are solutions,

$$y = \frac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2}$$

is a solution.

$$y = \lim_{m_1 \to m_2} \frac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2}$$

is a solution for $m_1 = m_2$.

Using L'Hôpital's Rule

$$y = \lim_{m_1 \to m_2} \frac{x e^{m_1 x}}{1} = x e^{m_2 x}$$

is a solution, and $y = c_1 x e^{m_2 x} + c_2 e^{m_2 x}$ is the general solution.

Consider Euler's differential equation $a_2 x^2 y'' + a_1 x y' + a_0 y = 0$. Assuming a solution of the form $y = x^m$ we obtain roots of the auxiliary equation as $m = m_1, m_2$ and the general solution is $y = c_1 x^{m_1} + c_2 x^{m_2}$ if $m_1 \neq m_2$.

Case II. $k_2 > k_1$. k_2 will give a particular solution. If k_2 differs from k_1 by an integer, k_1 may then give another solution, or $\lim_{k\to k_1} (k-k_1)F(x, k) = F(x, k_2)$.

$$y = \lim_{k\to k_1} \frac{(k - k_1)F(x, k) - F(x, k_2)}{k - k_1}$$

is a solution, or

$$y = \lim_{k\to k_1} \frac{\frac{\partial}{\partial k}[(k - k_1)F(x, k)]}{1},$$

and

$$y = c_1 \lim_{k\to k_1} \frac{\partial}{\partial k} [(k - k_1)F(x, k)] + c_2F(x, k_2).$$

Under Case II, for the solution of Bessel's differential equation, where $y = J_n(x)$ with n an integer and $J_n(x) = (-1)^n J_{-n}(x)$, we write

$$y_n(x) = \lim_{n\to\text{integer}} \frac{(\cos n\pi)J_n(x) - J_{-n}(x)}{\sin n\pi}$$

to obtain the solution of the second kind. This again can be evaluated by L'Hôpital's Rule.

My thesis is that if one can use the same property throughout the work it is easier for the student to follow, and to remember, the basic principles. Trick devices are interesting but do not lend themselves to an extension of the theory. The above method may be used when two particular solutions become identical in the limit and when the equation is homogeneous and linear.

ON THE SLOPES OF PERPENDICULAR LINES

S. LEADER, Rutgers University

The usual proof that the slopes of lines which meet at right angles are negative reciprocals is based upon a trigonometric identity. A geometric proof can be given using the theorem that the interior altitude of a right triangle is the geometric mean of the segments into which it divides the hypotenuse.

Let O be the intersection of the lines l_1 and l_2 which meet at right angles. We exclude the case in which either line is vertical by assuming both lines have slopes. Draw a horizontal segment OA extending one unit to the right of O. Draw a vertical line through A meeting l_1 at B_1 and l_2 at B_2. Then the slopes m_1 and m_2 are given respectively by the directed vertical segments AB_1 and AB_2. Since these segments have opposite directions, the product m_1m_2 is negative. Now OA is the interior altitude of the right triangle B_1OB_2. Therefore, $|OA|^2 = |AB_1|\,|AB_2|$. That is, $1 = |m_1m_2|$. Finally, since m_1m_2 is negative, $m_1m_2 = -1$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1326. *Proposed by P. L. Chessin, University of Maryland*

With n straight line segments of lengths $1, 2, 3, \cdots, n$, how many nondegenerate triangles can be constructed?

E 1327. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

A confirmed angle-watcher marked the angle α formed by the hands of a clock. Some time later he noticed that the hands trisected α. In how short a time could this have happened? How soon after 3:00 could one start such an experiment?

E 1328. *Proposed by Winton Laubach, Colorado School of Mines*

A ray from the origin intersects the circle $\rho=1$ at C and the spiral $\rho=e^{\theta}$, $\theta>0$, at S. Tangents to the circle at C and to the spiral at S intersect at P. Identify the locus of P.

E 1329. *Proposed by Jose Gallego-Diaz, Vanderbilt University*

A spherical square is a spherical quadrilateral whose four sides are equal and whose four angles are equal. If we let a, b, c denote the areas of the spherical squares constructed on the legs and hypotenuse of a right spherical triangle, show that

$$\tanh^{-1}(\sin c/4) = \tanh^{-1}(\sin b/4) + \tanh^{-1}(\sin a/4).$$

E 1330. *Proposed by J. L. Pietenpol, Columbia University*

Find the limit of the infinite product $\prod_{n=1}^{\infty}(1+1/a_n)$, where $a_1=1$, $a_n=n(a_{n-1}+1)$.

SOLUTIONS

A Generalization of E 1242

E 1296 [1958, 42]. *Proposed by Beckham Martin, Owens-Illinois Glass Co., Toledo, Ohio*

Problem E 1242 [1957, 433] says that the circle orthogonal to the circles (A'), (B'), (C') inscribed in the squares constructed exteriorly (or interiorly) on the sides of a triangle ABC is concentric with the nine point circle of triangle $A'B'C'$. Show, more generally, that all circles cutting the circles (A'), (B'), (C') under angles α, β, γ, respectively, where

$$\cos\alpha\sin A = \cos\beta\sin B = \cos\gamma\sin C,$$

are concentric with the nine point circle of triangle $A'B'C'$.

Solution by Victor Thébault, Tennie, Sarthe, France. Let a, b, c denote the sides BC, CA, AB of triangle ABC, R the circumradius of triangle ABC, N' the nine point center of triangle $A'B'C'$, (ω) the circle with center ω and radius r which cuts (A'), (B'), (C') under the angles α, β, γ such that

$$\cos\alpha\sin A = \cos\beta\sin B = \cos\gamma\sin C = k,$$

and M one of the points of intersection of (A') and (ω). By the law of cosines, applied to triangle $A'\omega M$, we have

$$(A'\omega)^2 = (\omega M)^2 + (A'M)^2 + 2(A'M)(\omega M)\cos\alpha,$$

which gives

$$(A'\omega)^2 - a^2/4 = r^2 + ar\cos\alpha = r^2 + 2Rr\cos\alpha\sin A = r^2 + 2kRr.$$

In this way we find

$$(A'\omega)^2 - a^2/4 = (B'\omega)^2 - b^2/4 = (C'\omega)^2 - c^2/4 = r^2 + 2kRr. \tag{1}$$

Now, if r' denotes the radius of the circle having center N' and orthogonal to (A'), (B'), (C'), we have

$$(A'N')^2 - a^2/4 = (B'N')^2 - b^2/4 = (C'N')^2 - c^2/4 = r'^2. \tag{2}$$

Subtracting (2) from (1) we find

$$(A'\omega)^2 - (A'N')^2 = (B'\omega)^2 - (B'N')^2 = (C'\omega)^2 - (C'N')^2,$$

from which it follows that ω coincides with N'.

Also solved by D. R. Brillinger and D. C. B. Marsh.

Concerning the Product of a Number by Its Reversal

E 1297 [1958, 42]. *Proposed by D. C. B. Marsh, Colorado School of Mines*

In Solution II of Problem E 1243 [1957, 434] occurs the following conjecture for proof or disproof: When an integer and its reversal are unequal, their product is never a square except when both are squares.

Show that for any $n>2$ there is a nonsymmetric, nonsquare, n digit integer whose product with its reversal is a square.

Solution by Joe Lipman, University of Toronto. If n is odd we may take the number as $2(10^{k-1}+2)^2$, where $k=(n+1)/2$; if n is even we may take the number as $11(10^{k-1}+2)^2$, where $k=n/2$.

Also solved by Merrill Barnebey, D. M. Brown, Michael Goldberg, Naoki Kimura, M. S. Klamkin, W. M. McKeeman, Otto Mond, C. S. Ogilvy, Benjamin Sapolsky, W. B. Stovall, Jr., and the proposer.

Editorial Note. Many solvers gave numbers with unorthodox reversals, such as $2(10^{2k})$ and $11(10^{2k})$.

An Infinite Nesting of Parentheses

E 1299 [1958, 43]. *Proposed by P. L. Chessin, Westinghouse Electric Corporation*

Find the limit of $1+x(1-x[1+x(1-x[1+\cdots])])$, for $|x|<1$.

I. *Solution by T. H. Slook, Temple University.* Removal of parentheses gives

$$1+x-x^2-x^3+x^4+x^5-\cdots,$$

which converges absolutely for $|x|<1$. We may then regroup terms to obtain

$$(1+x)(1-x^2+x^4-x^6+\cdots),$$

which, for $|x|<1$, represents $(1+x)/(1+x^2)$.

II. *Solution by Joe Lipman, University of Toronto.* The given series is absolutely convergent for $|x|<1$, therefore it has a limit L, which then must satisfy $L=1+x(1-xL)$. Hence $L=(1+x)/(1+x^2)$.

Also solved by J. L. Alperin, A. G. Anderson, Philip Bacon, Edward Barbeau, Merrill Barnebey, A. P. Boblétt, Julian Braun, D. A. Breault, D. R. Brillinger, E. W. Brown, C. N. Campopiano, A. E. Danese, T. W. Daniel, J. E. Darraugh, R. S. Dinsmore, E. S. Eby, S. J. Einhorn, E. L. Ellis and D. L. Muench (jointly), G. V. Emerson, G. W. Erwin, W. V. Gamzon, H. M. Gehman, Michael Goldberg, Ronald Gordon, R. E. Graf, Donald Greenstein, Emil Grosswald, J. W. Haake, R. H. Hou, A. R. Hyde, N. S. Kandalgaonkar, Seymour Kass, M. A. Kirchberg, M. S. Klamkin, J. D. E. Konhauser, Morton Kupperman, P. S. Landweber, R. L. London, W. M. McKeeman, D. C. B. Marsh, C. T. Molloy, Jr., Otto Mond, Morris Morduchow, J. B. Muskat, D. E. Myers, C. S. Ogilvy, F. D. Parker, D. J. Peterson, C. F. Pinzka, L. A. Ringenberg, Jeff Ritterman, D. A. Robinson, H. D. Ruderman, D. L. Schell, Francis Sevier, Paul Shaefer, Arnold Singer, D. L. Smith, O. E. Stanaitis, W. B. Stovall, Jr., Donato Teodoro, C. W. Trigg, Chih-yi Wang, Walter Wiebenson, B. G. Willis, Clement Winston, David Zeitlin, and the proposer.

A Class of Determinants

E 1300 [1958, 43]. *Proposed by D. A. Robinson, University of Wisconsin*

Let D_1, D_2, $D_3 \cdots$ be the determinants

$$\begin{vmatrix} 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{vmatrix}, \cdots,$$

respectively. Find the value of D_n for any positive integer n.

Solution by A. R. Erskine, Pennsylvania State University. D_n may be described as follows: The first row consists of $n+1$ ones followed by $n-2$ zeros. The next $n-2$ rows are the cyclic permutations of this row which introduce successively $1, 2, \cdots, n-2$ zeros on the left. The nth row consists of the integers from 1 to n followed by $n-1$ zeros. The remaining $n-1$ rows are the corresponding cyclic permutations of the nth row.

To evaluate D_n, from row $k+n-1$ subtract the sum of the rows indexed from k to $k+n-2$ inclusive, giving k successively the values $1, 2, \cdots, n$. This will produce a triangular matrix. The first n elements on the main diagonal will be 1; the last $n-1$ elements will be $n+1$. Hence D_n has the value $(n+1)^{n-1}$.

Also solved by A. G. Anderson, Edward Barbeau, Merrill Barnebey, W. J. Blundon, D. R. Brillinger, E. W. Brown, A. E. Danese, T. W. Daniel, J. W. Gammill, W. V. Gamzon, Michael Goldberg, Emil Grosswald, J. W. Haake, B. A. Hausmann, S.J., M. S. Klamkin, J. D. E. Konhauser, Morton Kupperman, Joe Lipman, D. C. B. Marsh, Otto Mond, C. S. Ogilvy, Jeff Ritterman, Benjamin Sapolsky, Nathan Schwid, O. E. Stanaitis, C. W. Trigg, L. H. Tulloch, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4798. *Proposed by John McCarthy, Dartmouth College*

Let $\alpha_1, \cdots, \alpha_n$ be algebraic numbers linearly independent over the rationals.

Show that there is a positive constant C and an integer N such that if $m_1, \cdots, m_n$ are rational integers not all zero then

$$|m_1\alpha_1 + \cdots + m_n\alpha_n| \geqq \frac{C}{\{|m_1| + \cdots + |m_n|\}^N}.$$

This generalizes Liouville's theorem on the approximation of algebraic numbers by rationals.

4799. *Proposed by K. L. Chung, Syracuse University*

Find a function continuous in $(0, 1]$ without any interval of constancy which takes on rational values almost everywhere.

4800. *Proposed by N. S. Mendelsohn, The University of Manitoba*

Let $\{u_n\}$ be a sequence with a recurrence formula of the form

$$u_{n+1} = a_0u_n + a_1u_{n-1} + \cdots + a_ku_{n-k},$$

where $a_0, \cdots, a_k$ are real constants. Show that the sequence obtained by taking every rth term of this sequence satisfies a recurrence of the same length; *i.e.*, if $v_n = u_{rn}$ for $n = 1, 2, \cdots$, there exist constants $A_0, \cdots, A_k$ such that for $n = k+1, k+2, \cdots$,

$$v_{n+1} = A_0v_n + A_1v_{n-1} + \cdots + A_kv_{n-k}.$$

4801. *Proposed by S. W. Golomb, Pasadena, California*

Let $R(n)$ denote the minimum number of terms required in representing n as a sum of squares. If n is restricted to perfect numbers, show that

$$R(n) = \begin{cases} 2 & \text{for odd } n, \text{ if any;} \\ 3 & \text{for } n = 6; \\ 4 & \text{for even } n \neq 6. \end{cases}$$

4802. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let $A = (a_{ij})$ be a real symmetric matrix of order $n(\geqq 3)$ and of rank $\leqq 3$. If A is positive semi-definite and if $a_{ii} = 1$ $(1 \leqq i \leqq n)$, then

$$\operatorname*{Max}_{i \neq j} a_{ij} \geqq \tfrac{1}{2} \operatorname{cosec}^2 \omega_n - 1,$$

where $\omega_n = n\pi/6(n-2)$.

4803. *Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Let $\sum a_nx^n$ be analytic at 0 and suppose $a_n > 0$, $a_{n+1}a_{n-1} > a_n^2$. Prove that the expansion $1/\sum a_nx^n$ has all negative coefficients (except for the constant term).

SOLUTIONS

Feuerbach Hyperbola Tangent to Circumcircle

4751 [1957, 509]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the Feuerbach hyperbola of a triangle ABC, with orthocenter H and incenter I, is tangent to the circle ABC, then it is also tangent to one of the circles BCH, CAH, ABH, and its focal axis is the perpendicular bisector of IH. (The Feuerbach hyperbola is the equilateral hyperbola determined by the four points, A, B, C, and I.)

Solution by the proposer. Since the Feuerbach hyperbola $\mathcal{H}$ passes through A, B, C, it must be tangent to circle ABC at one of these points, say A. Now, under the isogonal transformation set up by triangle ABC, an equilateral hyperbola corresponds to some line through the circumcenter O of ABC. It follows that $\mathcal{H}$, being an equilateral hyperbola passing through the fixed point I, must correspond to the line OI. Since the isogonal conjugate of the ray through A tangent to circle ABC at A is parallel to BC, it follows that OI is parallel to BC.

Under the isogonal transformation set up by triangle HBC, hyperbola $\mathcal{H}$, since it also passes through H, corresponds to a diametral line of circle HBC. This line must have the same orientation as OI, and hence is parallel to BC. It now follows that $\mathcal{H}$ is tangent to circle HBC at point H.

By elementary geometry it is easy to show that the tangent to circle HBC at H is parallel to the tangent to circle ABC at A. Hence A and H are diametrically opposite points on $\mathcal{H}$, and the center of $\mathcal{H}$ is the midpoint ω of AH. Since the center of $\mathcal{H}$ is the internal Feuerbach point of triangle ABC, we see that $\omega I=r$, the radius of the inscribed circle of triangle ABC. Also, if M is the midpoint of BC, $AH=2OM=2r$. It follows that $\omega I=\omega H$, and the focal axis of $\mathcal{H}$ is thus the perpendicular bisector of IH.

Also solved by J. W. Clawson, who used complex coordinates.

Editorial Note. The figure is rich. One may show, among other things, that AIH is a right triangle, $\mathcal{H}$ is tangent to OI at I, $OI=(b-c)/2$, $r=R\cos A$ (where R is the circumradius of triangle ABC), $8Rr=4R^2-(b-c)^2$, $\cos B+\cos C=1$. For an analytical study of the isogonal transformation, see, *e.g.*, D. M. Y. Sommerville, *Analytical Conics*, London, 1924.

Quadrics with Common Points

4754 [1957, 596]. *Proposed by T. G. Room, University of Sydney, Australia*

Given the four quadrics

$$L \equiv x^2 + dyz - cyt + bzt = 0, \qquad M \equiv y^2 + dzx + cxt - azt = 0,$$
$$N \equiv z^2 + dxy - bxt + ayt = 0, \qquad K \equiv t^2 - ayz - bzx - cxy = 0.$$

Show that if $1+a^3+b^3+c^3+d^3+3abcd=0$ then they have six points in common, and otherwise, none.

Solution by the proposer. If the four quadrics have six common points, then, included in the triply infinite (∞^3) system

$$\phi \equiv \alpha L + \beta M + \gamma N + \delta K = 0,$$

there is a net (linear ∞^2 system) of which the base is the unique twisted cubic which passes through the six points. Conversely, if there exists such a net, since ϕ is completely determined by this net and one other quadric, and this quadric meets the twisted cubic in six points, it follows that all the quadrics pass through these six points.

Write $\Delta = 1 + a^3 + b^3 + c^3 + d^3 + 3abcd$,

$$A = a^2 + bcd, \qquad B = b^2 + cad, \qquad C = c^2 + abd, \qquad D = d^2 + abc.$$

It is to be proved that the linear ∞^2 system ψ determined in ϕ by the parameters which satisfy

$$A\alpha + B\beta + C\gamma + D\delta = 0$$

consists, if $\Delta = 0$, of quadrics with a common twisted cubic, and the main theorem is then proved.

All quadrics of the pencil (linear ∞^1 system)

$$\alpha L + \beta M + \gamma N = 0, \qquad A\alpha + B\beta + C\gamma = 0$$

contain the line $x/a = y/b = z/c$, and meet therefore residually in a twisted cubic. This line meets a general quadric of ψ in the two points $(a, b, c, \pm d')$, where

$$d'^2 = d^2 + 4abc.$$

We find similarly for ψ altogether eight base points, namely

$$(a, b, c, \pm d'), \quad (\pm a', b, -c, d), \quad (-a, \pm b', c, d), \quad (a, -b, \pm c', d).$$

It is easily proved that in general no four of these points are coplanar, so that the base of the system ψ is either this set of 8 (associated) points, or a twisted cubic. None of these 8 points lies on any other quadric of ϕ, so that unless ψ has a base twisted cubic, the four original quadrics have no common points.

The system ψ will have a twisted cubic for base only if there is a cone of the system with its vertex at each base point of ψ, and it will be sufficient if we can find cones for two such points.

The point (a, b, c, d') is the vertex of a cone in the system ψ if

$$\begin{aligned}
\alpha 2a + \beta c(d + d') + \gamma b(d - d') &= 0\\
\alpha c(d - d') + \beta 2b + \gamma a(d + d') &= 0\\
\alpha b(d + d') + \beta a(d - d') + \gamma 2c &= 0\\
\delta 2d' &= 0\\
\alpha A + \beta B + \gamma C + \delta D &= 0.
\end{aligned}$$

The last equation is linearly dependent on the others (with $\delta = 0$ and multipliers $a, b, c, 0, -2$); the first four can be solved for $\alpha, \beta, \gamma, \delta$, if

$$\begin{vmatrix} 2a & c(d+d') & b(d-d') \\ c(d-d') & 2b & a(d+d') \\ b(d+d') & a(d-d') & 2c \end{vmatrix} = 0,$$

i.e., if $abc\Delta = 0$.

In the same way we find that each of the 8 points is the vertex of a cone in the system ψ if $\Delta = 0$. Two of these cones meet, apart from the line joining their vertices, in a twisted cubic which passes through all 8 points, and which therefore lies on all quadrics of ψ.

A Permutation Problem

4755 [1957, 596]. *Proposed by Chandler Davis, Institute for Advanced Study*

In how many ways can the first n positive integers be arranged in alternately increasing and decreasing order? That is, how many permutations π: $\pi(1), \cdots, \pi(n)$ are there such that the quantities $(-1)^k\{\pi(k+1)-\pi(k)\}$, for $k=1, \cdots, n-1$ have all the same sign?

Solution by W. J. Blundon, Memorial University of Newfoundland. Let P_n be the required number of arrangements of the first n positive integers, under the restriction that the common sign of the stated quantities is negative. The integer n, being the largest in the set, is necessarily of the form $\pi(2i)$, $i=1, 2, \cdots, [n/2]$. The integers to the left of n can be chosen in $\binom{n-1}{2i-1}$ ways, and each such selection can be arranged in P_{2i-1} ways. The integers to the right of n can be arranged in P_{n-2i} ways. Hence

$$P_n = \sum_{i=1}^{[n/2]} \binom{n-1}{2i-1} P_{2i-1}P_{n-2i}, \qquad n = 1, 2, \cdots,$$

where, for convenience, we define $P_0 = 1$. Putting $P_n = n!Q_n$, we have

$$Q_0 = 1; \qquad nQ_n = \sum_{i=1}^{[n/2]} Q_{2i-1}Q_{n-2i}, \qquad n = 1, 2, \cdots.$$

Define $f(x) = \sum_{n=0}^{\infty} Q_n x^n$. Then it is easily verified that

$$\{f(x)\}^2 = -1 + 2f'(x).$$

The solution of this differential equation gives

$$\sec x + \tan x = f(x) = \sum_{n=0}^{\infty} P_n x^n/n!.$$

From the well-known expansions of sec x and tan x, we have

$$P_n = \begin{cases} B_n & n \text{ even}, \\ \dfrac{2^{n+1}(2^{n+1}-1)}{n+1} B_n & n \text{ odd}, \end{cases}$$

where the B's are alternately Bernoulli numbers and Euler numbers $(1/6, 1, 1/30, 5, \cdots,)$.

We now remove the restriction of the first sentence of this solution. Then, by symmetry, the required number of arrangements is $2P_n$, (except when $n=1$, when the restriction has no meaning). The number of arrangements for small n is given by

n	1	2	3	4	5	6	7	8	9	10
$2P_n$	1	2	4	10	32	122	544	2770	15872	101042

Also solved by W. H. Furry, E. C. Milner, L. E. Clarke, A. van Heemert, and the proposer.

Editorial Note. Using known expansions for the tangent and secant, Furry gets the result in the form

$$P_n = 2(2/\pi)^{n+1}n!\{1 + (-3)^{-n-1} + 5^{-n-1} + (-7)^{-n-1} + \cdots\}.$$

For large n the convergence is rapid, and the first term provides a good asymptotic expression for P_n.

The proposer notes an obvious connection with his problem no. 4714 [1957, 679–680]. He notes also that, because of the uniqueness of the Taylor expansion, P_n is the nth derivative of $\sec x + \tan x$, evaluated at $x=0$.

Real Functions

4756 [1956, 596]. *Proposed by J. L. Massera, Institute of Mathematics and Statistics, Montevideo, Uruguay*

Let $p(x_1, \cdots, x_n)$, $q(x_1, \cdots, x_n)$ be two real functions of n real variables x_i, defined and continuous in a parallelotope R: $0 \leqq x_i \leqq a_i < \infty$. Assume that $p(x_1, \cdots) = q(x_1, \cdots) = 0$ whenever $x_1x_2 \cdots x_n = 0$, and that $p(x_1, \cdots) > 0$, $q(x_1, \cdots) \geqq 0$, when $x_1x_2 \cdots x_n \neq 0$. Prove that there exists a real function $h(u)$ of a real variable u, defined, continuous and strictly increasing for $u \geqq 0$, $h(0)=0$, such that throughout R

$$h\{q(x_1, \cdots)\} < p(x_1, \cdots).$$

Solution by Neill McShane, Yale University. Let $\mathbf{x} = (x_1, \cdots, x_n)$. Since R is a compact domain, the continuous function $r(\mathbf{x})$ defined implicitly by $q(\mathbf{x}) = r(\mathbf{x})p(\mathbf{x})$ is bounded above uniformly by some number r_0. Assume $q(\mathbf{x})$ not identically 0, since otherwise the problem is trivial. Then $r_0 > 0$. $h(u) = u/2r_0$ satisfies the demands of the problem, for

$$h(q(\mathbf{x})) = q(\mathbf{x})/2r_0 = r(\mathbf{x})p(\mathbf{x})/2r_0 < p(\mathbf{x}).$$

Also solved by J. Horváth and by the proposer.

Identity Related to the Beta- Function

4757 [1957, 596]. *Proposed by O. P. Aggarwal, University of Washington*

Prove for every integer $n \geqq 0$, and for any positive c,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(c+k)^2}=\frac{\Gamma(c)\Gamma(n+1)}{\Gamma(c+n+1)}\sum_{k=0}^{n}\frac{1}{c+k}\,.$$

Solution by Tien Chi Chen, IBM Research Center, Ossining, N. Y. Both sides of the equation are equal to $-d\beta(c, n+1)/dc$. This is because

$$\beta(c,n+1)=\int_0^1 t^{c-1}(1-t)^n dt=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{c+k},$$

by expanding $(1-t)^n$ and integrating. Differentiating with respect to c and changing sign, we have the left-hand side of the equation. For the right-hand side we obtain

$$\beta(c,n+1)=\frac{n!}{c(c+1)\cdots(c+n)},$$

$$-\frac{d}{dc}\beta(c,n+1)=\frac{n!}{c(c+1)\cdots(c+n)}\sum_{k=0}^{n}\frac{1}{c+k}\,.$$

Also solved by R. G. Buschman, L. Carlitz, J. E. Darraugh, Emil Grosswald, Irwin Guttman, Peter Henrici, R. D. James, M. S. Klamkin, Margaret M. LaSalle, D. C. B. Marsh, F. D. Parker, A. K. Rajagopal, B. E. Rhoades, D. A. Robinson, S. C. Saunders, E. M. Scheuer, Blagovest Sendov, R. E. Simmons, Chih-yi Wang, Morgan Ward, David Zeitlin, and the proposer.

Editorial Note. As noted by James, a sequence of similar formulas may be derived by comparing higher derivatives of $\beta(c, n+1)$ or by noting, for example, that $\beta(c, n+1)-\beta(c+1, n+1)=\beta(c, n+2)$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

SELECTED MATHEMATICS BOOKS FOR HIGH SCHOOL LIBRARIES

The National High School and Junior College Mathematics Club, Mu Alpha Theta, which is sponsored by the Mathematical Association of America, has prepared a list, containing five sections of about twenty-five dollars valuation each, of selected mathematics books for high school libraries. This list will provide a well-balanced mathematics library on the high school and junior college level. The list is selective rather than extensive. Copies may be secured by sending a self-addressed, stamped envelope to Richard V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, requesting the high school book list.

Reason and Chance in Scientific Discovery. By R. Taton, Translated by A. J. Pomerans. Philosophical Library, New York, 1957. xii+171 pp. $10.00.

Korzybski in his general semantics defines plants as energy binders, animals as space binders, and man as a time binder. One is tempted to extend this epigrammatic definition to that rare subspecies of Homo sapiens, the geniuses, as idea binders. Taton's book is a survey of the subtlest form of idea-binding: invention and discovery.

What leads to an invention? Can we discover the laws of discovery? These are much-mooted questions, and the answers vary widely. Some writers maintain that "the act of discovery is purely an accident and owes nothing to reason or logic" (Nicolle). This view is rejected by Taton. Especially with regard to mathematical discovery he follows Poincaré and Hadamard in assuming the existence of two more-or-less distinct types of mind: the logical and the intuitional, or the analyst and the geometrician. The former proceeds by sustained and methodical effort, while the latter, as described vividly in Poincaré's self-analysis, may arrive at his results in three stages: first, conscious but often fruitless *concentration* ("stirring of ideas"), followed by a period of rest and *incubation* (subconscious processing of ideas), which stage may end happily by sudden *illumination* and insight, requiring merely verification for its completion.

In the observational and experimental sciences, however, the role of chance in discovery cannot be denied, but even here Taton holds with Pasteur that chance helps those whose minds are well prepared for it. Another point that he develops and illustrates convincingly by historical examples is the fact that a discovery must be attributed not only to its author, but also to its epoch. Scientific discoveries usually appear at the right time and proper background. Premature discoveries are not appreciated by their contemporaries, and even their authors are frequently unable to perceive their full significance. Still another fact: error has occasionally played a paradoxically fruitful role in discovery.

Taton concludes his compact and erudite survey with a word of warning and apprehension regarding the increasingly collective character of modern research. While the advantages of this development are unquestionable, the dangers of putting the accent on the immediately profitable and of reducing individual initiative are undeniable.

The book is handsomely illustrated and the translation reads smoothly, disregarding an occasional solecism like "this phenomena."

PINCUS SCHUB
University of Pennsylvania

An Analytical Calculus. Vol. IV. By E. A. Maxwell. Cambridge University Press, New York, 1957. ix+288 pp. $4.00.

This is the final volume in a series of classroom textbooks by the author covering the calculus and early stages of analysis. The volume under review is divided into three sections: 1) Ordinary differential equations, 2) Functions defined by infinite series and integrals, and 3) La Place and related equations.

The theme of the book is set forth in the first section and it is directed towards enabling the student to develop skill and confidence in solving differential equations. Among the assets of the book are 1) its spirit of liveliness and intellectual honesty, 2) selection of topics, 3) careful statements of the conditions under which the techniques used are correct, 4) numerous examples worked out in detail to illustrate when the techniques succeed or fail, and 5) an abundance of exercises for the student. His proof of theorems, when given, are well within the understanding of the calculus student. The emphasis on the first section is on the nth order differential equation with constant coefficients; in the second section, which is the largest, on a) handling of series including termwise integration and differentiation, b) improper integrals and the differentiation and integration of functions defined by integrals, and c) an introduction to Fourier series; the third section contains an excellent introductory treatment of the La Place, heat, and wave equations, Jacobians, and spherical harmonies. Special functions such as the Bessel, Beta, Gamma, and Legendre functions are discussed. The book is distinctive and well written.

PASQUALE PORCELLI
Illinois Institute of Technology

The Fascination of Numbers. By W. J. Reichmann. Essential Books, Fair Lawn, New Jersey, 1957. 176 pp. $4.00.

This work is intended for the general reader, and purports to show in a simple way the behavior and structure of numbers and their relation to each other.

Many of the recreational aspects of numbers are discussed, magic squares, puzzles, Russian multiplication and so on. Many of the famous problems accessible to the lay reader are presented, and simply, too.

But the main intent, it seems, is to lay a foundation for serious thought and study on numbers. In this the author has been successful, indeed. Unlike the writer of a text, he has taken time to write out sequences of numbers for his discussion so that the lay reader or curious young student can watch the development of the theory. This approach suggests a method of investigation to the nonprofessional. Many things are proved, but the author leaves much that is not. Still, the numerous specific examples suggest proofs which a student of some maturity can devise, or find.

A good short treatment of the positional representation is followed by two chapters on arithmetic series, which together with the chapter on primes would alone justify the book for high school students. There is a fine discussion on perfect numbers which includes the fact that the known perfect numbers are triangular numbers. The chapter on irrational numbers develops into a gratifying elementary discussion of continued fractions. There are interesting appendices, the last being an introduction of the geometric series as a tool of additive number theory.

Perhaps the strongest contribution the book makes is to demonstrate the

fundamental and far-reaching character of the arithmetic series. One wishes that an equal treatment had been given the geometric series at, say, the expense of magic squares. Even so, this little book should be very stimulating to high school and elementary mathematics teachers who are trying to revitalize their students' interest in numbers.

ED WALTERS
Wm. Penn High School

An Introduction to Probability Theory and Its Applications, Vol. I, 2nd Ed. By William Feller. Wiley, New York, 1957. xv+461 pp. $10.75.

The first edition of this book (Wiley, 1950, reviewed in this MONTHLY, vol. 59, 1952, p. 265) is so well-known that the only pertinent topic for the present review is the extent of the changes.

Feller has long been interested in the probable vagaries of a single sequence of trials as contrasted with the mean behavior of an aggregate of sequences. In this connection he has inserted a new chapter (Chapter III) in which he gives a very elegant, yet elementary, derivation of some rather startling theorems on coin tossing.

The major change is that the theory of recurrent events has been pushed forward so as to permeate the entire book, whereas in the first edition serious consideration of this theory began in Chapter 12. The result of this revision is a remarkable improvement in organization.

The preface to the second edition mentions "space saved by streamlining," but it must be emphasized that the streamlining is in organization, not in exposition. Explanations and examples have been expanded so that what was already an outstanding bit of exposition has been noticeably improved. Type has been completely reset so that these improvements occur on nearly every page. Kudos to Wiley for agreeing to scrap so many costly plates! The result is well worth the increase in price.

One big disappointment to the reviewer: failure to distinguish between a function and one of its values makes the introductory discussion of random variables confusing. Instead of clearing this up, Feller has deleted the one sentence in the first edition that hinted at the explanation. Had he only chosen to expand this now-missing sentence to two pages in his usual expository style, it would have been a real service to the teaching of mathematics in general.

M. E. MUNROE
University of Illinois

BRIEF MENTION

A Freshman Honors Course in Calculus and Analytic Geometry. By Emil Artin. Committee on the Undergraduate Program, Mathematical Association of America, 1957. 126 pp. Free.

The Committee on the Undergraduate Program of the Mathematical Association of America has prepared this edition of Seligman's notes on Artin's

course which is taught to the upper ten or fifteen per cent of the freshmen students enrolled in mathematics at Princeton University. It should certainly be on the active bookshelf of every mathematician interested in teaching gifted students. The Committee on the Undergraduate Program deserves the sincere thanks of mathematicians everywhere for making these notes available without charge. Copies may be obtained by writing directly to Professor H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Logical Design of Digital Computers. By Montgomery Phister, Jr. Wiley, New York, 1958. xvi+408 pp. $10.50.

This is not a mathematical book even though it contains two chapters on Boolean algebra, including the use of Veitch diagrams to simplify Boolean polynomials. It provides an excellent illustration of the extreme importance of modern abstract algebra in the engineering worlds of today and tomorrow.

Mechanical Resolution of Linguistic Problems. By Andrew D. Booth, L. Brandwood, and J. P. Cleave. Academic Press, New York, 1958. vii+306 pp. $9.80.

The results of the Birkbeck College Computational Laboratory on the application of digital computers to language translation problems are presented without reference to detailed programming. This book belongs in the library of every computing center as well as in departments of modern language.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor F. L. Wolf, Carleton College, represented the Association at the Observance of the Centennial of the Founding of Shattuck School at Faribault, Minnesota, on Friday, June 6, 1958.

Professor R. E. Wheeler, Head of the Department of Mathematics of Howard College, represented the Association at the inauguration of Henry King Stanford as President of Birmingham-Southern College on Friday, April 11, 1958.

Professor Morris Kline, New York University, has received a Fulbright award to lecture at Technische Hochschule in Aachen, Germany, during the academic year 1958–59.

Alabama Polytechnic Institute: Mr. P. W. Lindsey, Jr., has been promoted to Assistant Professor; Mr. S. M. Lukawecki has been appointed Instructor.

University of Detroit: Assistant Professor B. V. Ritchie has been promoted to Associate Professor; Mrs. Natalie Frazis, Mr. J. F. Lanahan, Mr. M. A. Laframboise, Mr. J. A. Mansour, and Mr. J. G. Sowul, have been promoted to Assistant Professors; Professor L. E. Mehlenbacher was a Visiting Professor and Assistant Director of the Summer Institute for High School Teachers of Science and Mathematics at the University of North Dakota during the summer of 1958.

Emory University: Associate Professor Trevor Evans has been promoted to Professor; Assistant Professor Henry Sharp, Jr., has been promoted to Associate Professor; Mr. B. K. Youse has been promoted to Assistant Professor; Mr. G. R. Flowers and Mrs. Peggie W. Wiegand have been appointed Instructors; Mr. F. L. Hardy begins a leave of absence in September in order to continue his graduate training.

University of Hawaii: Associate Professor S. B. Townes has been promoted to Professor; Assistant Professor Paola Comba has been promoted to Associate Professor; Captain J. H. Spiller, USN (retired), and Mr. I. C. Young, University of Colorado, have been appointed Assistant Professors.

Hobart and William Smith Colleges: Associate Professor R. L. Beinert has been promoted to Professor; Assistant Professor Abigail M. Mosey has been promoted to Associate Professor.

Illinois Institute of Technology: Associate Professor L. R. Wilcox has been promoted to Professor; Assistant Professors Pasquale Porcelli and R. J. Silverman have been promoted to Associate Professors; Dr. M. A. McKiernan and Dr. H. L. Pearson have been promoted to Assistant Professors.

Institute for Advanced Study, 1958–59: Dr. E. H. Batho, on leave from the University of Rochester, NSF grant; Associate Professor Alice T. Schafer, on leave from Connecticut College, NSF Science Faculty fellowship; Professor R. D. Schafer, on leave from the University of Connecticut, NSF Senior Postdoctoral fellowship.

Massachusetts Institute of Technology: Professor N. C. Ankeny was awarded a Guggenheim fellowship and will be on leave 1958–59 at Cambridge University; Professor G. W. Whitehead will be on leave as a visiting professor at Princeton University; Assistant Professor I. M. Singer has been promoted to Associate Professor; Assistant Professor J. K. Moser, New York University, visiting assistant professor during the current year, has been appointed Associate Professor; Dr. G. E. Backus, and Dr. F. P. Peterson, Princeton University, have been appointed Assistant Professors; Dr. A. P. Mattuck, Lecturer, who is spending the current semester in England, has been promoted to Assistant Professor; Doctors R. M. Peterson, New York University, E. R. Rodemich, Stanford University, and J. W. Smith, Columbia University, have been appointed C. L. E. Moore Instructors; Dr. A. W. Adler, Princeton University, has been appointed Lecturer; Dr. Shoshichi Kobayashi, Institute for Advanced Study, has been appointed Research Associate; Dr. James Yeh, University of Minnesota, has been appointed Instructor; Dr. R. J. Gribben, University of Manchester, England, and Dr. Azriel Levy, Hebrew University, have been awarded Sloan Foreign Post-doctoral fellowships in the School for Advanced Study at M.I.T.

St. Louis University: Associate Professor John Elder has been promoted to Professor.

University of Wisconsin, Mathematics Research Center, U. S. Army: Professor E. H. Zarantonello has returned to his post at the University of Cuyo, Mendoza, Argentina; Professor Zdenek Kopal is on leave, and has resumed his post at the University of Manchester, England; Professors H. F. Weinberger, University of Maryland, H. M. Schaerf, Washington University, and Fritz Oberhettinger, American University and National Bureau of Standards, are visiting members on leave from their Universities.

Dr. A. G. Anderson, General Tire & Rubber Co., has been appointed Professor and Head of Department, Western Kentucky State College.

Associate Professor D. F. Atkins, University of Richmond, has been appointed Assistant Professor at Eastern Illinois University.

Dr. J. L. Bailey, Michigan State University, has been appointed Assistant Professor at Case Institute of Technology.

Dr. Jeremiah Certaine, Howard University, has accepted the position of Manager, Department of Mathematics, Nuclear Development Corporation of America, White Plains, New York.

Mr. Willard Draisin, Lincoln Laboratories, has been appointed Instructor at Connecticut College.

Professor M. M. Day has been appointed Head, Department of Mathematics, at the University of Illinois, to succeed Professor S. S. Cairns.

Mr. E. S. Eby, University of Illinois, has accepted a position as Mathematician with U. S. N. Underwater Sound Laboratory, New London, Connecticut.

Mrs. Pauline P. Edwards, Los Alamos Scientific Laboratory, is now a Mathematician at the Aeronautical Chart & Information Center, St. Louis, Missouri.

Mr. J. R. Eno, Jr., University of Idaho, has been appointed Instructor at Whittier College.

Dr. R. E. Fagen, Bell Telephone Laboratories, has accepted a position as a member of the technical staff at Hughes Aircraft Company, Culver City, California.

Dr. H. H. Goldstine, Institute for Advanced Study, has been appointed to the staff of the I. B. M. Research Center, Yorktown, New York.

Mr. R. M. Gordon, Electro Data Division, Burroughs Corporation, Pasadena, California, has been promoted to Manager, Publications and Training.

Mr. R. J. Graham, Eglin AFB, Florida, is now a Mathematician with ABMA, Redstone Arsenal, Huntsville, Alabama.

Assistant Professor Bernard Greenspan, Drew University, has been promoted to Associate Professor. As a recipient of a National Science Foundation Faculty fellowship, he will be on leave at the University of California, Berkeley, 1958–59.

Mr. J. W. Haake, Armour Research Foundation, is now a Research Engineer with Convair, San Diego, California.

Mr. D. I. Hammer, Montclair State Teachers College, has been appointed Assistant Professor at Adelphi College.

Miss Virginia S. Hanly, Ohio State University, is now a Research Engineer at North American Aviation, Columbus, Ohio.

Professor F. F. Helton, Central College, Fayette, Missouri, has been appointed Chairman of the Division of Mathematics and Natural Sciences.

Dr. H. G. Hertz, U. S. Naval Observatory, is now with the Army Map Service, Washington, D. C.

Dr. T. P. Higgins, Boeing Airplane Company, is now a Senior Scientist with Dalmo-Victor, Belmont, California.

Dr. John Hilzman, Oregon State College, has been appointed Assistant Professor at Harpur College.

Professor J. H. Hodges, University of Buffalo, has accepted a position as Mathematician with Cornell Aeronautical Laboratories, Buffalo, New York.

Mr. Edgar Karst, International Business Machines Corporation, Endicott, N. Y., has been appointed Assistant Professor at Brigham Young University.

Dr. L. D. Kovach has been appointed Professor and acting head of the Department of Mathematics and Physics at George Pepperdine College. He will retain his position as design specialist at the Douglas Aircraft Company, El Segundo, California.

Dr. Joseph Lehner, Visiting Professor at Michigan State University, has been appointed Professor.

Dr. M. E. Levenson, Brooklyn College, has been promoted to Assistant Professor.

Dr. Eugene Levin, Ramo-Wooldridge Corporation, is now a Mathematician with the RAND Corporation, Santa Monica, California.

Professor G. G. Lorentz, Wayne State University, was Visiting Professor at the

University of Tübingen, Germany, during the second part of the Summer semester of 1958.

Mr. F. L. Lynch, Jr., Madison High School, Madison, New Jersey, is now Associate Professor at Seton Hall University.

Mrs. Matilde C. Macagno, formerly Professor on leave, University of Cuyo, San Juan, Argentina, is now Research Associate, Iowa Institute for Hydraulic Research, State University of Iowa.

Professor K. O. May, Carleton College, Northfield, Minnesota, will be on sabbatical leave in western Europe during the first semester of the academic year 1958–59.

Mr. R. W. McChesney, University of Rochester, has been appointed Instructor at Albion College, Michigan.

Mr. M. S. Moheban, Montana State University, is now a teacher at Park Senior High School, St. Paul, Minnesota.

Mr. M. G. Ossesia, University of Pittsburgh, has been appointed Instructor at Duquesne University.

Mr. G. P. Paternoster, Roosevelt University, is now Structural Detailer and Checker with Johnson & Johnson Engineers and Architects, Chicago, Illinois.

Dr. Sara L. Ripy, Vassar College, has been appointed Assistant Professor at Agnes Scott College.

Mr. Jack Roseman, D. I. C. Staff, Massachusetts Institute of Technology, has been appointed Instructor at the University of Massachusetts.

Mr. S. J. Scott, Vitro Corporation of America, is now Operations Analyst with North American Air Defense Command, Colorado Springs, Colorado.

Mr. R. T. Seeley, Massachusetts Institute of Technology, has been appointed Instructor at Harvey Mudd College.

Assistant Professor W. A. Small, Grinnell College, has been appointed Chairman of the Department of Mathematics. He was also Visiting Professor for the 1958 summer session at Nebraska State Teachers College, Chadron.

Mr. Irwin Stoner, Cornell Aeronautical Laboratories, is now Assistant Section Head, Raytheon Mfg. Co., Bedford, Massachusetts.

Professor D. D. Strebe, Teachers College at Oswego, State University of New York, has been appointed Associate Professor at the University of South Carolina.

Dr. Walter Bartky, University of Chicago, died March 19, 1958. He had been a member of the Association for thirty-one years.

Professor Daniel Block, Yeshiva University, died February 28, 1958. He had been a member of the Association for ten years.

Mr. W. M. Bullit, Louisville, Kentucky, died on October 3, 1957. He had been a member of the Association for thirty-two years.

Associate Professor Emeritus C. F. Lewis, Kansas State College, died on April 4, 1958. He had been a member of the Association for twenty-six years prior to his retirement.

Mr. P. W. A. Raine, Newport News High School, died October 14, 1957. He had been a member of the Association for eighteen years.

Professor Emeritus G. E. Ramsdell, Bates College, died February 2, 1958. He was a charter member of the Association.

Professor S. J. Smith, State Teachers College, Lock Haven, Pennsylvania, died January 21, 1958. He had been a member of the Association for twenty-seven years.

Dr. Marion B. White of Pasadena, California, died January 30, 1958. She was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1958 by a mail vote of the membership of the Association in the Sections indicated:

Kansas	R. G. Smith, Kansas State College, Pittsburg
Missouri	W. R. Utz, Jr., University of Missouri
New Jersey	William Feller, Princeton University
Northeastern	F. M. Stewart, Brown University
Ohio	G. M. Merriman, University of Cincinnati
Pacific Northwest	A. T. Lonseth, Oregon State College
Southeastern	G. B. Huff, University of Georgia
Southwestern	Charles Wexler, Arizona State College at Tempe
Upper New York State	H. S. M. Coxeter, University of Toronto

The highest percentage of votes cast was in the Kansas Section, where votes were received from more than 52% of the membership.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 198 persons have been elected to membership by the Board of Governors on applications duly certified.

ROBERT D. ADAMS, B.S. (Northwestern) Teaching Asst., University of Minnesota.

JOHN J. AEBERLY, C. E. (Illinois Tech.) Chief, Bureau of Heating, Ventilation and Industrial Sanitation, Chicago, Illinois.

WALTER A. ALBRECHT, JR., Ph.D. (Ohio S. U.) Asso. Professor, Long Beach State College, California.

GARY E. ANDERSON, Student, Abilene Christian College.

MARYSE BADER, B.A. (California) Instr., Seattle University; Student, University of Washington.

MERVIN R. BARNES, JR., Student, University of California, Riverside.

MARGARET M. BASKERVILL, Ph.D. (Alabama Poly.) Professor, Shorter College.

CHARLES BATCHLOR, M.S. (Howard) Mathematician, U. S. Army Map Service, Washington, D. C.

JOHN E. BEAM, Student, University of Kansas.

ROBERT C. BELSCAMPER, M.S. (Iowa S.C.) Mathematician, Remington Rand Univac, St. Paul, Minnesota.

PHILLIP R. BENDER, B.S. (Purdue) Asst. Professor, Milwaukee School of Engineering, Wisconsin.

WILLIAM H. BENSON, M.S. (M.I.T.) Private, United States Army, Ft. Lee, Virginia.

PAULINE J. BIGGS, Student, University of Oklahoma.

ROBERT L. BLEFKO, M.A. (Penn. S.U.) Instr., Pennsylvania State University.

WILLIAM F. BLOSE, Student, Oklahoma State University.

SAMUEL B. BLUMERT, Student, Cooper Union.

JOHN M. BOSSERT, Student, University of California, Los Angeles.

GARY T. BOSWELL, Student, Texas Christian University.

GERALD A. BOTTORFF, Student, Seneca High School, Louisville, Kentucky.

BEN B. BOWEN, A.B. (U.C.L.A.) Teacher, Vallejo Junior College, California.

JOAN E. BRENIZER, M.A. (Texas) Instr. Lamar State College of Technology.

MRS. MARY L. BRIGHAM, M.A. (Baylor) Instr., Baylor University.

ARCHIE D. BROCK, B.S. (Northwestern S.C., Oklahoma) Grad. Student, University of Oklahoma.

ROBERT R. BROWN, M.A. (Southern California) Manager, Special Projects, I.B.M. Beverly Hills, California.

WILLIAM J. CAHILL, M.A. (Harvard) Senior Engineer, Boeing Airplane Co., Patrick A.F.B., Florida.

MELVAN D. CARTER, Student, University of Oklahoma.

NED Y. H. CHANG, Student, Washington & Jefferson College.

BRUCE L. CHILTON, Student, University of Buffalo.

PETER J. CHRISLER, M.A. (California) Teacher, Sacramento Junior College.

LESLIE C. COLLINS, A.F.R. Aero S. (Royal Aero. Soc.) Design Engineer, Sikorsky Aircraft, Bridgeport, Connecticut.

PETER S. COOK, B.S. (Nebraska S.T.C., Chadron) Asst. Professor, Scottsbluff College.

RICHARD W. COTTLE, B.A. (Harvard) Teach. Fellow, Harvard.

JOHN F. CUNNINGHAM, Student, St. Johns College, Jamaica, New York.

ALLEN S. DAVIS, Student, University of Oklahoma.

NATHAN L. DAVIS, B.S. (Ohio) Instr., Lancaster High School, Lancaster, Ohio.

LAURENCE A. DRYDEN, M.S. (Fort Hays Kansas S.C.) Instr., Fort Hays State College.

JOHN R. DURBIN, B.A. (Wichita) Teach. Fellow, University of Wichita.

EDWARD G. ELDER, Student, University of Oklahoma.

EUGENE FAMOLARI, JR., Student, Massachusetts Institute of Technology.

LEONARD FELDMAN, Ed.D. (California, Berkeley) Asst. Professor, San Jose State College.

MRS. E. MURIEL FERGUSON, Ph.D. (Washington, St. Louis) Asst. Professor, Washington University; Senior Research Asst., Social Science Institute, St. Louis.

ALVARO FERLINE, M.S. (Chicago) Asst. Professor, New Haven State Teachers College.

REV. JAMES J. FISCHER, S.J., M.S. (Fordham) Instr., Fordham University.

HENRY E. FLEMING, Student, Harpur College.

PHILIP FRANDINA, Student, University of Buffalo.

STUART M. FRIEDMAN, M.A. (Columbia) Chairman of Department, Los Angeles Harbor Junior College, Wilmington, California.

ROBERT H. FRYLING, M.S. (Pittsburgh) Asst. Professor, Gettysburg College.

JOHN N. FUJII, M.A. (Columbia, T.C.) Instr., Oakland Junior College (Merritt).

CHARLIE W. GAMMILL, B.S. (Delta S.C.) Acting Instr., Mississippi State College.

EDITH T. GARDNER, B.S. (Ft. Valley S.C.) Grad. Student, University of Chicago.

FRANK C. GERMAN, M.A. (Illinois) Professor, Kansas State Teachers College, Pittsburg.

RICHARD R. GERO, Student, Hofstra College.

NANCY L. GETCHELL, Student, University of Maine.

QAMARUDDIN K. GHORI, M.S. (Pakistan) Teaching Asst., University of British Columbia.

SCOTT K. GIBSON, M.S. (M.I.T.) Instr., Florida Southern College.

CHARLES H. GLADMAN, M.A. (Ohio S.U.) Asso. Professor, Texas Western College.

MICHAEL GOLDSTEIN, Student, Polytechnic Institute of Brooklyn.

PAUL M. GRABARKEWITZ, M.S. (N. Dakota S.C.) Instr., Bemidji State College.

MRS. MILDRED L. GROSS, B.S. (Nebraska S.T.C., Kearney) Grad. Asst., University of Nebraska.

STANLEY J. GUBERUD, B.S. (Wisconsin S.C., Eau Claire) Asst. Professor, Milwaukee School of Engineering.

IRVING HABER, Student, Cooper Union.

HWA S. HAHN, M.S. (Oregon) Grad. Student, University of Illinois.

CHARLES A. HALIJAK, Ph.D. (Wisconsin) Asso. Professor, Kansas State College, Manhattan.

ROBERT B. HALL, B.S. (Arizona S.C.) Teaching Asst., Arizona State College.

AUGUST J. HALTMAIER, M.A. (Boston) Teacher, Oceanside High School, New York.

LT. NEIL K. HANSEN, B.A. (Washington) Instr., U. S. Naval Academy, Annapolis.

DELBERT R. HARPER, B.S. (Tulsa) Research Physicist, Carter Oil Co., Tulsa, Oklahoma.

WILLIAM E. HARTNETT, Ph.D. (Kansas) Asst. Professor, College of the Holy Cross.

HARRY S. HAYASHI, M.A. (Southern California) Instr., Los Angeles City College.

D. FAYE HENDRIX, M.A. (Geo. Peabody) Instr., Lamar State College.

DONALD G. HESS, M.A. (Long Beach S.C.) Instr., Whittier High School, La Habra, California.

ROBERT HEYNEMAN, Ph.D. (California, Berkeley) Instr., Cornell University.

ROY H. HOLMGREN, M.A. (California) Instr., College of San Mateo.

ALBERT JAQUA, Ph.D. (U.C.L.A.) Los Angeles, California.

HELENE S. JOHNSON, Student, Texas Christian University.

THOMAS J. JOHNSON, III, Student, University of Oklahoma.

DAVID W. JONAH, M.S. (Purdue) Grad. Asst., Brown University.

WILLIAM B. JONES, M.A. (Vanderbilt) Grad. Fellow, Vanderbilt University.

STANISLAUS C. KARPACZ, Student, St. John's University, Jamaica, New York.

ALAN M. KATCHER, Student, Cooper Union.

ROBERT KATZ, B.S. (Chicago) Instr., Tufts University.

LEE H. KENNEDY, Student, Texas Christian University.

JOHN V. KENNY, B.C.E. (Glasgow) Senior Engineer, New Jersey State Highway Dept., Rockaway, New Jersey.

KENNETH P. KIDD, Ph.D. (Geo. Peabody) Professor, University of Florida.

CHARLES H. KING, Student, San Jose State College.

KERRY W. KING, Student, Abilene Christian College.

TRUMAN S. KLEIN, M.A. (Maryland) Supervisor of Secondary Education, Board of Education, Upper Marlboro, Maryland.

LADIS D. KOVACH, Ph.D. (Purdue) Design Specialist, Douglas Aircraft Co., El Segundo, California.

JESSE D. KREHBIEL, M.S. (Iowa S.C.) Instr., University of Wichita.

ARLENE S. KROCHMAL, A.A. (Geo. Washington) Student, George Washington University.

ORVILLE C. KRUSCHWITZ, M.A. (Geo. Peabody) Instr., Texas Western College.

WILLIAM R. LACOUR, Student, McNeese State College.

MRS. BETTY P. LAMKIN, Student, University of Oklahoma.

NEAL F. LANE, Student, University of Oklahoma.

PHILIP W. LATIMER, M.S. (N. Texas S.) Asso. Professor, Lamar State College.

ROBERT W. LAWRENCE, Student, University of Oklahoma.

SOLOMON LEADER, Ph.D. (Princeton) Asst. Professor, Rutgers University.

GEORGE F. LEHMAN, B.A. (Minnesota) Employment Interviewer, Iowa State Employment Service, Davenport, Iowa.

JAMES B. LEWIS, M.A. (Mississippi) Asst. Prof., McNeese State College.

DON R. LICHTENBERG, B.A. (Iowa S.T.C.) Grad. Student, University of Wisconsin.

WILLIAM M. LINDSTROM, B.S. (Kansas) Grad. Student, University of Kansas.

M. ALBERT LINTON, JR., A.M.T. (Harvard) Head of Department, Wm. Penn Charter School, Philadelphia, Pennsylvania.

BIRGER LÖVGREN, Fil. Mag. (Stockholm) Grad. Student, Cornell University.

EUGENE T. LUCAS, M.A. (Denver) Asst. Professor, Oklahoma Baptist University.

EUGENE M. LUKS, Student, The College of the City of New York.

JOHN J. LUND, Ph.D. (Chicago) Head of Department, Lassen Junior College.

ROBERT J. LYON, Student, Colorado State College.

KENNETH A. MACDONALD, B.A. (Arizona) Grad. Asst., University of Vermont.

MRS. VIVIENNE M. MAYES, M.A. (Fisk) Instr., Paul Quinn College.

GEORGE S. MCCARTY, JR., B.A. (California, Berkeley) Grad. Student, University of California, Los Angeles.

ROBERT D. MCCOY, Student, University of Oklahoma.

JOSEPH C. MCCULLY, Ph.D. (Michigan) Asst. Professor, Western Michigan University.

ROSS M. MCDONALD, B.S. (Tulsa) Design Engineer, Douglas Aircraft Co., Tulsa, Oklahoma.

WILLIAM C. MCINTOSH, M.A. (California) Instr., California State Polytechnic College.

ALEXANDER H. MCLEOD, M.A. (N. Carolina) Mathematics Editor, The Macmillan Co., New York.

HAROLD MECHANIC, M.A. (Columbia) Mathematician, Nuclear Development Corp. of America, White Plains, New York.

VALERIE MIKÉ, B.A. (Manhattanville) Tech. Staff Asso., Bell Telephone Labs., New York.

FRANK L. MILLER, M.S. (Oklahoma) Teaching Asst., University of Oklahoma.

HARLAN D. MILLS, Ph.D. (Iowa State) Research Asso., Princeton University.

LLOYD E. MISH, Student, Arizona State College.

GUY D. MOFFETT, B.S. (Mississippi Southern) Instr., Perkinston Junior College, Mississippi.

ERROL L. MURPHY, Student, Notre Dame High School, Sherman Oaks, California.

ANTHONY Q. MUSTOE, M.S. (Florida) Asst. Professor, Florida Southern College.

LEROY B. NEGUS, B.A. (N.Y.S. College for Teachers) Teacher, Gilboa Central High School, New York.

HARRY L. NELSON, M.A. (Kansas) Asst. Instr., University of Kansas.

CAPT. DONALD O. NORRIS, B.S. (Chicago) Mathematician and Pilot, USAF, Wright-Patterson AFB, Dayton, Ohio.

ROBERT F. NORRIS, Student, Arizona State College.

NORBERT L. J. OLDANI, Student, University of Detroit.

MRS. KERAMANEH P. OSCHWALD, M.A. (Vanderbilt-Peabody) Instr., Northwest Missouri State College.

DONALD A. PARKER, B.S. (St. John's, New York) Asst. Tech., Western Electric Co., New York.

THOMAS C. PATTEN, Student, University of Oklahoma.

DAVID M. PERKINS, Student, Pennsylvania State University.

RICHARD E. PETERSON, M.S. (Purdue) Instr., University of Dayton.

PAUL E. PFEIFFER, Ph.D. (Rice Institute) Asso. Professor, The Rice Institute.

ROBERT PHILLIPS, Student, Wm. L. Dickinson High School, Jersey City, New Jersey.

MRS. BERNICE V. H. POLLYEA, B.S. (Chicago) Teacher, Fenger High School, Chicago, Illinois.

CAPT. WILLIAM T. POWELL, JR. (retired) B.S. (U. S. Naval Academy) United States Navy.

CLARENCE E. PRINCE, JR., B.S. (Southern Methodist) Instr., Taejon Presbyterian College, Taejon, Korea.

MRS. CANDIDA K. REES, M.S. (Virginia Poly. Inst.) Instr., University of Illinois.

CHARLES A. REIHER, Student, St. John's University, Jamaica, New York.

JIMMY M. RICE, M.S. (Eastern New Mexico) Instr., Fort Hays Kansas State College.

MELVIN B. RICHINS, B.S. (Arizona S.C.) Asst. Instr., Arizona State College.

ARTHUR M. RIEHL, B.S. (Indiana) Grad. Asst., Purdue University.

LELAND F. RITCEY, Ph.D. (Chicago) Chairman of Department, University of Manitoba.

ROBERT A. ROTHSTEIN, Student, Massachusetts Institute of Technology.

ANDREW W. ROW, M.A. (Michigan) Instr., University of Florida.

WILLIAM B. RUNDBERG, Student, San Jose State College.

DONALD E. RYERSON, B.Com. (Toronto) Teacher, Toronto Board of Education.

DON E. RYOTI, Student, Northwestern University.

HANS SEGAN, Ph.D. (Vienna) Asso. Professor, University of Idaho.

THOMAS W. SAMPSON, M.A. (Western Ontario) Head of Department, Blakelock High School, Oakville, Ontario.

ROBERT L. SANDERS, Student, St. Marys University, Texas.

JULIUS SASLOW, Student, Cooper Union.

RICHARD P. SAVAGE, M.S. (Oklahoma S.U.) Instr., University of Tennessee.

FRED W. SCHLECHT, Student, Rutgers University.

BEAL A. SCHREINER, Clerk-Dispatcher, United States Government, Marine City, Michigan.

CALVIN F. SCHWARTZKOPF, Student, University of Wichita.

GEORGE E. SCHWEIGERT, Ph.D. (Johns Hopkins) Asso. Professor, University of Pennsylvania.

JAMES B. SERRIN, Ph.D. (Indiana) Asso. Professor, University of Minnesota.

JAMES F. SHEPPARD, Student, University of Oklahoma.

RICHARD L. SHOBE, B.S. (Alabama Poly. Inst.) Teach. Fellow, Alabama Polytechnic Institute.

JOHN W. SHUE, B.S. (Colorado) Instr., University of Colorado.

DONALD M. SILBERGER, B.S. (Harvard) Teacher, Shaw High School, Cleveland, Ohio.

JACK H. SILVER, Student, Montana State University.

RODNEY W. SIMISTER, B.S. (Queens) Mathematician, Bendix Aviation, Teterboro, New Jersey.

SISTER M. PETER (STENGL) B.A. (Holy Family Coll.) Holy Family College, Wisconsin.

LAWRENCE B. SKLAR, M.S. (Pennsylvania S.U.) Grad. Student, Rutgers University.

MRS. JANET SMASHEY, Student, Butler University.

KENNETH V. SMITH, Student, University of Texas.

SAM SPEED, JR., Student, Maryville College, Tennessee.

ROBERT L. STANLEY, Ph.D. (Harvard) Asst. Professor, Washington State College.

DALE W. STEIN, B.S. (Carnegie Tech.) Senior Scientist, Litton Industries, Beverley Hills, California.

DAVID J. STEINDLER, Student, Wesleyan University.

BENJAMIN STERN, M.S. (California Inst. Tech.) Instr., University of Omaha.

RICHARD H. STEVENS, M.A. (Cincinnati) Asst. Prof., University of Cincinnati.

RICHARD R. SWIFT, M.S. (Marquette) Racine, Wisconsin.

KENNETH W. THIESSEN, M.Ed. (Western Washington College) Chairman of Department, Skagit Valley College.

ROBERT J. THOMAS, M.S. (Indiana) Grad. Asst., University of Illinois.

GEORGE TIPTON, SR., M.S. (Colorado) Head of Department, Henderson County Junior College, La Rue, Texas.

LINCOLN H. TURNER, Ph.D. (Purdue) Asst. Professor, Purdue University.

BILL L. TURNEY, D.Ed. (Houston) Asst. Professor, McNeese State College.

FREDERICK R. URBANUS, M.S. (DePaul) Scientist, Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

ROBERT E. WALDEN, Student, Hendric College.

MRS. LINA R. WALTER, M.A. (Columbia) Asso. Prof., New Jersey State Teachers College, Paterson.

CHUNG-LIE WANG, B.S. (Ordnance Eng. College, Taiwan) Grad. Student, University of South Carolina.

PAUL M. WEICHSEL, M.S. (New York) Teach. Asst., California Institute of Technology.

DAN B. WELLS, B.S. (Western Carolina) Instr., University of Kentucky.

DAVID B. WELLS, B.S. (Nebraska) Instr., University of Nebraska.

BRO. THOMAS C. WESSELKAMPER, B.S. (Dayton) Instr., Chaminade High School, Dayton, Ohio.

MRS. LYLE M. WHEELER, M.A. (Oregon S.C.) Head of Department, Astoria High School, Oregon.

ROBERT I. WHITE, B.A. (Pennsylvania) Chairman of Department, Cheltenham High School, Philadelphia, Pennsylvania.

JOHN E. WHITESITT, Ph.D. (Illinois) Asso. Professor, Montana State College.

JAMES L. WIDMAIER, Student, Otterbein College.

DOUGLAS W. WILLETT, Student, South Dakota School of Mines and Technology.

RALPH C. WILLIAMS, M.S. (California Inst. Tech.) Teacher, Pasadena City College.

MRS. VELMA M. WILLIAMS, M.S. (DePaul) Instr., Wiley College.

WILLIAM S. WORLEY, JR., Student, Phillips University.

ROBT. H. WORONUK, Student, University of British Columbia.

NSF GRANTS TO THE ASSOCIATION

The Association has recently received the following grants from the National Science Foundation.

A grant of $7,500 has been received for a conference to review the program of the Association and to formulate a plan of action. This conference, which was held in Washington, D. C. on May 16–18, will be reported in more detail in the October issue of the MONTHLY.

A grant of $8,000 was awarded for the support of a preliminary study of nonteaching mathematical employment. An advisory committee for this survey has been appointed, consisting of Morris Ostrofsky (Chairman), Wallace Givens (Acting Chairman), Paul Armer, T. E. Caywood, Churchill Eisenhart, Z. I. Mosesson, G. B. Thomas, Jr.

A grant of $47,700 has been received for the support of a pilot program of visiting lectureships to secondary schools for a two-year period. To administer this program, the following Committee on Secondary School Lecturers has been appointed: J. R. Mayor, Chairman (1958–1960), Roy Dubisch (1958–1960), W. E. Ferguson (1958–1961), B. W. Jones (1958–1959), Mrs. Marie S. Wilcox (1958–1961).

A grant of $49,000 has been received for the support of a program of production of films for improving collegiate mathematics.

THE NINETEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The nineteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, November 22, 1958. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly-enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed out early in September to the regular mailing list. If an application blank is not received by September 20, you may secure one by writing the director, Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 1, 1958. For further details of the examination and the list of prizes (including the $2500 scholarship at Harvard) see the announcement which will be mailed out along with the application blank.

Reports of the eighteen previous competitions and the examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August–September 1947; December 1948; August–September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August–September (announcement of winners) and November (questions and solutions) 1957; and this issue, pp. 515–516.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 22, 1958 at the University of Michigan, Ann Arbor, Michigan in conjunction with the meeting of the Michigan Academy of Science, Arts and Letters. Professor A. E. Lampen of Hope College presided at both the morning and afternoon meetings and at the luncheon business meeting. Total attendance was 108, including 70 members of the Association.

The nominating committee, consisting of Professors P. S. Dwyer, Chairman, H. D. Larson and S. D. Conte, proposed for Chairman, Professor G. Y. Rainich, University of Michigan; Vice-Chairman, Professor W. D. Baten, Michigan State University; and Secretary-Treasurer, Professor F. A. Beeler, Western Michigan University. The slate was elected unanimously.

The Michigan Mathematics Prize Competition Committee, consisting of Professors R. H. Oehmke, A. J. Lohwater, A. W. Jacobson and F. L. Celauro gave a report of its activities during the past year. The Committee, with the approval of the Executive Committee, decided to go ahead with plans for a competitive examination for High School students this Spring. This examination will be given March 27, 1958 in 315 high schools to 6100 students. The Committee felt that it would be better to write its own examination and also to score the papers. The examination, based on the subject matter of the four-year high school mathematics curriculum, is in two parts. The first part, a multiple-choice type examination, is designed to test the general mathematical back ground of the contestant. The second part is of a written character and is designed to measure the

mathematical maturity of the contestant. Such factors as originality and clarity of expression will be weighed. Medals, certificates, prizes and scholarships will be awarded to the highest scorers. These are given by the sponsoring organizations; colleges, universities and industries of the area. The Section passed a resolution to sponsor such an examination.

Professor Leo Goldberg, Head of the Department of Astronomy at the University of Michigan, was the special guest speaker at the meeting. Speaking on the topic "Solar Physics," Professor Goldberg explained various physical phenomena associated with the sun and methods used in investigating these phenomena. His lecture was illustrated with slides. At the conclusion of the lecture a special motion picture film, prepared by a group of European astronomers, was shown.

The following papers were presented:

1. *A particular affinely-connected manifold*, by Dr. N. J. Hicks, University of Michigan, introduced by the Secretary.

The purpose of this note is to exhibit a simply-connected C^∞ manifold M on which is defined a complete affine connection satisfying the following property: for any integer $n\geqq 0$ there are two points p and q in M such that any broken geodesic from p to q must contain at least n breaks. The manifold M is not compact; however, for each n there is a compact simply-connected manifold for which one must allow at least n breaks in order to connect all pairs of points with broken geodesics.

2. *The solution of certain differential equations by Laplace transforms*, by Professor H. E. Stelson, Michigan State University.

The solution of linear differential equations with polynomial coefficients is considered. Properties of the differential operator $\overline{D}$, $[X^mD^m=\overline{D}(\overline{D}-1)(\overline{D}-2)\cdots(\overline{D}-m+1)]$ are used as an aid in the solution. Especially the relation $\phi(\overline{D})X^n=\phi(n)X^n$ is used. The reducibility of differential coefficients is facilitated by using the operator $\overline{D}$. The property of Laplace transforms, $L(\overline{D}^ny)=(-1)^n(d^n/dp^n)[p^nL(y)]$ is used in the solution.

3. *Stieltjes product integrals*, by Professor T. H. Hildebrandt, University of Michigan.

While the Stieltjes integral $\int_a^b fd\alpha$ for $f(x)$ continuous on (a, b) and $\alpha(x)$ of bounded variation exists as the limit of the sums $\sum_{i=1}^n f(x_i')[\alpha(x_i)-\alpha(x_{i-1})]$ as the maximum length $|x_i-x_{i-1}|$ of the intervals of the subdivision $a=x_0<x_1<\cdots<x_n=b$, with $x_{i-1}\leqq x_i'\leqq x_i$, approaches zero, this does not hold for the product $\prod_{i=1}^n\{1+f(x_i')[\alpha(x_i)-\alpha(x_{i-1})]\}$ connected with the Stieltjes product integral $(P)\int_a^b[1+f(x)d\alpha(x)]$ if $\alpha(x)$ is discontinuous. However, the convergence is valid in the sense of successive subdivisions. While $(P)\int_a^b[1+f(x)dx]$ is equal to $\exp\int_a^b f(x)dx$ if $f(x)$ is Riemann integrable, and $(P)\int_a^b[1+d\alpha(x)]=\exp[\alpha(b)-\alpha(a)]$ if $\alpha(x)$ is continuous and of bounded variation, terms which are not exponential in character appear if $\alpha(x)$ is discontinuous. Moreover, when $\alpha(x)$ is continuous $(P)\int_a^x[1+d\alpha(x)]$ satisfies the integral equation analogous to a linear differential equation *viz.* $y(x)=\int_a^x[d\alpha(x)y(x)+1]$, but this is no longer always true if $\alpha(x)$ is discontinuous. Similar remarks hold when $f(x)$ and $\alpha(x)$ are replaced by matrices with matrix multiplication.

4. *Stone-Weierstrass theorem and Komolgorov's consistency theorem of statistics*, by Professor Shu-Teh C. Moy, Wayne State University.

From the point of view that a probability measure defined for Borel subsets of a compact Hausdorff space Ω is a positive linear functional on the Banach space $C(\Omega)$ of continuous functions on Ω, the Komolgorov's consistency theorem of statistics is derived from the Stone-Weierstrass theorem.

5. *Parabolic analogues on theorems on harmonic functions*, by Dr. F. W. Gehring, University of Michigan, introduced by the Secretary.

Let $u=u(x, t)$ be a function defined in a domain D in the xt-plane. If u has continuous second partial derivatives and if $u_t=u_{xx}$, then u is in H over D. If $u=u_1-u_2$ where u_1 and u_2 are non-

negative and in H, then u is in H^{Δ}. Functions in H have many properties in common with harmonic functions. Analogues of the maximum and of simple Phragmén-Lindelöf extensions are mentioned. When u is in H^{Δ} over the strip $0<t<c$, there is an important representation theorem due to Widder. Various forms of the Fatou theorem and its converse are discussed for such functions along with some uniqueness theorems and some parabolic analogues of the second theorem of Harnack and the Vitali convergence theorem.

6. *The effect of digital computers on university mathematical education*, by Professor J. W. Carr III, University of Michigan.

Digital computers are having a revolutionary impact on society. They will have a revolutionary impact on the teaching of mathematics. Since their use emphasizes the *generation of the algorithm* to solve, rather than the *actual solution* of the problem, much of the present rote solution of problems should be eliminated. Emphasis must be more on an abstract algebraic approach to teaching. The use of *command languages*, basic to computer use, should be considered, rather than the present descriptive languages. The actual use of computers in numerical and analytical (differentiation, integration, decision) problems, must be included in the standard texts.

7. *A computational algorithm for logical analysis*, by Mr. R. R. Korfhage, University of Michigan.

In order to make use of such logical methods as *reductio ad absurdum* an electronic computer must be able to derive the structure of any given logical expression. An algorithm is presented enabling the computer to do this for expressions written in the Polish prefix notation. The procedure requires only two scans of the expression. It can be easily adapted to other computational schemes ranging from arithmetic to mechanical translation procedures, and analogous algorithms can be written for systems using other than the Polish notation.

8. *Some electrical examples to illustrate Stokes' theorem*, by Professor W. P. Reid, Michigan State University

With the aid of Maxwell's equation for the curl of the magnetic field, H, one may use Stokes' theorem to determine H_θ around a straight wire carrying a steady current; the field due to steady currents flowing in various figures of revolution; the electric and magnetic fields inside a charging condenser; and the electric and magnetic field outside a charging sphere. Also, if one assumes that the electric field due to an oscillating dipole or to certain electro-magnetic waves traveling along circular wave guides is known, then H_θ may be calculated for these cases.

9. *Retention of mathematics by college freshmen*, by Professor F. L. Celauro, Central Michigan College.

Retention of knowledge and skills in trigonometry was investigated in relation to elapsed time, amount learned, intelligence and general mathematical proficiency. Students were tested upon commencing the study of trigonometry. After completing the course they were tested for achievement, and tested again some weeks later for delayed recall. Retention in trigonometry falls at a decelerated rate, 37 per cent retained after thirty weeks. Retention varies with different topics, identities most easily forgotten. Generalizations depend upon definition of retention employed (amount versus per cent retained). Retention is aided by injecting meaning and unification into the subject and viewing it in relation to other fields.

10. *Convergence in the mean of Taylor series*, by Dr. D. S. Greenstein, University of Michigan.

Given $f(x)$, a c^∞ function all of whose derivatives belong to $L^p(-\infty, \infty)$ for some fixed $p \geqq 1$, the author considers representing the translates $f(x+h)$ by means of a "Taylor series in the mean." The class of functions so representable are generalizations to L^p of analytic function classes in L^2 studied by Paley and Wiener.

11. *A search for analogues of the Mathieu groups*, by Dr. E. T. Parker, University of Michigan and Mr. P. J. Nakolai, Ohio State University, presented by Dr. Parker.

The Mathieu groups of degrees 11, 12, 22, 23, and 24 are the only known simple finite groups contained in no known infinite system. Four of these a e the only known quadruply transitive permutation groups not symmetric or alternating. Degrees 11 and 23 are both of the form $p=2q+1$, with p and q primes. The analogues are simple transitive permutation groups of degree $p=2q+1$, of order $>p$ and $p!/2$. The authors first described pairs of generators of such groups, then wrote a program for UNIVAC Scientific Computer, Model 1103A. Groups of degree $p=2q+1$, $23<p\leqq 1823$, were examined exhaustively by the computer. No analogues exist in this range. Some six hours of machine time were required.

12. *The combination of $2\times n$ contingency tables*, by Dr. W. M. Kincaid, University of Michigan, introduced by the Secretary.

Let x and y be two experimental variables and let m trials, having outcomes "success" or "failure," be conducted under each of the kn conditions $x=x_i$, $y=y_j$, $i=1, \cdots, k$, $j=1, \cdots, n$; thus the results form a set of $2\times n$ contingency tables. The hypothesis that the probability of success is independent of y for each x may be tested by pooling the entries to form a single table and computing χ^2, but any dependence of the probability on x causes a spurious loss of significance. The purpose of the present paper is to show how this difficulty may be avoided.

(This work was partly supported by the Office of Naval Research.)

F. A. BEELER, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The forty-fifth meeting of the Iowa Section of the Mathematical Association of America was held at Drake University, Des Moines, Iowa, April 18, 1958. Professor A. H. Blue, Chairman of the Section, presided. The total attendance was 71, including 30 members of the Association. Routine business was considered during the afternoon meeting. As a result of the work by the Committee on Contests appointed a year ago, it was voted that the Committee be commended for their work and that the officers of the Section be empowered to go ahead with high school contests in mathematics on a limited basis.

It was agreed that the Iowa Section would meet jointly with the State University of Iowa's Annual Conference of Teachers of Mathematics in October, 1958.

The following officers were elected: Chairman, Professor E. N. Oberg, State University of Iowa, Iowa City, Iowa; Vice-Chairman, Professor R. S. Jacobsen, Luther College, Decorah, Iowa; Secretary-Treasurer, Professor E. L. Canfield, DrakeUniversity, Des Moines, Iowa.

The following papers completed the program:

1. *Report of the committee on the problem of contests*, by Professor I. H. Brune, Iowa State Teachers College, Chairman.

The following points were made in the report of the Committee on Contests: (1) Mixed reactions were noted by high school teachers, some in favor, some opposed to contests. (2) Many teachers reported contests were not permitted. (3) Members of the advisory board of the Iowa Association of Mathematics Teachers seemed not to be in favor of the contests. (4) A poll of teachers in convention at Des Moines resulted in 44 votes in favor and 13 votes against. (5) While the Committee itself saw merit in a talent search it doubted that a contest should in any way pit school against school or teacher against teacher. Rather the contest should be a search for talent, not a device for rating teacher efficiency. (6) The Committee recommended that the Iowa Section of Mathematical Association of America try the contest for one year, taking advantage of the offer of the National Committee to conduct the examination but that time remaining to prepare for a contest for 1958 is doubtful; consequently, 1959 would be a better beginning date.

2. *Some curious results in distance geometry*, by Professor L. M. Blumenthal, University of Missouri. (By invitation.)

The results discussed concerned (1) non-congruent simplices with the same edges, (2) simplex-producing combinations of simplices, (3) a mapping of a bipunctured n-sphere onto an $(n-1)$-sphere, (4) an uncountable class of nonrectifiable arcs of Hilbert space, and (5) metric continua without small acute triangles.

3. *Grading systems*, by Professor Fred Robertson, Iowa State College.

The author compared several grading or testing systems, in industry and schools, with the one commonly considered as the grading system.

4. *A note on defining an extension of a probability measure on subsets of function space by applying one of J. L. Doob's theorems*, by Professor W. A. Small, Grinnell College.

Let W be the set of real-valued functions, $w(t)$, of the real variable t. An extension (W, F_2, P_2) of a Fundamental Borel Probability Field (W, F_0, P_0) was defined by J. L. Doob and S. Kakutani. It may happen that an adjunction extension (W, F_0', P_0') of (W, F_0, P_0) exists through adjoining a subset W' of W to F_0. By applying one of Doob's theorems, the condition on the outer P_2 measure $P_2^*(W')=1$ is seen to be necessary and sufficient for the existence of the corresponding adjunction extension (W, F_2', P_2') of (W, F_2, P_2). If (W, F_0', P_0') is measurable, then so is (W, F_2', P_2').

5. *An unusual method of teaching logarithms*, by Professor Fred Robertson, Iowa State College.

The author stressed the laws of operation needed for computation. The use of the tables is taught as an entirely different phase of the work. The tables of natural logarithms may be used first. Then a suggestion is made to change the tables of common logarithms. The terms characteristic and mantissa are not introduced.

6. *Uniform convergence of the second differences*, by Mr. U. R. Kodres, Iowa State College.

A sequence of theorems, whose proofs were sketched, was used to characterize the class of functions whose second differences converge to zero uniformly.

7. *Exceptional values of metric density*, by Mr. N. F. G. Martin, Iowa State College.

The usual definition of the metric density of a measurable set in E_1 at a point of E_1 is given. Then for a given real number λ, $0<\lambda<1$, a set is constructed whose density exists at 0 and has the value λ.

8. *Note on the classical canonical form of a matrix*, by Mr. J. C. Mathews, Iowa State College.

A proof of the existence and uniqueness of the classical canonical form of a matrix is accomplished without the intervention of invariant factors, elementary divisors, or modules. Using simple induction proofs a matrix A defined over an algebraically closed field is reduced to an intermediate canonical form B. At this point a final similarity transformation R is constructed such that $R^{-1}BR$ is classical. The proof of the uniqueness of the classical form of A is done by comparing ranks.

9. *Functions whose second difference goes to zero*, by Professor S. D. Nolte, Iowa State College.

If $f(x)$ is defined on an open interval I, the second difference is defined to be $|f(x+h)-2f(x)+f(x-h)|$. If a sequence $f_n(x)$, converges uniformly on I to $f(x)$ and if $f_n(x)$ is such that $\lim_{h\to 0}|f_n(x+h)+2f_n(x)-f_n(x-h)|=0$ for all h and all x in I, then $f(x)$ also has this property.

Definition. A function $f(x)$ is said to satisfy a second difference Lipschitz condition of order α on I if there exists an M and a $\delta>0$ such that $|f(x+h)-2f(x)+f(x-h)|<M|h|^{\alpha}$ for all x in I and all $|h|<\delta$. If $f(x)$ in this definition is continuous at one point in I, and if $\alpha\geqq 1$, then $f(x)$ is continuous at every point in I.

10. *Maxima of functions*, by Mr. J. D. Miller, Iowa State College.

By first defining what is meant by a real-valued function of a real variable taking on a proper relative maximum or a nonproper relative maximum at a point, it is shown that a function can possess at most a denumerable number of proper relative maxima. Furthermore, if a function takes on a relative maximum at every point of its domain of definition, then the range of the function is at most denumerable.

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-third annual meeting of the Kansas Section of the Mathematical Association of America was held at Kansas State Teachers College, Emporia, Kansas, on April 12, 1958, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. There were 178 persons registered, including 52 members of the Association. Professor L. E. Laird, Chairman, presided at the sessions.

The following officers were elected for one year terms: Chairman, Professor P. S. Pretz, St. Benedict's College; Vice-Chairman, Professor J. D. Haggard, Kansas State Teachers College, Pittsburg; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State Teachers College, Pittsburg.

At the joint session, held in the morning, Dean Albert E. Meder, Jr., Rutgers University, delivered an address entitled "Modern Mathematics and Its Role in Secondary Education."

The following short papers were presented at the afternoon session:

1. *Statistics*, by Dr. Stanley Wearden, Kansas State College of Agriculture and Applied Science, introduced by the Secretary.

Rainfall information was obtained from the *United States Department of Commerce Weather Bureau Climatological Data*. Only data from the Hays, Kansas, reporting station was used. Median annual rainfall over the ninety years was 22.59 inches. Each year was classified as being above or below median rainfall, and the probability that the pluses and minuses occurred in an essentially random sequence was calculated. There were only thirty-six runs, or cycles, among the ninety years ($P=.034$). The correlations between monthly and annual rainfall were calculated. Ten correlations were significant ($P=.05$), and June rainfall explained over half the variability of annual rainfall.

2. *Braids*, by Professor J. C. Lillo, University of Kansas, introduced by the Secretary.

This paper consisted of expository comments on the theory of braids.

3. *What's wrong with mathematics*, by Professor C. B. Read, University of Wichita.

Without exhaustive or necessarily representative coverage, the paper presented some criticisms of mathematics which have appeared in professional periodicals. There was no attempt to give specific reference to any quotation, but it was suggested that listeners try to identify the person or group to which the point of view might be attributed. At the conclusion the fact was revealed that every quotation was written prior to 1932. The tentative conclusion was that, if, after twenty-five years, almost identical criticisms can be made, one wonders if mathematicians are making rapid progress towards revision and improvement.

4. *Calculus—a modern approach*, by Professor W. C. Janes, Kansas State College of Agriculture and Applied Science.

The idea of function as a consistent class of quantities with domain and range is developed early. A symbol for the identity function is helpful. Geometric considerations lead naturally to the fundamental theorem of the integral calculus. The derivative is defined as the limit of a difference quotient. Discussions of increments, infinitesimals, and differentials are almost nil, though in applications of integration the differential is retained as reminiscent of the type of product-sum whose limit is obtained. *Menger's Calculus—A Modern Approach* is an excellent book exemplifying the method.

5. *Are computing machines replacing mathematicians?* by Professor U. W. Hochstrasser, University of Kansas, introduced by the Secretary.

The capabilities and limitations of electronic computers are discussed and compared with the characteristics a mathematician is supposed to have. This comparison shows that the computer, at least in its present form, can not replace the mathematician and that its efficient and sensible use frequently depends essentially on the mathematical capabilities of its user. The widespread introduction of computers has accordingly not decreased the demand for mathematicians but caused a serious shortage of people with adequate mathematical training, a situation which can be only improved upon by offering a broader and better mathematical education at academic institutions.

HELEN KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The spring meeting of the Kentucky Section of the Mathematical Association of America was held on April 26, 1958 at the University of Kentucky, Lexington, Kentucky. Professor J. C. Eaves, University of Kentucky, presided at both the morning and afternoon sessions. There were 71 persons in attendance, including 49 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor W. J. Robinson, Centre College of Kentucky; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor Sallie E. Pence, University of Kentucky.

By invitation of the Committee, Professor Ernst Snapper of Miami University delivered an hour address at the afternoon session entitled, "Geometric Algebra." An abstract of this address follows:

The term "Geometric Algebra" was coined by Professor Emil Artin whose book by that title has recently appeared. One of the messages this book brings us is that the notion of a matrix, thought of as a rectangle of field elements, should be very much reduced in importance. Generally speaking, a matrix is an analytic representation of some geometric object, relative to a chosen coordinate system. If the same geometric object is represented by means of another coordinate system, we get another matrix. Consequently, if we treat a geometric object by means of matrices, we do not get an invariant theory. By now we have the experience that for the purposes of modern algebra invariant theories are the best. This change from "dealing with the geometric object through the intervention of a coordinate system" to "dealing with the geometric object directly, *i.e.*, doing geometric algebra," constitutes one of the major changes from the algebra of, say, 120 years ago to present-day algebra. As an example of geometric algebra, quadratic forms were discussed. A quadratic form was defined as a metric vectorspace, and the theory of metric vectorspaces was described through Witt's theorem.

The following papers were presented.

1. *The effects of prime characteristic in differential algebra*, by Professor Frank Levin, University of Kentucky.

In this talk the author cited various difficulties which arise in differential algebra when the groundfield has a characteristic $p\neq 0$.

2. *A coefficient problem for Laurent series*, by Miss Betty C. Detwiler, University of Kentucky.

In this paper the author studied analytic functions having a positive real part in an annulus. New proofs were obtained for well-known theorems due to Nehari and Robertson.

3. *Objectives of modern high school trigonometry*, by Professor Edith R. Schneckenburger, University of Buffalo.

Recommendations of the Commission on Mathematics were considered. The following objectives were suggested: mastery of subject matter needed for calculus and science courses, development of understanding of the nature of mathematics, appreciation of applications of trigonometry, improvement in the reading and writing of mathematics.

4. *Freshman mathematics at the University of Louisville*, by Professor W. L. Moore, University of Louisville.

A program was outlined with the objective of encouraging the more gifted students to take special courses in mathematics.

5. *The Mathematics of a wave tank*, by Professor W. J. Robinson, Centre College of Kentucky.

This paper presented the basic mathematics of a wave tank with a submerged transverse cylinder. Certain elementary properties were discussed, as well as one approximation to the true solution.

6. *Multiplication on the real line*, by Professor J. G. Horne, University of Kentucky.

All of the continuous associative multiplications of the real line which agree with the ordinary multiplication for the nonnegative reals are determined. In particular it is shown that Fawcett's characterization of ordinary multiplication on the closed-unit interval extends to a characterization of ordinary multiplication on the set of reals.

7. *On solving a certain nonlinear differential equation of 2nd order*, by Miss Dorothy I. Koehler, University of Kentucky.

The differential equation considered is taken from *On the solution of a differential equation with nonlinearity appearing in the second derivative of combined linear and cubic terms* by Chi-Neng Shen, Quart. Appl. Math., vol. 15, 1957, pp. 11–30. In particular we study the equation $d^2(y+\beta y^3)/dx^2+y=r$. The method of solution presented in this note varies from the method in the reference in that upon making the transformation $v^2=t$, the resulting equation is a first order linear differential equation. This is then solved by the usual methods.

8. *A network of checks for the multiplication of matrices*, by Professor J. C. Eaves, University of Kentucky.

In this paper the author indicated various checks that may be employed in the multiplication of matrices. This material is related to modern high speed computers.

V. F. Cowling, *Secretary*

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Spring Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Randolph-Macon Woman's College, Lynchburg, Virginia, on April 26, 1958, with 70 members in attendance. Professor R. P. Bailey, Chairman, presided.

The following officers were elected for the year 1958–59: Chairman, Professor M. W. Oliphant, Georgetown University; Vice-Chairmen, Professor Herta T. Freitag, Hollins

College and Professor Joseph Milkman, United States Naval Academy; Secretary, Professor D. B. Lloyd, District of Columbia Teachers College; Treasurer, Professor T. W. Moore, United States Naval Academy.

The following papers were presented at the meeting:

1. *Certain graphical solutions to the heat conduction equation for an insulated, infinite metal slab*, by Mr. W. H. Holter, Atlantic Research Corporation, Alexandria, Virginia.

The solution to the heat equation for the case of a metal slab separated from a transient flow of hot gases by a layer of insulation was presented. Simplified, approximate solutions were obtained at the two boundaries of the insulation. Graphical representations of these solutions provide a rapid method of determining the insulation required to maintain the temperature of the metal slab below a given level. The wide application of the graphs to various heating problems was illustrated.

2. *The solution of ordinary differential equations with constant coefficients by analog computer methods*, by Professor W. L. Fields, Hampton Institute.

The machine methods for addition, subtraction, multiplication, division, and integration were shown. These operations were combined to demonstrate the solution of differential equations.

3. *Advances in automatic programming for digital computers*, by Mr. R. F. Reiss, Babcock and Wilcox Co., Atomic Energy Division, Lynchburg, Virginia, introduced by the Secretary.

The translation of a mathematical problem into use for computer operation ordinarily requires much training and experience. To simplify the programming of such problems and to minimize the personnel training required, methods known as "automatic programming" have been devised. One of the more sophisticated systems, the FORTRAN formula translator, was described in detail and compared with ordinary programming techniques.

4. *Experiments in solving ordinary differential equations*, by Mr. W. Timlake, Babcock and Wilcox Co., Lynchburg, Virginia, introduced by Professor M. G. Humphreys.

Truncation error is usually a dominant factor in the choice of which numerical integrating scheme is applied to a set of ordinary differential equations. This paper presented a clear example of a system of differential equations in which stability (as defined by O'Brien, Hyman and Kaplan), and propagated error completely overshadow the problem of truncation. The difficulty of obtaining a "good" *a priori* bound on Δt was also indicated.

5. *A study of variations in the viewing of a picture*, by Professor Herta T. Freitag, Hollins College, and Mr. A. H. Freitag, Hollins College School, Hollins College, Virginia, presented by Professor Freitag.

The well-known calculus problem of determining the position from which a picture appears "at its best" leads to an interesting consideration of other related factors. Arbitrarily varying the observer's height, his position, the height of the picture, the height at which it is hung—keeping any two of these parameters constant, while varying the others, leads to six cases. In each case the resulting functional relationships are conic sections—parabolas, hyperbolas, or even degenerate conics.

6. *Cardioids and rolling polygons*, by Professor R. C. Yates, College of William and Mary. (By invitation.)

Professor Yates discussed the properties and applications of the cardioid, based upon its fundamental geometric structure. He interpreted the curve as a *roulette*, a *conchoid*, a *pedal*, a *caustic*, and an *envelope of circles*. A unique feature involved the determination of arc length and area by means of rolling polygons. This was based upon formulas for sums of sines and cosines of integral multiples of an angle of roll, sums which have limits as the number of sides of the polygon

increases. He pointed out the generalization of this method to all closed cycloidal curves. The lecture was illustrated with colored drawings, which were distributed to the audience.

D. B. LLOYD, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-fourth annual meeting of the Nebraska Section of the Mathematical Association of America was held on April 19, 1958, at the University of Nebraska, Lincoln, Nebraska, in conjunction with the meetings of the Nebraska Academy of Sciences. Professor Edwin Halfar presided. There were 35 persons in attendance, including 25 members of the Association.

The following officers were elected for 1958–1959: Chairman, Professor J. F. Wampler, Nebraska Wesleyan University; Vice-Chairman, Professor Edwin Halfar, University of Nebraska; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

The following papers were presented:

1. *The icosahedron*, by Professor W. G. Leavitt, University of Nebraska.

Consider a cylinder with cones of the same radius at both ends, the radius of the cylinder being equal to the distance between the bases of the cones, and the angle between the axis and the slant of the cones being a little less than 60°. Five equally spaced points are marked around the base of each cone in such a way that if P is a point on the upper cone, the point P' directly below P bisects the arc between two of the points marked on the lower cone. These ten points together with the apexes of the cones form the twelve vertices of the icosahedron.

(Analysis of method discovered by a local machinist.)

2. *Integral means and subfunctions*, by Professor L. K. Jackson, University of Nebraska.

Let $f(x)$ be a bounded integrable function defined on an interval. The integral mean $F_h(x)$ is one degree "smoother" than $f(x)$, for example, if $f(x)$ is continuous, $F_h(x)$ is differentiable. Furthermore, the integral-mean operation preserves linear functions and transforms convex functions into convex functions. In this paper is discussed the problem of extending these results to more general differential equations. Specifically, given a differential equation, a smoothing operation is sought which preserves solutions of the differential equation and transforms subfunctions into subfunctions. A solution is given for linear second order differential equations.

3. *Hausdorff separation and compactness*, by Professor Edwin Halfar, University of Nebraska.

The fact that compact subsets of a Hausdorff space are closed suggests the problem of determining restrictions on a space so that the property of being Hausdorff is equivalent to the property that compact subsets are closed. A simple theorem of this type is proved, and an example illustrating the necessity of some such restriction is given.

4. *First Nebraska (Ninth National) Mathematics Contest*, by Professor H. M. Cox, University of Nebraska.

1685 contestants from 130 high schools were enrolled in the Mathematics Contest held March 27, 1958. Scoring was provided by the University of Nebraska's Bureau of Instructional Research. Norms were prepared on 1405 contestants from 121 high schools. (Mimeographed report is available for distribution.)

5. *A report on the use of television for stimulating more interest in mathematics*, by Professor W. E. Mientka, University of Nebraska.

During 1957, "Fun with Figures" was presented in eighteen half-hour programs on KOLO-TV (Reno, Nevada) and in eleven half-hour programs over KUON-TV (University of Nebraska, Lincoln, Nebraska). Each program consisted of three parts: (1) an introduction to some mathe-

matical problem by means of a model (Mobius strip, curves of constant breadth, towers of Hanoi, *etc.*); (2) a discussion of various mathematical topics, not ordinarily considered in usual mathematics courses (tests for divisibility, the conic sections, the Königsberg bridge problem, *etc.*); (3) a problem of the week. The verbal and written response indicated that the objectives of the series were achieved.

6. *Types of errors in Dickson's "History of the Theory of Numbers,"* Vol. 2, by Mr. H. W. Becker, Radio Engineering Institute, Omaha, Nebraska.

Typical of the three hundred errors discovered are the failure to tell a triangle from a tetrahedron (p. 210), whether two problems are the same, or different (p. 194 and p. 210), whether two solutions are the same or different (p. 210, p. 212, p. 213), to recognize a general solution (all three references), and to see through a fallacious proof even when only a few lines long (p. 502).

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The forty-second annual meeting of the Ohio Section of the Mathematical Association of America was held at Denison University, Granville, Ohio, on Saturday, April 26, 1958. Professor Sam Selby, Chairman of the Section, presided at the morning and afternoon sessions. There were 117 persons registered in attendance, including 97 members of the Association.

Officers elected for the coming year are: Chairman, Professor L. E. Bush, Kent State University; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor Melvin Bloom, Miami University. The members of the Program Committee are: Professor H. E. Tinnappel, Bowling Green State University, Chairman; Professor R. W. Shoemaker, University of Toledo; Professor C. W. Topp, Fenn College.

The following papers were presented:

1. *Critique on preparation of elementary and secondary school teachers of mathematics*, by Professor Nathan Lazar, The Ohio State University. (By invitation.)

2. *Advanced slide rule techniques* (*cubic equations, addition and subtraction*), by Dr. B. L. Schwartz, Battelle Memorial Institute, Columbus, Ohio.

a. Any cubic equation is readily reduced to the form $y^3+Py=Q$. This equation can be solved with one setting of any slide rule having the usual A, B, C, and D scales.

b. Normally there is no special advantage in performing addition on a slide rule. However, there may be one when a formula combines additions with normal slide rule operations, *e.g.*, multiplications. The method presented requires as many operations as a multiplication followed by a division. For use with rules with other scale arrangements than the K & E series 4080, the formula the process solves is given by: $u+v=[\sqrt{u}/\sin \text{arc tan } (\sqrt{u}/\sqrt{v})]^2$.

3. *The modern approach to LR^2H*, by Professor R. F. Rinehart, Case Institute of Technology.

How writers of literature might impart new austerity—and hence beauty—into their works by employing modern mathematical language was illustrated by discussion of the story of Little Red Riding Hood (LR^2H) in modern mathematical parlance. For example, the grandmother, G, could be denoted by $f(f(LR^2H))$, where f is the Murrow mapping (*i.e.*, person-to-person) "daughter" into "mother."

4. *The derivative as a front of expansion*, by Professor Wayne Dancer, University of Toledo.

To give the derivative meaning within the students' experience let us visualize it as a "front of expansion" along which a geometric figure grows into a larger but similar figure. A square of area x^2 grows into a larger square by expansion along two sides, $2x$, and a circle expands along the

circumference, $2\pi r$. These are the derivatives of the variable areas. The standard "area under a curve," bounded laterally by a fixed ordinate and a variable ordinate, can expand only along the variable ordinate, *i.e.*, $dA/dx=y$. Similarly, the front of expansion of a variable sphere is the surface of the sphere, expressed: $d(4\pi r^3/3)/dr=4\pi r^2$. A right circular cylinder with variable radius and altitude has two fronts of expansion: the top, πr^2, and the lateral surface, $2\pi rh$. These are the partial derivatives of the volume with respect to h and r respectively.

5. *The new Dartmouth mathematics curriculum*, by Professor J. L. Snell, Dartmouth College. (By invitation.)

A mathematics curriculum in a liberal arts college must fulfill a variety of needs. It must provide an opportunity for the average liberal-arts student to learn of the nature of advanced mathematics. It must provide specialized training for various scientists who will use mathematics as a tool. It must make provision for the man who wants to specialize in mathematics, whether he be of the caliber of a creative mathematician or of average ability. And finally, it must provide information to the entire undergraduate body on modern mathematics as a living and creative field, with applications in the physical, biological and social sciences. The speaker described the way in which Dartmouth College has attempted to fulfill these goals.

6. *Report from the Commission on Mathematics of the College Entrance Board*, by Professor E. R. Ranucci, Newark State College, introduced by the Secretary. (By invitation.)

The Commission on Mathematics of the College Entrance Examination Board grew out of the concern of the Mathematics Examiners that the Board's tests were not reflecting fully and appropriately the emerging programs of mathematics instruction in forward-looking college preparatory schools, both public and private, and moreover that the standard curriculum taught in most secondary schools was sadly out of date.

The Commission accepts the following theses: (1) In the past thirty years the nature of mathematics as a subject has been substantially altered by the results of mathematical research. (2) Despite these developments, school and college mathematical curricula have been largely unaltered. (3) It is both possible and desirable to rectify this situation.

7. *Independent studies at Oberlin College*, by Professors John Baum and Wade Ellis, Oberlin College, given by Professor Ellis.

Four of Oberlin College's eight sections of regular freshman mathematics were organized, in 1957–1958, as two control and two experimental independent study classes. Professors Baum and Ellis each taught one control and one experimental section. The control sections went through the usual course experience. The experimental sections attended classes from September 17 through December 14 (about 13 weeks) and from April 8 through May 24 (about 7 weeks), except for one week in November. During the remainder of the school year, they were out of contact with their instructors.

Tests administered on April 8, and covering material studied during the out-of-class period by both experimental and both control groups, failed to reveal any significant difference between the two types of classes insofar as mastery of subject matter and facility with techniques were concerned.

8. *The "Cincinnati Experiment,"* by Professor Gaylord Merriman, University of Cincinnati.

The speaker reported the fate of three experimental courses which had been outlined previously (Oberlin meeting, 1956): (a) a course, for secondary school teachers, in modern concepts being urged for inclusion in the high school programs; (b) a course of the Kemeny-Snell-Thompson type for majors in certain social sciences; (c) a course of tutorial readings primarily for advanced placement students. All three had been successful. The first two are now integrated into the regular

program. The last is being abandoned because of large curricular changes which accommodate students into a more flexible program. The talk concluded with a detailed outline of this new curriculum—a continuing "experiment."

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE OKLAHOMA SECTION

The semi-annual meeting of the Oklahoma Section of the Mathematical Association of America was held at Central State College, Edmond, Oklahoma, on Saturday, April 19, 1958. Professor W. A. Rutledge, Chairman of the section, presided. There were 52 persons in attendance, including 41 members of the association.

Several committee reports presented at the business meeting are summarized with the abstracts of the papers presented. The section voted that the sum of fifteen dollars be given to the Committee on the Oklahoma Visiting Lecture Service from the section treasury. The committee was instructed to determine the type of lecture program desired by Oklahoma and Arkansas high schools, if any, and if possible, to implement the program before the fall meeting. It was further voted that the committee be specifically instructed to secure lecturers, and that both the lecturers and their topics be subject to the approval of the committee.

The following papers and reports were presented:

1. *Some properties of connectivity maps*, by Professor O. H. Hamilton, Oklahoma State University of Agriculture and Applied Science.

According to Nash, a connectivity map from a space A into a space B is a transformation T such that the induced transformation g of A into $A \times B$ defined by $g(p) = p \times T(p)$ transforms connected subsets of A onto connected subsets of $A \times B$. Examples of connectivity maps which are not continuous were given and some simple properties of connectivity maps were derived and compared with different characterizations of continuous transformations.

2. *Two theorems with respect to sequences of Banach spaces*, by Mr. Robert Welland, student, University of Oklahoma, introduced by the Secretary.

The norm for a Banach space often determines the set of points to be in the space. This situation leads one to ask how the spaces vary if the norm is changed. Unions and intersections of ascending and descending sequences of Banach spaces were considered.

3. *Evidence that $2^{8191}-1$ is prime*, by Mr. J. W. Sehestedt, Hoyt, Oklahoma.

The purpose of this paper is to present evidence that $2^{8191}-1$ is prime, as conjectured by E. Catalan. The evidence is dependent on facts verifying that a Mersenne prime used as P value in his formula will always produce a perfect number. It can be shown that $2^{8191}-1$ has characteristics identical to all known Mersenne primes, in relevance to perfect numbers. Computation is greatly reduced and simplified by using the sequence of a few digits at the end of the number to identify it. A graphic presentation in which all numbers are positionized was also demonstrated.

4. *Certain nonaffine projective transformations in 4-space*, by Professors Simon Green and W. A. Rutledge, University of Tulsa, presented by Professor Green.

This paper is an extension of the work done by J. S. Taylor, St. Kwietniewski, K. Kommerell and others on the problem of representation of equations in two complex variables by surfaces in a real 4-space. The problem of determining the nonaffine projective transformations in 4-space that leave invariant the set of regular, tangent planes to a surface is studied. It is found that all such transformations are degenerate.

5. *Restriction of acreage with allowance for variation of productive capacity*, by Mr. K. C. Cartwright, Vandervoort, Arkansas.

When farm acreage was restricted to limit production, the farmer utilized his labor and better

methods on the lesser area to the effect of getting more production than he did before, nullifying the assignment of acreage. The method of reduction given is linked with the best production that has been made in the area. This makes it difficult for even one farmer to excel this best record. For all of them to excel it is virtually impossible. The remarks deal with the application of the same type of function to the initial aspect of the solubility of gases in liquids.

6. *Summability operators in topological groups*, by Mr. John Thomas, student, University of Oklahoma.

Summability operators are defined in a first-countable, Hausdorff topological group and a set of 3 sufficient conditions given for an operator to be regular. It is shown that 2 of the conditions are always necessary, and the third is also, where G is the additive group of a Banach space over a field complete in some (not necessarily Archimedean) valuation. When the Banach space is a complete field, the theorem is shown to be equivalent to the classical Toeplitz theorem and its recent extension to non-Archimedean fields.

7. *Some optimization problems related to rocket propulsion*, by Mr. G. M. Ewing, U.S.A. Artillery and Missile School, Fort Sill, Oklahoma. (By title.)

The paper formulates a number of maximum and minimum problems involving the choice of three time-functions termed the thrust program, the thrust attitude program, and the staging program, each subject to appropriate restrictions. Attention is confined to the simplest mathematical model, *viz.* the point mass, flat stationary earth, constant gravity idealization. The problems are related to, but distinct from, classical nonparametric problems of the calculus of variations. The only results known to the author are existence theorems together with complete solutions for trivial cases.

8. *The cross ratio in n-dimensional space*, by Mr. Robert Strong, student, University of Oklahoma, introduced by the Secretary.

By considering one form of the cross ratio one arrives at a generalization to a cross ratio of four points with respect to $n-1$ points in an n-dimensional space. From this expression one can show the equivalence of the definition of the cross ratio of four hyperplanes of a pencil and of the definition given by N. A. Kolmogorov in terms of hypervolumes.

9. *A dispersion concept pertaining to the damage potential of an artillery projectile*, by Mr. O. S. Spears, U.S.A. Artillery and Missile School, Fort Sill, Oklahoma.

When the vulnerable area of a target is known, the density and angular distribution of effective fragments about a bursting projectile are used to determine the probability of required damage from various positions. If $g(x, y)$ denotes the probability of damage to a target centered at the point (x, y), then a measure of effectiveness ϕ is defined as follows: $\phi=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x, y)dxdy$.

While these integrals have the dimensions of area, they are ordinarily not discussed with respect to any particular size and shape. On the other hand, when a number of projectiles are fired into a specific target area, it is often necessary to consider some specific configuration attached to the numbers ϕ. A simple approach to this matter is presented in this paper.

10. *Desmic systems of tetrahedrons*, by Professor N. A. Court, University of Oklahoma. (To be published in this MONTHLY under Mathematical Notes.)

11. *Report from Committee on Oklahoma Visiting-Lecture Service*, by Professor Katherine Mires, Northwestern State College, Chairman.

The committee has considered several possible types of visiting-lecture programs suitable for high schools, but found itself stopped from further action by lack of time and funds. It is hoped the high school lecture service will have been implemented before the October meeting of the section.

12. *Report from the Committee on High School Mathematics Contest*, by Professor H. N. Carter, University of Tulsa, Chairman.

Although the committee was late in forming, and met with several difficulties, the contest was held, thanks to the assistance of Professor Fagerstrom.

13. *Report from the advisory committee of Junior Academy of Science*, by Professor R. B. Deal, Oklahoma State University of Agriculture and Applied Science.

Over 1900 science and mathematics teachers in Oklahoma have been invited to present academic papers (in contrast with the usual "model" or "gadget" used in science fairs). Forty-seven papers were submitted for consideration, from which the program for the May 10th meeting will be selected.

R. V. ANDREE, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico, Albuquerque, New Mexico, on April 11–12, 1958. Dr. R. C. Hildner, Chairman of the Section, presided at the afternoon session on April 11, and also at the morning session on April 12. There were 63 persons in attendance, including 47 members of the Association.

The following officers were elected: Chairman, Professor J. H. Butchart, Arizona State College at Flagstaff; Vice-Chairman, Professor R. B. Crouch, New Mexico College of Agriculture and Mechanic Arts; Secretary-Treasurer, Professor Deonisie Trifan, University of Arizona.

The following papers were presented:

1. *A note on cancellation of groups of rank one*, by Professor E. A. Walker, New Mexico College of Agriculture and Mechanic Arts.

This note is concerned with the following question: if G and H are Abelian groups, if F is an Abelian group of rank one, and if $F\oplus G\cong F\oplus H$, is $G\cong H$? It has been shown that if F is infinite cyclic then $G\cong H$ (E. A. Walker, Proc. Amer. Math. Soc., Oct. 1956). Also if F is isomorphic to the additive group of rational numbers, then $G\cong H$. For arbitrary groups of rank one the following hold: (a) G is isomorphic to a subgroup of H and H is isomorphic to a subgroup of G; (b) If G is torsion free and almost locally pure (for definition of almost locally pure, see abstract 542–108 in the Notices of the Amer. Math. Soc.) then $G\cong H$.

2. *Problems in diophantine approximation*, by Professor G. M. Petersen, University of New Mexico.

Let $\{s_n\}$ be a sequence for which $0<s_n<1$ for all n, and I any interval (a, b) in $(0, 1)$. If $I(x)=I$, $x\in(a, b)$, and $I(x)=0$, $x\notin(a, b)$ then the sequence $\{s_n\}$ is well distributed if and only if $\lim_{p\to\infty}[I(S_{+n1})+\cdots+I(S_{n+p})]/p=b-a$ uniformly in p, for all intervals (a, b). This definition is a modification of that of Weyl for uniformly-distributed sequences. Different cases of when uniformly-distributed sequences are also well distributed are discussed.

3. *Cascading solutions of a nonhomogeneous equation of Mathieu-Hill type*, by Professor R. M. Conkling, New Mexico College of Agriculture and Mechanic Arts.

The differential equation $\ddot{\theta}+A\dot{\theta}+(B-C\cos\omega t)\theta=D\cos\omega t$ becomes a nonhomogeneous Mathieu equation when the first derivative is removed by the usual substitution. Upon replacing $\cos\omega t$ by a square wave $S(t)$, the method of Louis Pipes (J. App. Phys. vol. 24, July 1953, pp. 902–910) can be adapted to give the solution as a matrix equation.

4. *Government research*, by Professor Charles Wexler, Arizona State College, Tempe.

This was a proposal that the government gradually disband its own groups doing theoretical research not requiring specialized and extensive equipment, and parcel out the research on a part-time contract basis to appropriate college teachers. The following advantages are seen: (1) An influx of qualified technical personnel into college teaching where they are sorely needed. Their teaching salaries would be supplemented by the part-time contracts. (2) A better intellectual atmosphere to work in, unhampered by civil service restrictions and red tape. (3) More first-class minds contributing to the research. (4) As much or more research as is done now, at less cost.

5. *The ubiquitous integral equation of Abel*, by Dr. G. M. Wing, Los Alamos Scientific Laboratory.

The integral equation $(*) F(t)=\int_0^t (t-s)^\mu G(s)ds$, $\mu>-1$, and its generalizations are considered. Various classical instances in which (*) arises are discussed. Two recent problems at the Los Alamos Scientific Laboratory have led to equations of type (*). One has to do with an x-ray technique for determining the density ρ of material in a sphere, where ρ depends upon the radial position only. The other is an attempt to determine whether the neutrons observed in an experiment related to the controlled thermonuclear reaction arise from such a reaction or whether they have a spurious source.

6. *Some remarks on splitting theorems for monomial groups*, by Professor R. B. Crouch, New Mexico College of Agriculture and Mechanic Arts.

Let H be a group, U a set and $\Sigma(H; B, d, C)$ the monomical group of H over U. Here $o(U)=B$ and $d\leqq C\leqq B^+$. The normal subgroups of Σ are known (R. B. Crouch and W. R. Scott, *Normal Subgroups of Monomizal Groups*, Proc. Amer. Math. Soc., vol. 8, 1957, pp. 931–936). Let N be a normal subgroup of Σ contained in the basic group V. The question then arises, does Σ split over N? Denote by G the subgroup of H from which the factors of multiplications of N may be chosen. Let the subgroup G_2 of products of factors of multiplication of N be G. Assume that H splits over G and $H=GK$. It can then be shown that Σ splits over N, $\Sigma=NR$ and R is the weak direct product of K unioned with the symmetric group $S(B, C)$.

7. *A set theoretic counterexample*, by Dr. W. W. Bledsoe, Sandia Corporation, introduced by the Secretary.

Let F be a family of sets and $S=\bigcup B \in FB$. DEFINITION. *Borel* F = *The smallest* σ-*field containing* F. Due to A. P. Morse is the following DEFINITION. *Bor* $F=EA$ (A *is* ϕ-*measurable whenever* ϕ *measures* S *and* $F\subset$ *measurable* ϕ). THEOREM. *Borel* $F\subset$*Bor* F. Well-known is THEOREM A. *Borel* $F=\bigcup G\subset F$; G *is countable Borel* G. It is shown by counter-example that the result of Theorem A no longer holds whenever "Borel F" is replaced by "Bor F."

8. *Determination of atmospheric density from satellite observations*, by Dr. Herbert Knothe, Holloman Air Development Center, introduced by the Secretary.

9. *Properties of the tractrix and catenary*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

Using synthetic methods, Professor Butchart showed that the catenary is the evolute of the tractrix, that the tractroid of revolution has constant negative total curvature, that the catenoid of revolution has less area than any other surface joining two circles on the same axis, and that this surface has zero mean curvature. He also obtained the derivatives of the inverse trigonometric functions directly from diagrams and showed that the locus of the focus of a parabola rolling on a line is the catenary.

10. *Mathematical problems arising in engineering analyses of heat transfer processes*, by Dr. Knox Millsaps, Holloman Air Development Center, Holloman, New Mexico.

The mathematical problems that arise in the solution to the partial differential systems which

describe the heat transfer from a solid boundary to a moving fluid are outlined. The particular case of the heat transfer to Hagen-Poiseuille flows is given in some detail as a typical example of the mathematical analysis. Possible generalizations to other physical situations are also discussed.

11. *$\sum(H, A, C, D)$ does not split over $\sum(H, A, D, d)$*, by Mr. A. B. Gray, Jr., New Mexico College of Agriculture and Mechanical Arts, introduced by the Secretary.

It is well known that $\sum(H, A, D, d)$ is a normal subgroup of $\sum(H, A, C, B)$ with $D \leqq C$, $d<B$. The group $S(A, B)$ does not split over $S(A, d)$. (This result was communicated to me by W. R. Scott.) If $\sum(H, A, C, B)=\sum(H, A, D, d) \cup L$ and $\sum(H, A, D, d) \cap L$ is the identity then let T be the set of all s such that vs is in L where v is a multiplication with less than C nonidentity entries. The set T is a subgroup of $S(A, B)$, $S(A, B)=S(A, d) \cup T$ and $S(A, d) \cap T$ is the identity. Thus $S(A, B)$ splits over $S(A, d)$ contradicting the above so $\sum(H, A, C, B)$ does not split over $\sum(H, A, D, d)$.

12. *Science, society, and survival*, by Dr. J. W. McRae, President of the Sandia Corporation.

13. *Loci associated with families o plane osculants*, by Mr. Louis Child, New Mexico College of Agriculture and Mechanic Arts.

At P_0 of a plane curve Γ, a member f_4 of the 5-parameter family F_4 of 4-pointic cubics meets each member γ_4 of the 1-parameter family C_4 of 4-pointic conics in 4 points at P_0 and ordinarily at 2 additional points P_1, P_2 whose abscissas x_1, x_2 are the roots of $\psi_0 x^2+2\psi_1 x+\psi_2=0$ where the ψ_i are polynomials of degree 4 in λ, the parameter of C_4. The roots $\lambda_1, \cdots \lambda_8$ of $D=\psi_0\psi_2-\psi_1^2=0$ then determine points $P_1, \cdots P_8$ at which f_4 doubly contacts the corresponding conics. The Halphen point $H(f_4)$ for $P_1, \cdots P_8$ is thus unique for f_4. The set $H(F_8)$ of all such Halphen points for the members f_8 of the 1-parameter subfamily F_8 of 8-pointic cubics forms a Halphen curve associated with P_0, and the point $H(f_9)$ on this curve corresponding to the ordinarily unique 9-pointic cubic is a second Halphen point associated with P_0.

14. *Some formulas related to least squares smoothing and predicting formulas*, by Mr. W. L. Shepherd, Texas Western College.

Consider the points $(0, z_0)$, $(1, z_1)$, $\cdots$. By suitable translation we have

$$(1) \qquad V\sum_{i=0}^{p-1} z_{i+k} = m_k\sum_{i=0}^{p-1} i + b_k\cdot p, \qquad\qquad (2) \qquad \sum_{i=0}^{p-1} z_{i+k}\cdot i = m_k\sum_{i=0}^{p-1} i^2 + b_k\sum_{i=0}^{p-1} i$$

as normal equations for the linear least squares fit to the p points (k, z_k), $\cdots$, $(p+k-1, z_{p+k-1})$, yielding a predicted value, say $\bar{z}_{p+k}$, for z_{p+k}, given by $\bar{z}_{p+k}=m_k p+b_k$. The formulas

$$(3) \qquad \sum_{i=0}^{p-1} z_{i+k+1} = \sum_{i=0}^{p-1} z_{i+k} + z_{k+p} - z_k,$$

$$(4) \qquad \sum_{i=0}^{p-1} z_{i+k+1}\cdot i = \sum_{i=0}^{p-1} z_{i+k}i + p\cdot z_{k+p} - \sum_{i=0}^{p-1} z_{i+k+1}$$

are derived and used to obtain $\bar{z}_{p+k+1}$. (3) and (4) are generalized.

15. *Finite divisibility in Abelian groups*, by Professor D. L. Boyer, New Mexico College of Agriculture and Mechanic Arts, introduced by the Secretary.

Some comments on a paper presented to the American Mathematical Society (abstract number 542-108) by D. L. Boyer and E. A. Walker.

16. *An iterative method for determining a differential equation*, by Dr. J. A. Ward, Holloman Air Force Base, Holloman, New Mexico.

The purpose of this paper is to give an iterative method for finding the coefficients of a differential equation if the form of the equation and some points on its solution are known.

The procedure applies for variable or constant coefficients. The method was developed on a digital computer for "real time" computation and for data that includes "noise."

17. *On a decomposition of convex sets due to N. Aronszajn*, by Professor K. R. Lucas, New Mexico College of Agriculture and Mechanic Arts.

18. *On the three-choice problem in dynamic programming*, by Professor I. I. Kolodner, University of New Mexico.

Let for $i=1, 2, \cdots m$, $j=1, 2, \cdots n$: (1) $0 \leqq p_i < 1$, (2) $0 \leqq c_{ij} \leqq 1$, $c_{ij} = 1 - c_{ij}$, (3) $x_j \geqq 0$, (4) $T_i[f] = p_i(\sum_j c_{ij}x_j + f(c_{i1}x_1, c'_{i2}x_2, \cdots, c'_{in}x_n))$, (5) $T[f] = \max_i T_i[f]$. The optimal expected payoff $f(x_i, x_2, \cdots, x_n)$ of an m-choice discrete decision process is the solution of $f = T[f]$. R. Bellman showed that this equation has a unique solution which is continuous but not necessarily differentiable. For $m=2$, $n=2$, $c_{12}=c_{21}=0$ (solution f_2) a satisfactory resolution method has been found. THEOREM. *Let* $a_{ij} = p_i c_{ij}(1 - p_i c'_{ij})^{-1}$, $\bar{a}_j = \max_i a_{ij}$, $a_j = \min_i a_{ij}$, $\bar{f} = \sum_j \bar{a}_j x_j$, $f = \sum_j a_j x_j$. Then: (1) $f = \lim T^k[\bar{f}] = \lim T^k[f]$, furthermore, the first (second) sequence is antitone (isotone) and $f \leqq f \leqq f$; (2) f is an increasing function of the p_1; (3) if for some s, $\bar{a}_j = a_{sj}$ then $f = \bar{f}$; (4) for $m=3$, $n=2$, $c_{12}=c_{21}=0$ (three-choice problem, solution f_3), $f_2 \leqq f_3 \leqq \bar{f}$.

D. TRIFAN, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The annual spring meeting of the Allegheny Mountain Section of the Mathematical Association of America was held on May 3, 1958 at Washington and Jefferson College, Washington, Pennsylvania. The Section Chairman, Professor I. Dee Peters of West Virginia University, presided at the morning session. Professor W. G. Brady of Washington and Jefferson College presided at the afternoon session. There were 64 persons registered, including 28 members of the Association.

At the business meeting, the by-laws of the section were amended to define the geographical boundaries of the section and to provide for biennial election of officers. Professors E. E. Posey and J. H. Neelley, co-chairmen of the Section Committee on High School Contests, reported that 4220 students, representing 168 of the 400 high schools invited, participated in the 1958 contest. Professor J. K. Stewart of West Virginia University described their plans for a summer 1958 institute for high school teachers. It is intended to teach specific subject matter in mathematics and in physical science.

The following short papers were presented:

1. *Some remarks on derivations of Lie algebras*, by Professor G. F. Leger, University of Pittsburgh, introduced by Professor J. C. Knipp.

This paper gives a discussion of some results on derivations of Lie algebras and culminates in an example of a Lie algebra L with Levi decomposition $L = S + R$ such that $D(L)$ splits over $I(L)$ but $D(R)$ does not split over $I(R)$. This settles a question left open by the author in an earlier note (*Proc. Amer. Math. Soc.* vol. 4, 1953, pp. 511–514).

2. *Inequalities on the gravest tone of a vibrating string*, by Professor R. A. Moore, Carnegie Institute of Technology.

The lowest eigenvalue of the differential system $y'' + \lambda p(x)y = 0$, $y(0) = y(a) = 0$, is discussed. Upper and lower bounds for λ_1 are obtained for positive continuous functions $p(x)$, normalized by $\int_0^a p(x)dx = w$, in each of the following classes: (i) $p(x) \leqq m^2$, (ii) $p(x)$ concave, (iii) $p(x)$ convex. In case (i) the result is a generalization of Liaponnoff's inequality: $\lambda_1 \geqq 4/(aw - w^2/2m^2)$.

3. *Some algebraic, analytical and topological properties of algebraic curves*, by Professor Mario Benedicty, University of Pittsburgh, introduced by Professor J. C. Knipp.

Exposition of some modern developments of the most important branches of algebraic geometry, such as theory of equivalence on an algebraic manifold, fields of algebraic functions, abelian functions and group varieties, and their relationships with classical properties of algebraic

curves, in particular of elliptic cubics, such as linear series, algebraic functions of a point, elliptic integrals, birational transformations of the curve onto itself.

4. *What must be done about a new type of freshman*, by Professor J. H. Neelley, Carnegie Institute of Technology.

This paper opposes the teaching of analytic geometry and calculus by high schools. It points out that there is no credit that can be given by colleges for such courses; that many high school teachers are not properly prepared to teach these subjects; that most failures in freshman mathematics can be traced to algebraic ignorance. Hence, it advises more algebra instead of college courses in high school.

5. *Airy's Differential Equation and Airy Functions*, by Professor William Laird, University of Pittsburgh.

A brief historical account of the researches of Sir G. B. Airy is presented together with the investigations leading to certain improper integral functions which satisfy the equation $y''-xy=0$. Properties of these solutions, called Airy Functions, are described together with a discussion of the differential equation and its application.

6. *An inverse Sturm-Liouville problem*, by Professor A. D. Martin, Carnegie Institute of Technology.

The *focal function* $f(a, \lambda)$ of a Sturm-Liouville equation $(ry')'+py=0$ is defined to be the first zero to the right of a of the unique solution $y=u(x, \lambda)$ such that $u(a, \lambda)=1$ and $r(a)u'(a, \lambda)=-\cot\lambda$. It is then proved that if two equations $(r_iy')'+p_iy=0$, r_i, p_i continuous functions on $(-\infty, \infty)$ while $r_i>0$ there for $i=1, 2$, have focal functions f_1 and f_2 respectively such that for each x, $f_1(x, \lambda)=f_2(x, \lambda)$ for a sequence of values of λ which tend to zero, then $r_1(x)\equiv r_2(x)$ and $p_1(x)\equiv p_2(x)$.

7. *Experimental science, mathematics, and the undergraduate*, by Dr. D. H. Shaffer, Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania.

It is becoming increasingly important for the worker in the physical sciences and engineering to recognize the existence of variability and chance occurrences in experimental work and to learn of the methods available for making sensible inferences from data in their presence. For such an awareness to be realized, the basic concepts and techniques of statistics must be taught (1) as a branch of mathematics, (2) to all students of science and engineering, and (3) as early as possible in the academic program. The teachers of mathematics should assume the responsibility of leading these students in this fascinating but underemphasized science.

B. H. MOUNT, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-seventh annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois College, Jacksonville, Illinois, on May 8 and 9, 1958. Professor C. T. McCormick, Chairman of the Section, presided at all sessions. There were 66 persons in attendance, including 50 members of the Association.

At the business meeting on Friday afternoon the following officers were elected to serve for the coming year: Chairman, Professor A. E. Hallerberg, Illinois College; Vice-Chairman, Professor B. K. Brown, Milliken University; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University.

Following the brief welcome by Professor E. B. Miller of Illinois College, the following program was presented:

1. *A study of the difficulties in algebra*, by Professor G. H. Miller, Western Illinois University, introduced by the Secretary.

This study was based on the results of an analysis of survey forms given to 389 students in two universities and two junior colleges in California. The recommendations based on the con-

clusions of the study were: (1) That more time be allotted to the topic of exponents and their interelationships with related topics in algebra such as logarithms, annuities, binomial expansion, and progressions; (2) the development of better teaching techniques for the difficult topics in algebra, *i.e.* probability and statement problems; (3) the suggestion that a study be made to determine the most vital topics for instruction of higher mathematics and science.

2. *A letter of Philip Melanchthon*, by Professor Marian Moore, Southern Illinois University.

Philip Melanchthon, 16th century educator and religious reformer, wrote many texts, edited many classics, and furnished introductions to others. In particular he wrote an introduction to the 1537 Basle edition of Euclid, which was a reissue, with slight changes, of the 1516 Paris edition. Melanchthon's introduction, *A Letter to Young Students*, is chiefly a plea for humanism in education. The notion of an axiomatic system being chosen for its usefulness was evidently unknown to Melanchthon, who apparently agreed with an ancient view that geometry was handed down from Olympus.

3. *The use of semigroups in the teaching of modern algebra*, by Professor A. O. Lindstrum, Jr., Knox College.

The author defined a semigroup and illustrated it by using a mapping of a set into itself so that many important and fundamental concepts such as the associative law, the commutative law, and the cancellation law could be made clear. The use of semigroups in defining factoring, in stating multiplicative properties of ideals, and in constructing linear associative algebras was also briefly discussed.

4. *Peeks behind the Curtain*, by Professor S. S. Cairns, University of Illinois.

This nonmathematical after-dinner speech compared Soviet with American educational philosophies, problems and procedures.

5. *UICSM course in deductive plane geometry*, by Professor H. E. Vaughan, University of Illinois.

The paper dealt with a set of postulates adequate to a rigorous development of Euclidean plane geometry.

6. *Evaluation of National Science Foundation Summer Institutes*, by Professor M. Anice Seybold, North Central College.

The author recounted some of her experiences as an evaluator of National Science Foundation Summer Institutes in Science and Mathematics during the summer of 1957. She discussed physical facilities, students, teachers and courses, illustrating how they varied in the institutes she visited. New kinds of courses and new kinds of teaching are evolving as a result of these institutes. There is some indication that a new master's degree in the teaching of subject matter may develop.

7. *An experiment with modern mathematics*, by Professor Flora Dinkines, University of Illinois, Chicago Undergraduate Division.

The paper presented a report of noncredit lectures provided twice a week in modern mathematics.

8. *A new approach to the teaching of intermediate mathematics*, by Professor Karl Menger, Illinois Institute of Technology.

Summer Institutes initiating high school teachers into concepts of modern algebra and ideas of topology undoubtedly are stimulating experiences to many participants. But it is not apparent how the teachers might extensively utilize those ideas in their own teaching of intermediate mathematics. The main difficulties that beset their students concern the antiquated conceptual and

symbolic frame in which intermediate mathematics and its applications are being presented: the discrepant meanings of the term *variable;* the anonymity of the identity function; the lack of articulate rules for the use of important symbols; and, especially, the inconsistent uses of the letters x and y. What high school teachers ought to be trained in is, above all, the clarification of those obscure ideas.

A. W. McGaughey, *Secretary*

MAY MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at the University of Missouri, Columbia, Missouri, on Saturday, May 3, 1958. Professor J. D. Elder, Vice-chairman of the Section, presided at the morning session, and Professor H. D. Brunk, Chairman of the Section, presided at the afternoon session. In attendance were 52 persons, including 34 members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor Francis Regan, St. Louis University; Vice-Chairman, Professor H. M. MacNeille, Washington University; Secretary-Treasurer, Professor Louise Beasley, Lindenwood College. The recommendation of the Committee on High School Contests, namely, that the Missouri Section initiate and conduct the M.A.A. High School contests in the high schools of Missouri in 1959, was voted upon favorably.

The program was as follows:

1. *Convergence of a certain continued fraction*, by Professor David Dawson, University of Missouri.

H. von Koch (*Ofersigt af Kongl. Vetenskaps-Akad. Forhandlinger*, vol. 52, 1895) showed that the continued fraction $1/1+a_1/1+a_2/1+\cdots$ converges in case infinitely many of the a_p are distinct from zero and $\sum a_p<1$. Dennis and Wall (*Duke Math. J.*, vol. 12 (1945), pp. 255–273) improved this result by use of some general convergence theorems. The purpose of this paper is to show that the continued fraction converges in case $\sum a_p \leqq 1$ and three of the a_p are distinct from zero. The proof is elementary, involving only the continued fraction algorithm. If $a_1=a_2=-\frac{1}{2}$, $a_p=0$ for $p>2$, then the continued fraction does not converge.

2. *A natural metric group associated with a metric space*, by Professor J. W. Riner, St. Louis University.

Let (X, d) be a metric space and let H be the free group with free basis X satisfying the relations $xy=yx$ and $x_2=1$ for all x, y in X. Let G be the set of all elements in H of the form: $x_1x_2\cdots x_{2n}=u$. Then G is a subgroup of H. Let $N(u)=N(x_1x_2\cdots x_{2n})=\min.\sum_{i=1} d(x_{v_{2i-1}}, x_{v_{2i}})$ where $(v_1, \cdots, v_{2n})$ is a permutation of the integers $(1, \cdots, 2n)$ and the min. is taken over all such permutations. The function $\partial(u, s)=N(us)$, for u and s in G, is a metric making G a metric group. X is homeomorphic to a subset of G, and if X is connected, then G is connected.

3. *Summer institutes*, by Professor C. A. Johnson, Missouri School of Mines and Metallurgy.

The National Science Foundation Summer Institute program for high school teachers has grown rapidly since its inception in 1953. The Missouri School of Mines Institute for mathematics, physics, and chemistry teachers represents a typical program for improvement of high school science teaching. The Institute gives teachers an opportunity to learn implications of developments in modern mathematics for the secondary program. Teachers who participate in the Institute will be in a better position to carry out the revision in curricular offerings suggested by the Commission on Mathematics.

4. *Nonassociative algebras*, by Professor L. A. Kokoris, Washington University.

The author's recent work on nodal noncommutative Jordan algebras is used to illustrate some concepts of nonassociative algebras.

5. *A representation symbol applied to Waring's theorem, modulo p*, by Professor J. D. Elder, St. Louis University.

The author gave an expository account of the representation symbol, $[a, b, c]$, introduced by Sr. M. F. Torline, C.S.J. in her doctoral dissertation. Implicative properties were given, and their uses in problems connected with Waring's theorem were discussed.

6. *Electronic computers, information and education*, by Professor P. C. Hammer, University of Wisconsin. (By invitation.)

If it is agreed that a proof is in a chain of symbols, machines can and do prove theorems. More generally, they answer questions. The principal role of computing machines is to prove propositions and answer questions. It is an obstacle to the use of machines that mathematicians consider they are dealing with infinite processes, which they say cannot be done by machine. While it may be debated whether there are infinite processes, there is no doubt that no one deals with any process with infinite means. Hence, there are no stated proofs which could not be duplicated on a computer.

MARY L. CUMMINGS, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-first annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado State College, Greeley, Colorado, on Friday afternoon and evening and Saturday forenoon, May 9 and 10, 1958. Professor D. O. Patterson, Chairman of the Section, presided at all three sessions. On Saturday morning the Section held a joint meeting and luncheon with the Colorado Council of Teachers of Mathematics.

There were 107 persons registered for the meeting, including 67 members of the Association. Officers elected at the meeting for 1958–1959 were: Chairman, Professor N. C. Hunsaker, Utah State Agricultural College; Vice-Chairman, Professor J. W. Ault, United States Air Force Academy; and Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *Solving boundary value problems by use of Green's function in conjunction with the Laplace transform and separation of variables*, by Professor L. C. Barrett, South Dakota School of Mines.

The primary purpose of this paper is to point out how an influence function, *i.e.* Green's function, may be utilized together with the Laplace transform and separation of variables to facilitate a solution of boundary value problems of engineering and physics. Among the notable features of the method are: (a) Its capacity to yield the inverse of certain Laplace transforms without requiring recourse to complex variable theory. (b) The method enables one to escape the tedium of the step-by-step procedure, and subsequent use of superposition, usually followed in solving such problems by separation of variables. (c) Time-dependent boundary conditions present no special difficulty to the method.

2. *The radiation of waves from a point source*, by Professor R. W. McKelvey, University of Colorado, introduced by the Secretary.

The object of the paper is to obtain by a new method, a known expression for a *radiation solution* of the generalized wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} a_i \frac{\partial u}{\partial x_i} + au.$$

The coefficients a_{ij}, a_i, a are variable functions of space, but are constant in time. The matrix (a_{ij})

is positive-definite. [For an exact definition of a radiation solution, see Courant-Hilbert, *Mathematischen Physik II*, Berlin, 1937, p. 453]. The formula in question has been obtained by the methods of Hadamard. [*loc. cit.*, p. 154]. The procedure given here avoids many of the complications of those methods, while preserving the spirit. It consists of a construction process, resembling Hadamard's construction of the fundamental solution. [J. Hadamard, *Lectures on Cauchy's Problem*, New York.]

3. *Coset spaces in topological groups and their relation to the group*, by Mr. D. A. Ford, Graduate Assistant, University of Utah.

A topological group is an abstract group defined on the elements of a topological space where x^{-1} is continuous in x, and the product xy is continuous in x and y simultaneously. If H is a subgroup of a topological group G, then the space of left cosets G/H is a topological space and if H is invariant, G/H is a topological group. The paper deals with the relation of topological properties among the group, subgroup and coset space. For example, if H and G/H are compact, then G is compact.

4. *Geometries based on the undefined terms "sets" and "inclusion,"* by Professor Aboulghassem Zirakzadeh, University of Colorado.

E. V. Huntington, in 1913, introduced a set of axioms for Euclidean Geometry which was based on the undefined terms "set" and "inclusion." Using these undefined terms and changing the given axioms, it is possible to find other geometries, finite and infinite. A minimum set of axioms is given to insure that the resulting geometries are sufficiently regular. The consistency and independence of these axioms is proven.

5. *Variation of parameters in solving systems of difference equations*, by Professor L. C. Barrett and Professor F. E. Dristy, South Dakota School of Mines, presented by Professor Dristy.

The primary purpose of this paper is to illustrate how the method of variation of parameters, so familiar in finding particular integrals of nonhomogeneous linear differential equations, may be extended to determine a particular solution of a nonhomogeneous system of linear difference equations. At the same time a technique is developed for solving such a system which, in contrast to the usual procedure, may be applied directly to the system. Thus, the usual reduction of the system to a single difference equation before solving becomes unnecessary.

6. *Heat conduction with variable thermal conductivity in a sphere*, by Professor Nathan Schwid, University of Wyoming.

When the dependence of the diffusivity coefficient k/cp of the heat conduction equation upon the temperature is sufficiently significant to warrant its consideration, the equation becomes nonlinear. The quantity k, the thermal conductivity, is then a function of temperature with $c\rho$, the product of specific heat and density, constant. If we take $k/c\rho$ as $\alpha+\beta u$, where β/α is small, and u is the temperature, an approximation to the temperature in a sphere under simple boundary conditions is obtained which approximates the solution for substantial values of t.

7. *The geometry of* $f(n, \alpha) = \sum e^{ik\alpha}$, $k=0, \cdots, n$, by Professor Emeritus A. J. Kempner, University of Colorado.

An obvious vector construction is combined with the geometrical multiplication in the plane of complex numbers of all points on one curve by all points of a second curve to obtain results of which the following is representative: The "Wertevorrat" (Set of values assumed) of $\sum\sum e^{i(k_1\alpha_1+k_2\alpha_2)} = f(n_1, n_2)$, α_1/π, α_2/π irrational, k_1, k_2 independently over $0, 1, \cdots, n_1$, and $0, 1, \cdots, n_2$, respectively, $n_1, n_2 < \infty$ is given by the cardioid $a\rho \sin(\alpha_1/2) \sin(\alpha_2/2) = 1 - \cos(\theta+\alpha_1/2+\alpha_2/2)$. In this cardioid the functional values are distributed everywhere densely.

8. *Mathematics program for outstanding cadets at USAFA*, by Professor W. Milliken, United States Air Force Academy.

Three levels of mathematics have been established at the Air Force Academy. The regular course is the usual two-year engineering mathematics course with spherical trigonometry, some statistics and differential equations added. A cadet may advance from this course to the accelerated course at the beginning of the second semester. The accelerated course covers the same material plus a course in elementary statistics in a year and a half. The super-accelerated course covers this material in one year. Thus, there is extra time available for cadets in the faster programs to take additional advanced mathematics courses or other electives.

9. *Mathematical education in Europe, Britain, and the United States*, by Professor W. W. Rogosinski, King's College, Durham University, England; Visiting Professor, University of Colorado.

An attempt is made to point out and to explain the striking differences in mathematical education, both at high school and university level, as seen in Continental Europe, Britain, and the United States of America. The explanation is sought in a different philosophy of education in general which, in turn, is conditioned by different history and tradition: the scholastic idealism of Europe, its realistic variant in Britain, and the social (and materialistic) trend in American education.

10. *A sequel to Euclid*, by Professor H. S. M. Coxeter, University of Toronto. (Invited Address.)

11. *Coaxal circles and inversion*, by Professor H. S. M. Coxeter, University of Toronto.

Any two given circles belong to a pencil of coaxal circles consisting of all the circles orthogonal to any two circles, α and β, orthogonal to the two given circles. The arbitrariness of α and β is established by inverting the two given circles into straight lines or concentric circles. When inverted with respect to a sphere whose center is outside the plane, two orthogonal pencils of coaxal circles yield sections of a sphere (the inverse of the plane) by pencils of planes through two polar lines. Such a pencil of circles is hyperbolic (*i.e.*, intersecting), parabolic (touching), or elliptic (disjoint) according as the common line of the planes is a secant, a tangent, or an exterior line.

12. *Oscillation and non-oscillation of second order complex differential equations*, by Mr. R. W. Hunt, Graduate Assistant, University of Utah.

The primary object of this paper was to investigate the zeros on $a \leqq x < \infty$ of solutions of the differential equation $(py')' + fy = 0$, with p and f complex-valued continuous functions of the real variable x. By the use of an associated system of two real, second-order equations obtained by writing p, f, and y in polar form, two sufficient conditions for disconjugacy (at most one zero on $a \leqq x < \infty$) of all nontrivial solutions were obtained. Then a special form of this equation, $(y'/q)' + \bar{q}y = 0$, q complex-valued, was changed to the first order system $y' = q\bar{z}$, $z' = -q\bar{y}$, with solutions $s(x) = s[a, x; q]$ and $c(x) = c[a, x; q]$ corresponding to the boundary conditions $y(a) = 0$, $z(a) = 1$. Finally $s[a, x; q]$ and $c[a, x,; q]$ were shown to have two properties analogous to well-known properties of the real sine and cosine functions; namely, $|s|^2 + |c|^2 \equiv 1$ and, for $k = 1$, $s[a, x; kq] = ks[a, x; q]$, $c[a, x; kq] = c[a, x; q]$.

13. *A multiple integral approach to Taylor's theorem*, by Professor L. C. Barrett and Mr. D. W. Willett, Student, South Dakota School of Mines, presented by Mr. Willett.

This note presents several elementary geometrical considerations, involving lengths, areas, and volumes, which lead quite naturally to a multiple integral approach to Taylor's theorem.

14. *Evaluation of a limit from the theory of heat flow*, by Dr. H. R. Bailey, Mathematician, Ohio Oil Company Research Center, Littleton, Colorado, introduced by the Secretary.

The problem of heat conduction in an infinite homogeneous medium from the surface of a cylinder whose radius is increasing with time is solved by the Green's function method. The solution is obtained as an integral of the form $I=\int_0^t f(t, \tau)d\tau$. A method is given to obtain an explicit evaluation of this integral for $t\to\infty$ for the case of the cylinder radius increasing at a constant velocity. It is shown that the integral can be divided into two parts, $I=\int_0^{t/N} f(t, \tau)d\tau+\int_{t/N}^t f(t, \tau)d\tau$, where the last integral goes to zero as $t\to\infty$ and the integrand in the range $[0, t/N]$ can be replaced by an asymptotic expression which can be integrated explicitly as a function of N. Finally the desired limit is obtained by passing to the limit as $N\to\infty$.

15. *An application of the decomposition of a matrix into principal idempotents*, by Professor D. W. Robinson, Brigham Young University.

As a simple application of the decomposition of a (diagonable) matrix into principal idempotent elements, this note provides a proof of the following well-known result: if the nth derivative of a function f exists at α, then it can be computed as the limit of $h^{-n}\sum_{m=0}^{n}\binom{n}{m}(-1)^m f[\alpha+(n-m)h]$ as h approaches zero.

16. *Families of Sturm-Liouville systems*, by Mr. E. L. Dunn, Colorado State University, introduced by Professor F. M. Stein, Colorado State University.

A family of Sturm-Liouville systems is defined as the collection of all Sturm-Liouville systems whose equations can be obtained by repeatedly differentiating and integrating a Sturm-Liouville equation. For each system, similar boundary conditions apply to the same interval. In this paper the conditions are developed such that a Sturm-Liouville system may generate a family. It is shown that (1) if a Sturm-Liouville system generates a family, the kth derivatives and antiderivatives of its eigenfunctions form orthogonal sets, and (2) if the eigenfunctions of the generating system are not polynomials the sets of eigenvalues for all members of a family are identical. The case when the eigenfunctions are polynomials must be considered separately.

17. *Let's not go off the deep end!* by Professor A. W. Recht, University of Denver.

The general theme is in opposition to the idea of introducing Boolean algebra, sets, and similar types of theoretical mathematics into high school and elementary college courses. We are already teaching too much "gifted" mathematics to the general student, and not teaching successfully the kind of mathematics the 85 per cent or perhaps the 100 per cent, ought to have before they go into the so-called superior pure mathematics. Maybe we have an inferiority complex, and are running away from our real job, which is to bring up all people in our democracy to their full potentialities with a more democratic kind of mathematics.

18. *The work of the Commission on Mathematics*, by Professor Henry Van Engen, University of Wisconsin, Madison.

See this MONTHLY, Report of the May Meeting of the Wisconsin Section.

F. M. CARPENTER, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The twenty-sixth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Carroll College, Waukesha, Wisconsin, on May 3, 1958, Professor R. D. Wagner, Chairman, presiding. Sixty-nine attended the meeting, including forty members of the Association.

At the business meeting of the Section the following officers were elected for the coming year: Chairman, Prof. J. V. Finch, Beloit College, Beloit, Wisconsin; Vice-Chairman, Prof. C. B. Hanneken, Marquette University, Milwaukee, Wisconsin; Secretary-Treasurer, Sister Mary Felice, Mount Mary College, Milwaukee, Wisconsin.

Mr. J. W. Kennedy gave the following report of the 1958 high school mathematics contest: A preliminary contest was held on Feb. 27, in 237 schools in the state, with

9516 students participating. This contest was intended to help teachers choose students for the final contests. Scores on this preliminary test were sent to the committee who then established a score to cut off all but the top 1000 students. Teachers were free, however, to enter any student in the final contest according to their own discretion. The final contest was held at twenty-seven centers throughout Wisconsin, on April 12, with 950 participating from 172 schools.

After a short address of welcome by President Robert D. Steele of Carroll College, the following papers were presented:

1. *On linear inequalities*, by Prof. J. V. Talacko, Marquette University.

As the mathematization of behavioral and empirical sciences continues, we are concerned with solutions of systems of linear inequalities. One class of these problems, useful in Operational Research, Industry and Management, is known as the Linear Programming. The problem was formulated and, by comparison to solutions of a system of linear equations by inverse or unit matrix, the formulation of Simplex Matrix and the Simplex Algorithm was demonstrated as the computational procedure. Finally, a typical maximum mixture problem was formulated, analyzed and solved by simplex.

2. *Integration in finite terms*, by Dr. E. C. Posner, University of Wisconsin, Madison, introduced by Prof. J. V. Finch, Beloit College.

The purpose of this paper was to give some meaning to such oft-repeated statements as "e^x/x cannot be integrated explicitly," and to give some idea of their proof. Working over the complex numbers, we use functions exp and log; start with the algebraic functions, exp and log them, take all rational combinations, in fact, anything satisfying a polynomial equation with these as coefficients. Do this again and again. A function that can be obtained in this way is called elementary. THEOREM. *If* $\int e^{g(x)}y(x)dx$ *is elementary with* $g(x)$, $y(x)$ *rational, then this integral is* $e^{g(x)}w(x)+C$, *for some rational* $w(x)$. COROLLARY. $\int e^x/x\,dx$ *is not elementary.*

3. *Queuing theory and applications*, by Dr. W. A. Golomski, Oscar Mayer & Co., Madison, Wisconsin.

In trying to minimize the waiting time in lines where the input and servicing time is known or can be approximated, a broad theory has been developed in terms of the various cumulative distribution functions involved as well as by differential-difference equations or integral equations. Approximations can be made in many cases in the meat packing industry so that problems of wrapping, hauling, weighing, unloading and loading can be solved by queuing theory techniques. This type of problem requires modifications of known techniques. Examples were given.

4. *A symposium on recent trends in mathematical education*, including:

a. A report on the work of the Commission on Mathematics of the College Entrance Examination Board, by Prof. Henry Van Engen, University of Wisconsin, Madison.

The Commission on Mathematics hopes to redirect the secondary mathematics program by (1) bringing in new mathematical ideas, (2) eliminating ideas which are not in the present stream of mathematical interest, and (3) providing an organization of content based on fundamental mathematical ideas.

In providing for a new high school program, the Commission has kept in mind that calculus must remain as a college subject, but that all work preparatory to a combined analytical geometry and calculus course should be considered as high-school work.

b. Report on the Summer Institute in Social Science for College Teachers of Mathematics, by C. J. Vanderlin, Wisconsin State College, Whitewater.

In addition to lectures on economics, psychology, sociology and social psychology, the members of this institute were given background materials in mathematics, primarily in the fields of

linear inequalities, linear programming, theory of games, and probability. The morning sessions were devoted to these lectures while the afternoon sessions were spent in informal discussion, talks by guest speakers, and "problem generating." An attempt was made to formulate a group of problems, with applications in the social sciences, which could be used in the traditional freshman and sophomore programs. The results were 91 problems presented by the members of the Institute, several of which were read by Mr. Vanderlin.

These papers were followed by lively discussion arising from questions raised particularly by high school teachers present.

SISTER MARY FELICE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Marian College, Indianapolis, November 7, 1958.

IOWA, State University of Iowa, Iowa City, October 17, 1958.

KANSAS

KENTUCKY, Centre College of Kentucky, Danville, April, 1959.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 6, 1958.

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University of Agriculture and Applied Science, East Lansing, March 28, 1959.

MINNESOTA, University of Minnesota, Duluth, Fall, 1958.

MISSOURI, Lindenwood College, St. Charles, Spring, 1959.

NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN, College of the Holy Cross, Worcester, Massachusetts, November 29, 1958.

NORTHERN CALIFORNIA

OHIO

OKLAHOMA, Oklahoma City University, October 24, 1958.

PACIFIC NORTHWEST, University of Oregon, Eugene, Oregon, June 19, 1959.

PHILADELPHIA, Lehigh University, Bethlehem, November 29, 1958.

ROCKY MOUNTAIN, Utah State University of Agriculture and Applied Science, Logan, Spring, 1959.

SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.

SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.

SOUTHWESTERN, Arizona State College, Tempe, Spring, 1959.

TEXAS, University of Texas, Austin, April, 1959.

UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

WISCONSIN, Wisconsin State College, Platteville, May, 1959.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

THE WASHINGTON CONFERENCE

A conference to review the program of the Association and to formulate a plan of action was held at the Burlington Hotel, Washington, D. C., from May 16 to 18, 1958. The expenses incurred in holding the conference and in publishing its proceedings have been paid by means of a grant from the National Science Foundation. Invitations to attend the conference were sent to the officers of the Association, to the members of the Committee on the Undergraduate Program in Mathematics and of the Committee to Review the Activities of the Association, and to representatives of other interested groups.

The following thirty-three persons attended the conference:

E. G. Begle	Yale University
R. C. Buck	University of Wisconsin
E. A. Cameron	University of North Carolina
W. B. Carver	Cornell University
L. W. Cohen	National Science Foundation
C. W. Curtis	University of Wisconsin
R. L. Davis	University of Virginia
W. L. Duren	University of Virginia
H. P. Fawcett	Ohio State University
H. M. Gehman	University of Buffalo
R. D. James	University of British Columbia
B. W. Jones	University of Colorado
J. R. Mayor	American Association for the Advancement of Science
E. J. McShane	University of Virginia
A. E. Meder	Rutgers University
L. J. Montzingo	University of Buffalo
L. W. Nelson	Fund for the Advancement of Education
G. B. Price	University of Kansas
A. L. Putnam	University of Chicago
Tibor Rado	Ohio State University
Mina Rees	Hunter College
G. A. Rietz	General Electric Company
R. D. Schafer	University of Connecticut
M. A. Shader	International Business Machines Corporation
M. F. Smiley	University of Iowa
Rothwell Stephens	Knox College
R. R. Stoll	Oberlin College
Patrick Suppes	Stanford University
G. B. Thomas, Jr.	Massachusetts Institute of Technology
A. W. Tucker	Princeton University
E. D. Vinogradoff	President's Committee on Scientists and Engineers
S. S. Wilks	Princeton University
Dael Wolfle	American Association for the Advancement of Science

The topics discussed at the conference were grouped under these five headings: I. What the Nation Needs in the Area of Responsibility of the M.A.A.; II. Programs Required to Fill These Needs; III. Modernization of the M.A.A.'s Organization, Staff, and Headquarters; IV. The Funds Required; V. A Plan of Action.

Members of the Conference were invited to prepare working papers on topics coming under the above headings. Most of these papers were mimeographed and distributed to participants in advance of the meeting. Summaries of the prepared papers and of some of the discussions are given herewith.

Price: Purpose of the Conference. The purpose of the Conference is to study some of the problems created by the revolution in mathematics and to seek solutions for them. The revolution in mathematics and the rapidly increasing enrollments in high schools and colleges have resulted in unprecedented needs for mathematics teachers in high school and college, mathematicians for industry, preparation of students for graduate work in mathematics, modernized college mathematics courses, undergraduate programs for the training of mathematicians, textbooks from which the modernized mathematics courses can be taught, development of film and television aids to teaching, more efficient methods of instruction (such as the teaching of large sections), and expositions designed to help assimilate mathematical research into the body of common knowledge.

Besides considering the nation's needs in mathematics and programs which will help to satisfy them, some means must be found for initiating, organizing, operating, and supervising these programs. The Conference is to consider also the role of the M.A.A. as a professional organization and an operating agency.

The rapid growth of the Association in recent years has emphasized its strength and the appeal of its program, but growth has also brought problems. There are now approximately 7500 members, with a membership of 12,000 to 15,000 indicated by 1970. Approximately 23% of the members are now employed in business, industrial, and government positions.

The Association's program and responsibilities have undergone explosive growth, but there has been no corresponding growth in its staff, facilities, and ability to deal with its problems. The transition from an organization operated by a volunteer staff in donated headquarters space to one operated by a paid professional staff in rented quarters has begun. Nevertheless the officers of the Association are for the most part still chairmen of departments of mathematics. The Association needs leaders who have the time required to organize new activities and to locate the personnel and funds to operate them.

This Conference should study the nation's needs in mathematics and plan a program for the Association, and it should also seek ways to strengthen the M.A.A. so that it is equal to the tasks which lie before it.

Jones: Visiting Lectureship Programs of the M.A.A. Since 1954, the Association has administered, with the support of the National Science Foundation, a Visiting Lectureship Program for colleges. Its general aims are to strengthen and stimulate the mathematics programs of the colleges, to provide personal contacts with productive and creative mathematicians to staff members and students in the colleges, and to aid in motivating able college students to consider careers in mathematics.

A feature of the program has been its adaptability to local needs. In addition to formal lectures, each lecturer has conferred with students on future opportunities for study and employment, discussed problems of teaching and curriculum with faculty, disseminated information about what is going on elsewhere in the field of mathematics, and talked with administrators.

During the past four years there have been 16 lecturers, spending a total of 34 months on tour, and visiting a total of 270 institutions. During 1958–59 there are plans for 8 lecturers for a total of 14 months.

Visiting lectureship programs for high schools have been sponsored by the Kentucky and Northern California Sections of the Association, resulting in increased interest in mathematics among high school students, teachers, and administrators. The Association expects to sponsor a secondary school lectureship program on a wider scale starting in September 1958.

Buck: Mathematical Competitions. The reasons for initiating high school mathematical contests on a statewide or national basis are: to give official recognition by awards, presented in the name of the M.A.A., to some of the better students now taking mathematics courses; to discover and encourage talented students; to motivate students to

take additional mathematics courses; to encourage highly-gifted students to consider mathematics as a career; to give tactful guidance to the high school curriculum by indicating the level of competence that could be expected of the better students.

Several different types of examinations have been used to achieve these objectives. Since students differ in the amount of preparation, one method is to offer the contest at two levels depending on courses completed. Another solution is to devise a test which could be passed successfully by a very able sophomore, but could not be passed by an *average* senior with four years of mathematics. The Wisconsin Section has chosen this solution.

Two types of examination have been used: the *achievement* type and the *aptitude* type. The achievement type is characterized as a multiple choice test which attempts to measure a wide sampling of basic concepts and skills. The questions are of varying difficulty and are drawn from standard subject matter. It can be machine graded and is especially suited to a large-scale contest. The aptitude type is marked by an emphasis upon originality and insight rather than upon routine competence. It does not seem possible to design a multiple choice test of this type. This poses a difficult problem in the administration and grading of such a test where large numbers of entrants are involved.

A modification of the current national competition is proposed under which the contest would have two stages. The Preliminary would be held in February and would be a multiple choice test given in the high schools and proctored by local teachers. Since this test is to serve as a selection screen for the Final, the level should be set to select approximately the top 5% of the entrants. Statewide and local winners might be announced on the basis of the Preliminary examination. The Final Contest would be held at certain centers scattered through each state, usually at colleges and universities. This examination would be of the aptitude type. National awards and prizes of substantial value would be awarded on the basis of the Final Contest.

A summary was given of the Mathematical Olympiads of the U.S.S.R., the Eötvös Competition in Hungary, the Stanford University Competitive Examination, and the contests sponsored by the Wisconsin Section and the Metropolitan New York Section of the M.A.A., and by the National Committee of the M.A.A. in 1958 with the cooperation of the Society of Actuaries.

Discussion: Professor Rado stated that the problems in the Eötvös Competition were set by research mathematicians and that great prestige was attached to winning an Eötvös examination.

The Putnam Competition for undergraduates in colleges and universities was described as an example of an aptitude type of examination.

Meder: Textbooks. There are three types of courses for which syllabi, textbooks, and other teaching materials are needed: college freshman courses, major courses for prospective high school teachers, and other unconventional undergraduate courses. It would be an appropriate activity of the Association to issue a report discussing possible syllabi for freshman courses and the relative advantages of each; to appoint a committee to study the proper content for courses for prospective teachers and to make recommendations and prepare teaching materials; and finally to suggest and outline various types of unconventional courses. There is needed an overall directing committee, working committees in each major area, and specific production groups. Actual publication of the books written by these groups might be undertaken by the Association.

Begle: Textbooks. A writing group to produce a series of sample high school texts will assemble at Yale University in the summer of 1958. The group will consist of both high school and university teachers. It is hoped that first drafts of the texts will be completed by the summer of 1959 and that these texts will influence commercially-published books thereafter.

It is planned also to publish monographs on topics not ordinarily taught in the high schools. They are to be written by eminent mathematicians and are intended for the

better students, their teachers and the educated public. Texts for junior high school pupils will probably be written later.

Discussion: The need for both new textbooks and teachers manuals at the high school level was mentioned. The need for better preparation of prospective teachers was emphasized, as well as for upgrading of present high school teachers. Professor Fawcett stated that teachers are sincere and earnest in wanting to study more mathematics, but need special courses which are not often available.

Price: Summer Institutes. The National Science Foundation is supporting a major program of summer institutes for high school and college teachers of mathematics. In some cases demonstration classes of high school students have been employed with great effectiveness in conjunction with the institutes for teachers.

The institutes have been highly successful in accomplishing the purposes for which they were established. The participants have been brought into contact with current mathematics, and the encouragement and recognition they have received as participants have been highly stimulating.

Two trends have been observed which should be reversed: first, summer institutes for college teachers of mathematics have been almost completely eliminated; second, the number of summer institutes in mathematics, relative to the number in other fields, has been decreased. Because of the large number of high school teachers of mathematics and the importance of the subject, the percentage of institutes in mathematics is obviously too small.

If there should be a significant increase in the number of summer institutes in mathematics in 1959, the Association should stand ready to give help and advice concerning the planning, organization, and operation of these institutes.

Mayor: Summer Institutes. It has been estimated that there are 150,000 secondary school teachers of science and mathematics and that more than half of these are teachers of mathematics. This situation should be recognized in planning for summer institutes, and a much larger part should be institutes for mathematics teachers alone.

While it is true that in a considerable number of summer institutes mathematics courses were offered along with science courses, it appears that in many institutes a first consideration was for science. While all would recognize that it is important for mathematics teachers to study science as well as mathematics, it appears more important, in this time of curriculum modernization, that secondary school mathematics teachers be provided the opportunity in a summer institute to give their full-time study to the field of mathematics.

Discussion: Mention was made of other N.S.F. institutes, such as Academic Year Institutes and In-Service Institutes, also of institutes sponsored by industry, such as General Electric and Shell. It was suggested that a series of institutes be established, each emphasizing the applications of mathematics to one of the social or physical sciences.

Suppes: Some Possible Revisions of Mathematics Curricula. At Stanford University a course in calculus, not including foundations or set theory, has been given for behavioral scientists for the past four years. Emphasis is placed on probability, which seems to be most useful to the students, and which provides many nontrivial examples of application. Possibly material in probability theory will be introduced into the regular calculus sequence. Material on applications is needed for a number of other courses, such as matrix theory, where the students are mainly electrical engineers, physicists, and psychologists.

The acceleration of high school mathematics programs raises the problem of advanced standing in college. On the other hand is the problem of the large amount of remedial mathematics now done in college, particularly at state universities.

Finally there is the problem of the mathematical education of gifted children. Experiments are under way in the teaching of mathematical logic to sixth, seventh, and eighth

graders and in the teaching of plane geometry constructions to first graders. The present mathematical curriculum realizes a pitifully small part of the mathematical potential of the abler students.

Cameron: Undergraduate Honors Programs in Mathematics. Honors programs constitute one of the ways in which American colleges and universities attempt to solve the difficult problem of providing appropriate training for the more able students in a system committed to the education of all the people. Traditionally, honors programs have been limited to the upper years of college and they have consisted principally of a considerable amount of solitary study on the part of the student. There appears to be an increased interest in a rather different form of honors work.

Swarthmore has long been known for its honors work in several fields. More recently, Carleton and Dartmouth Colleges have inaugurated some very exciting honors programs in mathematics. These programs start with the freshman year and provide special attention for superior students throughout the four years of college. In these programs special work is not limited to individual study. The top students are together in classes or seminars with their peers for much of their work.

The importance of starting special work for gifted students in the freshman year should be emphasized. Many promising mathematics students have been lost by being subjected to dull and uninspiring first-year courses.

The Advanced Placement Program should have a direct influence on undergraduate honors programs. Talented students entering college with advanced training in mathematics constitute likely candidates for honors work. The proper education of superior students, from which group the mathematicians of the future must come, is surely a matter of great concern to us all.

Rees: Recruiting of Teachers of Mathematics. The extent of the need for additional teachers of mathematics was discussed. A solution is to recruit into mathematics those who are committed to teaching as a career but in a field where the need for teachers is less urgent. The long range problem requires that we interest more students in mathematics as a career, and that we provide better training for these students. We need better-equipped teachers who are assisted throughout their careers to remain alert and interested.

To interest more high school students in the study of mathematics requires beneficial relations between college faculty members and high school teachers. Even more important are contacts with high school guidance counselors, who need to be equipped to give sound advice to young people about the nature of a career in mathematics.

Another plan is to use lecturers to promote an interest in mathematics, especially lecturers from industry. Books and monographs afford another method for arousing interest.

Thomas: Expanded Programs at Meetings of the M.A.A. Five years ago the typical summer and annual meeting of the Association consisted of two half-day sessions either preceding or following the meetings of the Society. Since 1957 the summer meeting has been scheduled for four days, from Monday to Thursday, overlapping the meetings of the Society but avoiding conflicts with the general sessions of the Society. Similarly the Annual Meeting of January 1958 consisted of three sessions. The change in time of the Annual Meeting from December to January did not discourage attendance.

The reasons for the expanded programs of meetings is due to the increase in the membership of the Association, the broadening of interests of the membership, and the increase in the activities in which the M.A.A. is engaged.

Meetings of Sections are either one-day or two-day meetings. The variety of topics covered roughly parallels that at national meetings, with closer cooperation between college and high school groups, and, in particular, attention to matters of local responsibility and activity.

Meetings of the Association provide us with a variety of experiences. We renew

acquaintances, we talk shop, we make new friends, we become informed of what the various committees and projects of the Association are doing, we have the opportunity to learn something new in mathematics, and we may pick up some pointers on curriculum or on teaching which will make us more effective teachers. The longer meetings provide increased opportunities for us to realize the benefits of the experience just described.

Curtis: Expository Writing. It is suggested that a new expository journal be established under the auspices of the Association, each number to consist of a single article, to be self-contained, and to include a reasonably comprehensive account of its subject. Some articles should be aimed at research mathematicians, some at undergraduates or high school teachers. Substantial royalty payments should be made to authors.

It was also recommended that the Association publish lists of recommended books and articles with brief critical comments.

Buck: Expository Writing. Three separate expository series of books are suggested: Monographs of high-level expositions directed toward the active research mathematician who desires a survey of the present state of a restricted area of mathematics; college-level expository articles; high-school-level series.

Discussion: The concensus was favorable toward the idea of a journal or a series of books of the types discussed, but the difficulty in obtaining authors for the present Carus Monographs and Slaught Papers seemed to make it unwise to attempt to expand the present program of expository publications of the Association. It was pointed out that the most economical method of putting mathematical material into the hands of a large group was through publication in the MONTHLY rather than through the establishment of new periodicals or series of books.

In connection with expository writing, attention was called to the Chauvenet Prize awarded every three years by the M.A.A. for an outstanding expository article.

James: Publication Program of the M.A.A. The official journal of the Association, the American Mathematical Monthly, is issued ten times a year. Usually there is a supplement in the form of a Slaught Paper, although in 1957 there were two supplements, a Slaught Paper and the Dunkel Problem Book. The total number of printed pages (not including advertisements) was 760 in 1956 and 778 in 1957.

There is a waiting period of about a year for all accepted articles. This time interval is too long. Since the volume of mathematical material will increase, some thought should be given to the possibility of eliminating or reducing some departments of the MONTHLY.

The section on News and Notices, and the department of Official Reports and Communications of the M.A.A. might be eliminated or reduced in size. There is nothing in the by-laws that makes it mandatory to publish either the names of new members or reports of Section meetings. The question of the publication of book reviews that may be duplicated in other journals is a more difficult one.

The Association might very well consider increasing publication to twelve issues a year and including more 96-page issues. (The normal size is 80 pages.)

For expository articles, it would seem desirable to concentrate on expanding the series of Slaught Papers. It is difficult to persuade mathematicians to write expository articles, and it is by no means certain that payment for writing is the answer.

The editor is responsible for the acceptance of main articles, and the section editors for the material in their respective sections. It is clear that eventually the Association should have a central editorial office where the routine work of marking manuscript and reading proof can be done. Such an office will probably be required before the end of the term of office of the present editor.

Price: Facilities for Departments of Mathematics. The revolution in mathematics has greatly increased the needs of departments of mathematics for adequate department offices, offices for staff members, secretarial assistance, and special equipment and special facilities of various kinds. Buildings for departments of mathematics are still a rarity, and the problem of facilities for departments of mathematics is largely ignored.

Departments of Mathematics are asked to teach and to produce research. Research reports must be prepared and issued. Summer institutes must be organized and operated. Courses must be revised and text materials must be written and reproduced for class use. Staffs have become large and the administration of departments of mathematics has become burdensome. Proper facilities of all kinds would greatly increase the effectiveness of the nation's limited supply of mathematicians.

Davis and Tucker: Committee on the Undergraduate Program in Mathematics. The present committee has asked to be discharged as of September 1, 1958. It is expected that the committee will be reorganized on a larger scale.

During the five years of its existence, the committee has published four volumes of experimental textbook material: *Universal Mathematics* (two parts) and *Modern Mathematical Methods and Models* (two volumes). *Universal Mathematics* is now out of print but has been rewritten and will be available before the end of 1958. Artin's notes on *A Freshman Honors Course in Calculus and Analytic Geometry* have also been published by the C.U.P.

Gehman: Brief History of the M.A.A. The Mathematical Association of America was organized in December 1915 to cultivate collegiate mathematics (as contrasted with secondary mathematics and with mathematical research). It adopted as its official journal the American Mathematical Monthly, founded by B. F. Finkel in 1894. The object of the Association is, briefly, to assist in promoting the interests of mathematics. Membership is open to anyone interested in the field of mathematics.

With a Board of Governors of 42 persons, and 27 Sections with three officers each, there are (in spite of duplications) over 100 persons officially connected with the government of the Association. This is desirable; it encourages the feeling of loyalty that most members have for the Association.

Sections should be established in individual states, rather than having a Section include several states. As the number of Sections increases, a change in the form of government may be needed with an enlarged Executive Committee acting as a legislative body, and a Board of Governors consisting of sectional representatives and members-at-large acting merely in an advisory capacity.

It is recommended that the By-laws of the Association be amended as soon as possible to provide for the separation of the offices of Secretary and Treasurer. In time the Association will need a full-time paid Business Manager or Executive Director. In the case of the Editor, the Secretary, and the Treasurer, financial arrangements should be made with his institution to relieve him of part of his teaching load.

The Association holds various permanent funds, largely as bequests, whose income is used to support the publication program of the Association. The General Fund, amounting to $40,000, represents the surplus which has accumulated since 1915 because the cost per member was for many years less than the amount charged for dues. The cost of operation has been low because of the contributions of time and money made by many mathematicians. The magnitude of the operations now conducted by the Association requires more time on the part of its officers and hence requires greater financial support.

Mayor: Paid Executive Staff and Office for M.A.A. The recommendation that the M.A.A. establish a permanent, paid executive staff functioning under elected officers and a permanent national office in Washington is based on the need for a national organization, which will have as a principal purpose the improvement of mathematics education at the college level, and the need for a person in a full-time administrative position in Washington to represent mathematics. Many meetings concerned with science and mathematics education during the past three years in Washington have had no official representative of mathematics. This has resulted in mathematics failing to receive its needed share of attention in the development of programs.

Discussion: Wilks described the organization of the Washington office of the Ameri-

can Statistical Association. Cohen urged the establishment of an office of the M.A.A. in the Washington area. It was pointed out that the matter of public relations might be cared for through the establishment of a Washington office of the Conference Board of the Mathematical Sciences.

Stephens: The Role of the Sections in the M.A.A. A program by which the central organization might strengthen the Sections would include the establishment of a Vice-President for Sectional Affairs whose sole function would be to promote the welfare of the Sections. The MONTHLY should carry reports on important activities of Sections. Funds available to Sections should be increased. Each Section should be urged to establish a Committee to Study the Activities of the Section. The Association should recommend to the Sections speakers who are informed on and active in the Association's work.

Many of the Association's current activities have had their origins in the Sections, such as the contest for high school students, and the visiting lectureship program for high schools. But more can be done. Sections should interest themselves in providing increased facilities in mathematics for the gifted student. They can urge the establishment of courses, assist with Science Fairs in making mathematics an integral part of the exhibit, provide awards for outstanding college undergraduates, and can speed revision of the curriculum by sponsoring intensive courses on phases of mathematics not generally known by a good segment of the teaching profession.

Stoll: What is the Role of the Sections of the M.A.A.? At the organization meeting of the Association in 1915 three Sections were organized. Within ten years, seventeen Sections were in operation. At present there are 27 Sections covering all of the U. S. and Canada.

Section meetings show a great diversity in nature. The majority tend to offer a balance of talks on new mathematics, collegiate mathematics, and applied mathematics. Currently there is great interest and concern with high school mathematics and related subjects. A Section might indulge in self-help by providing that its members be well informed on the current emphasis in mathematical research and new applications of mathematics. It might help others by assuming an active role in the revolution that is currently underway in the high school mathematics curriculum. The program now under way in certain Sections to stimulate interest in mathematics and to encourage exceptional students in high schools could be greatly extended.

There are opportunites for a Section to perform a service at the state level by cooperating with state agencies and other scientific societies in overcoming existing deficiencies in the training of secondary teachers.

Mayor: State Programs for the Improvement of Science and Mathematics Education. It is probable that Federal funds will become available during the next few years to assist State Academies of Science in the development of programs for the improvement of science education. It is important that such activities include mathematics. Since many of the Academies do not include mathematics sections, some planning should be done to provide for cooperative programs involving the State Academies and the Sections of the M.A.A. It is recommended that the Sections seek opportunities to work in cooperation with other scientific societies and with State Academies.

Mayor: M.A.A. and National Council of Teachers of Mathematics. It would be desirable that these two organizations sponsor a number of joint committees and that they publish jointly certain publications of interest to secondary and college teachers. At least one annual meeting should be a joint meeting of the two organizations, and within the states, one meeting of the Section of the M.A.A. and the affiliated group of the N.C.T.M. should be held jointly each year.

Mayor: Membership. Substantial gains in membership of M.A.A. have been made in the past five years. Ways and means should be investigated to bring about an increase in membership. A few possibilities are: organization of special campaigns to be conducted by the Sections of the Association, joint meetings with other groups, solicita-

tion of membership during summer and academic year institutes, and by asking each present member to obtain one additional member.

RESOLUTIONS ADOPTED BY THE WASHINGTON CONFERENCE

A. *Resolutions on Institutes*

1. The Secretary is instructed to transmit the following resolution to the appropriate committees of the Congress:

Whereas the National Science Foundation has found it possible to support institutes for college teachers of mathematics and science only to an extent completely inadequate to meet existing needs, and

Whereas there is great need for these institutes both to bring about improvement in collegiate instruction and to undergird the recent widespread activity directed toward the improvement of secondary school programs in mathematics and science, and

Whereas institutes for college teachers may be considered essential in effecting improved preparation of new teachers of mathematics and science on all levels, therefore

Be it resolved that the Congress be requested to increase the discretionary power of the National Science Foundation in allotting funds for institutes for college teachers of science and mathematics, and

Be it further resolved that the Congress be requested to increase the funds available to the National Science Foundation so that this program may be instituted without detracting from the secondary school institute program, and

Be it further resolved that the Washington Conference support the request of the National Science Foundation for support of the expanded programs of institutes for teachers of science and mathematics.

2. The Secretary is instructed to transmit the following resolution to Dr. Alan T. Waterman, Director of the National Science Foundation.

Whereas more than half of the approximately 150,000 science and mathematics teachers in secondary schools in the U. S. are teachers of mathematics, and

Whereas, the needs of mathematics teachers are often inadequately met by combined science-mathematics institutes, and

Whereas, in the summer of 1958 only 13 out of 125 institutes are in mathematics, therefore

Be it resolved that the Washington Conference recommends that at least one-third of all institutes in the summer of 1959 provide complete programs in mathematics, under the direction of mathematicians, for mathematics teachers.

3. This Conference recommends that the President and the Secretary-Treasurer of the M.A.A. urge chairmen of departments of mathematics to submit proposals to the National Science Foundation for summer institutes in 1959, and supply them with information concerning appropriate proposals, programs, and staff.

4. Whereas this Conference believes (a) that the effectiveness of the program of summer and other institutes can be greatly enhanced if there is reasonable diversity in the objectives and programs of the several institutes, and the groups to be served by them, subject always to the fundamental principles that the institute programs shall consist essentially of the study of mathematics, and (b) that it is therefore important that institutions submitting proposals should give careful attention to formulating specific objectives for the proposed institutes and to the suitability of the program and staff for attaining them, and that the reviewing panels give weight to these matters; therefore

Be it resolved that the officers of the Association call these considerations to the attention of colleges and universities planning the establishment of institutes and urge the National Science Foundation to continue to judge proposals in the light of their stated objectives and their probable success in achieving them for the specific groups of teachers the institutes are planned to serve, and, indeed, to place increased emphasis

upon these considerations.

5. This Conference recommends that the President of the Mathematical Association of America appoint a standing committee on summer, academic year, and in-service institutes for mathematics teachers.

6. This Conference recommends to the Association's Committee on Institutes (when established in accordance with the preceding resolution) that financial support from the National Science Foundation or other foundations be sought, so that the functions of the Committee can be carried out without exclusive reliance upon volunteer activity.

B. *Resolutions on Programs of the Mathematical Association of America*

1. This Conference commends the work of the Committee on the Undergraduate Program in Mathematics, and urges that its work be continued with all possible vigor and on an expanded scale.

2. This Conference recommends to the Board of Governors of the Association that vigorous steps be taken toward more extensive publication of an expository nature, possibly including reprints of selected articles appearing in former issues of the MONTHLY and elsewhere.

3. This Conference recommends to the members of the Committee on High School Contests that they consider seriously the detailed proposals made by Professor R. C. Buck in his report on mathematical competitions, and requests that the date of the contest be much earlier than at present, not later than February 15 if possible.

4. This Conference recommends that a committee of the Association be appointed to gather and disseminate information about honors programs in mathematics.

5. This Conference recommends that a committee of the Association be appointed to distribute information about mathematics and employment in the field of mathematics to teachers, counselors and students in high schools and colleges. Especially should this committee initiate contacts with national and, through the sections, with local guidance and counselor organizations. Not only should written information be supplied but speakers should be suggested for meetings of these organizations. The publication of future editions of "Professional Opportunities in Mathematics" should be the responsibility of this committee.

6. This Conference recommends that a committee of the Association be appointed to organize and operate a Speakers Bureau. Such speakers should be drawn from business, industry, and government as well as from academic institutions and should be available to speak at secondary schools, collegiate institutions, civic organizations, and meetings of Sections of the Association. This committee should cooperate with the Sections of the Association in establishing similar activities on a sectional basis.

7. This Conference recommends that the Secretary-Treasurer request the Sections of the Association to appoint state liaison officers to deal with mathematical matters within individual states or provinces. A matter requiring immediate attention is the National Science Foundation program of support to state academies. The Conference also recommends that the Sections be urged to offer their cooperation to the state and regional academies in planning programs for which NSF support will be sought.

8. This Conference recommends that the Joint Committee on Places of Meetings invite the National Council of Teachers of Mathematics to hold its annual meeting at the time and place of the winter meetings of the Society and Association in 1960 or as early as is feasible.

C. *Resolutions on a Plan of Action*

1. This Conference recommends to the Board of Governors of the Association that the By-Laws of the Mathematical Association of America be amended so as to provide for a Secretary and a Treasurer, and an officer to be known as Executive Director, who shall be a full-time paid employee of the Association.

2. This Conference recommends that the office of the Executive Director be located in Washington, D. C., that an appropriate person be appointed to this position, and that funds be sought to support the position for an initial period of five years. The duties of the Executive Director shall be to serve as the executive officer of the Association, to carry out its policies and programs, and to serve as a liaison officer in promoting its activities. He should inform the Board of Governors of the Association of current problems and needs, desirable programs for meeting these needs, and governmental decisions affecting mathematics.

3. The Washington Conference recommends to the Conference Board of the Mathematical Sciences that the Board look to the establishment of an office in Washington coupled with the appointment of a suitable officer based there to deal on a national scale with problems and questions involving mathematics as a whole.

4. Whereas government, foundations and industry have begun to launch vast new programs of mathematical education in the public interest, and

Whereas these national and continental programs of mathematical education can be expected to grow and to involve much greater amounts of money in the future, and

Whereas the foundations and government have called upon the Mathematical Association of America to organize certain of these programs, to bring them into being, and to insure that suitable mathematical quality is maintained in them, therefore

Be it resolved that this Conference recommends to the Board of Governors a reorganization of the Association to include the formation of departments of management with men and money adequate to carry out the responsibilities assigned to them.

Specifically, special departments of the Association might now be recognized in the following existing activities:

(a) A Department of Periodical Publication charged with editing and publishing the American Mathematical Monthly, the Slaught Papers, and the Carus Monographs.

(b) A Department of the Undergraduate Program, charged with the Curricular work of the Committee on the Undergraduate Program in Mathematics, including the preparation of source material on new undergraduate courses, the urgently needed new courses for teachers, expositions, honors programs, and the promotion of curricular studies.

(c) A Department of Institutes, Visiting Lectureships, and Speakers.

(d) A Department of Competitions, charged with preparing examinations for national prize contests and conducting a service for operating such contests.

(e) A Department of Professional Guidance and Mathematical Manpower charged with the preparation and publication of guidance material on mathematical manpower and the promulgation of information on mathematical vocations and training for them which is needed by school guidance officers and industrial employment agents.

The establishment of these departments of management should meet the following requirements.

(A) Each department should have a charter approved by the Board of Governors which defines its sphere of activity. Each department should be responsible to the Board of Governors through a committee with appropriate provision for limited terms of office and rotation, including representatives of Canadian collegiate mathematics where appriate.

(B) Each department should have a Chairman whose position is guaranteed by the Association to be stable, to have professional status, and to carry a full or part-time salary adequate to attract professional mathematicians of established reputation and ability.

(C) Each department should be financed in an adequate manner.

(D) Finally, the proposed departments are not intended to duplicate the responsibilities of the central organization of the Association as carried out by the home office and the proposed Washington liaison office of the Executive Director. Nor are they intended to duplicate the work on high school text materials now being organized in an

interorganization staff of all the mathematical organizations, in which the Association may cooperate.

D. *Resolutions of thanks*

1. Whereas, the Washington Conference has been of great value to the officers and members of the Mathematical Association of America by giving an opportunity to review its past activities and to plan its future activities, and

Whereas, such a conference was made possible by a grant from the National Science Foundation, therefore

Be it resolved that those in attendance at this Conference hereby express their sincere appreciation to the Foundation for making the Conference possible.

2. This Conference votes its sincere thanks to its chairman, Professor G. Baley Price, President of the Mathematical Association of America, for having arranged the program of the Conference and for having directed its actions in such an effective manner.

Concluding Note. The resolutions were duly transmitted by the Secretary-Treasurer of the Association to the persons concerned. In particular, about 500 letters were sent in June to chairmen of departments of mathematics urging them to submit proposals to the National Science Foundation for summer institutes in 1959. A memorandum of information on N.S.F. Summer Institutes prepared by Professor E. A. Cameron was enclosed, as were announcements of two current institutes. The various resolutions on Programs of the M.A.A. and on a Plan of Action are now under consideration by the Board of Governors.

HARRY M. GEHMAN, *Secretary-Treasurer*

ON A CERTAIN FAMILY OF ARITHMETIC FUNCTIONS

PAUL J. McCARTHY, Florida State University

1. Introduction. If r is a positive integer we define the arithmetic function $T_r(n)$ in the following way: $T_r(n)$ is the number of integers k such that $1 \leqq k \leqq n$ and the greatest common divisor (k, n) is not divisible by the rth power of any prime. $T_1(n)$ is Euler's totient function $\phi(n)$ and the function $\rho(n) = T_2(n)$ was introduced and studied by Haviland in [**3**]. In this paper it is our purpose to investigate some of the arithmetic and asymptotic properties of the functions $T_r(n)$. The methods used are similar to those used to treat $\phi(n)$ and the results we obtain show that the $T_r(n)$ behave in a manner somewhat similar to their better-known relative.

Haviland [**3**] defined $T_2(n)$ in terms of the properties of certain arithmetic progressions. The other $T_r(n)$ may be defined in a similar manner. We shall say that an integer is r-free if it is not divisible by the rth power of any prime.

THEOREM 1. *$T_1(n)$ is the number of arithmetic progressions $sn+k$, $k=1, \cdots, n$, which have infinitely many prime terms. For $r \geqq 2$, $T_r(n)$ is the number of these arithmetic progressions which have infinitely many r-free terms.*

When $r=1$ this statement is an immediate consequence of Dirichlet's theo-

rem on primes in an arithmetic progression which states that if $(n, k)=1$ then $sn+k$ is a prime for infinitely many values of s ([**4**], pp. 73–79). In general, let $d=(n, k)$, $n=n_1d$ and $k=k_1d$. Then $sn+k=d(sn_1+k_1)$ will be r-free for infinitely many values of s if and only if d is r-free, because by Dirichlet's theorem there are infinitely many values of s for which sn_1+k_1 is a prime which does not divide d.

2. Arithmetic properties. In this section we shall show that $T_r(n)$ is a multiplicative function for each r and obtain an explicit formula for $T_r(n)$. These results will be consequences of the theorem which we prove below, in which we denote by $T_r(x, n)$ the number of integers k such that $1 \leqq k \leqq [x]$ and (n, k) is r-free. Here $[x]$ is the greatest integer $\leqq x$, and $\mu(n)$ will denote the Möbius function.

THEOREM 2.

$$T_r(x, n) = \sum_{d^r|n} \mu(d)\left[\frac{x}{d^r}\right].$$

We shall first prove three lemmas which are known in the case $r=1$; statements and proofs of the second and third of these lemmas in this case can be found in [**5**].

LEMMA 1. *If $f(n)$ is a multiplicative arithmetic function and $F(n)=\sum_{d^r|n} f(d)$, then $F(n)$ is multiplicative.*

For, if $(m, n)=1$,

$$\begin{aligned} F(mn) &= \sum_{d^r|mn} f(d) = \sum_{d_1^r|m, d_2^r|n} f(d_1d_2) \\ &= \sum_{d_1^r|m, d_2^r|n} f(d_1)f(d_2) = \sum_{d_1^r|m} f(d_1) \sum_{d_2^r|n} f(d_2) \\ &= F(m)F(n). \end{aligned}$$

LEMMA 2. *Let $a_1, \cdots, a_s$ be any finite set of (not necessarily distinct) integers, and let $f(x)$ be a complex-valued function defined for these integers. Let S be the sum of all the $f(a_i)$ where a_i is r-free, and let S_d be the sum of all the $f(a_i)$ for which $d^r | a_i$. Then*

$$S = \sum \mu(d) S_d,$$

where the sum extends over all d such that d^r divides at least one a_i.

We first show that

$$\sum_{d^r|n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } r\text{-free}, \\ 0 & \text{otherwise}. \end{cases}$$

Now $\mu(n)$ is a multiplicative function and so, by Lemma 1, the left-hand side of this expression is a multiplicative function of n. Therefore, it is sufficient to show that this expression is true when $n=p^\alpha$, where p is a prime. But in this case the

left-hand side is equal to $\mu(1)=1$ if $r<a$ and to $\mu(1)+\mu(p)=0$ if $r\geqq a$.

We now have for any i,

$$f(a_i)\sum_{d^r|a_i}\mu(d) = \begin{cases} f(a_i) & \text{if } a_i \text{ is } r\text{-free,} \\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$S = \sum_{i=1}^{s} f(a_i)\sum_{d^r|a_i}\mu(d).$$

Now choose a d whose rth power divides some a_i. Then, collecting the coefficients of $\mu(d)$, we see that we have precisely S_d. This proves the lemma.

LEMMA 3. *Let n be a positive integer and let $g(m)$ be an arithmetic function. Let $b_1, \cdots, b_s$ be positive integers. Let S be the sum of all $g(b_i)$ for which (b_i, n) is r-free, and let S_d be the sum of all $g(b_i)$ for which $d^r|b_i$. Then $S=\sum_{d^r|n}\mu(d)S_d$.*

To prove this we set $a_i=(b_i,n)$ and $f(a_i)=g(b_i)$. Then we apply Lemma 2.

Now, to prove Theorem 2 we let $b_1, \cdots, b_s$ in Lemma 3 be the integers $1, \cdots, [x]$, and we set $g(b_i)=1$ for each i. Then S is the number of integers k such that $1\leqq k\leqq [x]$ and (k, n) is r-free, *i.e.*, $S=T_r(x, n)$. Let $d^r|n$. Then S_d is the number of integers k such that $1\leqq k\leqq [x]$ and $d^r|k$, *i.e.*, the number of multiples of d^r not exceeding x. But this is just $[x/d^r]$. This proves the theorem.

COROLLARY 1. *For all n,*

$$T_r(n) = n\sum_{d^r|n}\frac{\mu(d)}{d^r}\cdot$$

COROLLARY 2. *For each r, the function $T_r(n)$ is multiplicative.*

The second corollary follows from the first and from an application of Lemma 1 with $f(n)=\mu(n)/n^r$.

We can now determine $T_r(n)$ explicitly. Because of Corollary 2 we can do this by first determining $T_r(p^a)$, where p is a prime. If $a<r$ it is obvious that $T_r(p^a)=p^a$. Now suppose that $a\geqq r$. Then $T_r(p^a)$ is p^a minus the number of multiples of p^r which do not exceed p^a, *i.e.*, $T_r(p^a)=p^a-p^{a-r}$. Thus for an integer n,

$$T_r(n) = n\prod_{p^r|n}(1-p^{-r}).$$

3. Asymptotic properties. In this section we shall obtain some asymptotic results concerning the functions $T_r(n)$. We shall assume throughout this section that $r>1$, since the error term which we shall obtain in Theorem 3 is different from the error term which is obtained when $r=1$ (see p. 268 of [2]).

THEOREM 3. *Let* $r>1$. *If* $T_r^*(n)=\sum_{m=1}^n T_r(m)$, *then*

$$T_r^*(n)=\frac{n^2}{2\zeta(2r)}+O(n),$$

where $\zeta(s)$ *is the Riemann zeta function.*

Let $k=[n^{1/r}]$. Then

$$\begin{aligned}
T_r^*(n) &= \sum_{m=1}^n m\sum_{d^r\mid m}\frac{\mu(d)}{d^r}=\sum_{d^rd_1\le n} d_1\mu(d)\\
&=\sum_{d=1}^k \mu(d)\sum_{d_1=1}^{[n/d^r]} d_1\\
&=\frac{1}{2}\sum_{d=1}^k\mu(d)\left\{\left[\frac{n}{d^r}\right]\left(\left[\frac{n}{d^r}\right]+1\right)\right\}\\
&=\frac{1}{2}\sum_{d=1}^k\mu(d)\left\{\frac{n}{d^r}+O(1)\right\}\left\{\frac{n}{d^r}+O(1)\right\}\\
&=\frac{1}{2}\sum_{d=1}^k\mu(d)\left\{\frac{n^2}{d^{2r}}+O\left(\frac{n}{d^r}\right)+O(1)\right\}\\
&=\frac{1}{2}n^2\sum_{d=1}^k\frac{\mu(d)}{d^{2r}}+O\left(\sum_{d=1}^k\frac{n}{d^r}\right)+O(n^{1/r}).
\end{aligned}$$

Also

$$\begin{aligned}
\frac{1}{2}n^2\sum_{d=1}^k\frac{\mu(d)}{d^{2r}} &= \frac{1}{2}n^2\sum_{d=1}^\infty\frac{\mu(d)}{d^{2r}}-\frac{1}{2}n^2\sum_{d=k+1}^\infty\frac{\mu(d)}{d^{2r}}\\
&=\frac{n^2}{2\zeta(2r)}+O\left(\sum_{d=k+1}^\infty\frac{n^2}{d^{2r}}\right),
\end{aligned}$$

where we have used the fact that ([2], p. 250)

$$\frac{1}{\zeta(s)}=\sum_{d=1}^\infty\frac{\mu(d)}{d^s}\qquad (s>1).$$

We complete the proof by observing that

$$\sum_{d=1}^k\frac{n}{d^r}\quad\text{and}\quad\sum_{d=k+1}^\infty\frac{n^2}{d^{2r}}$$

are both $O(n)$.

Using this theorem we can prove

THEOREM 4. *The probability that the greatest common divisor of two integers be* r *free is* $1/\zeta(2r)$.

Our proof follows that given by Hardy and Wright on pages 268 and 269 of [2] for the case $r=1$. Our proof will be for $r>1$. If ψ_n is the number of pairs of integers a, b such that $1 \leqq a \leqq b \leqq n$, then $\psi_n = n(n+1)/2$. But the number of these pairs whose greatest common divisor is r-free is precisely $T_r^*(n)$. Hence the probability that the greatest common divisor of a and b be r-free is

$$\lim_{n\to\infty} \frac{T_r^*(n)}{\psi_n} = \lim_{n\to\infty} \left(\frac{1}{\zeta(2r)} + \frac{O(n)}{n^2}\right) = \frac{1}{\zeta(2r)}\,.$$

This result, for the case $r=2$, was obtained by Christopher in [1], and a result equivalent to this, when $r=2$, is contained in Haviland's paper [3].

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TRANSFORMATIONS OF INTEGRALS

IAIN T. ADAMSON, The Queen's University, Belfast

The aims of this note are first, to present the theory of "change of variable" in multiple integrals in the spirit of Menger's book on the calculus [1] and second, to sketch a completely analytic treatment of a subject which is frequently dealt with by more-or-less explicit appeals to geometrical intuition. In pursuing the second aim I have been greatly helped by Ostrowski's excellent book [2].*

We begin by describing the type of "change of variable" we shall allow and the type of sets over which our integrals are taken. Let E be a set of points in real n-space, R^n. If Φ is a function of E into R^n we define the *coordinate functions* of Φ to be the real-valued functions $\phi_1, \cdots, \phi_n$ of E given by

$$\phi_i(x) = i\text{th coordinate of } \Phi(x)$$

for all points x of E. We may write $\Phi = (\phi_1, \cdots, \phi_n)$. The *Jacobian* of Φ is the real-valued function $\mathcal{J}\Phi$ of E defined by setting $\mathcal{J}\Phi(x) = \det \mathfrak{D}_j\phi_i(x)$ for all points x of E. ($\mathfrak{D}_j\phi_i$ denotes the jth first-order partial derivative of ϕ_i.) The function Φ is called a *regular transformation* of E if

* The referee has kindly drawn my attention to the book Mathematical Analysis by T. M. Apostol which also gives a completely analytic treatment of the problem discussed here without, however, mentioning orientation explicitly.

1. Φ is a (1, 1) function of E onto the image set $\Phi(E)$;

2. the coordinate functions of Φ are continuous and have continuous first-order partial derivatives at every point of E;

3. the Jacobian of Φ vanishes nowhere in E.

The following remarks are simply translations into the present terminology of standard results (see, *e.g.*, Rudin [**3**], Chapter 9.)

REMARK 1. *If Φ_1, Φ_2 are regular transformations of E onto E_1, E_1 onto E_2 respectively, then $\Phi = \Phi_2\Phi_1$ is a regular transformation of E onto E_2 and $\mathcal{J}\Phi = \mathcal{J}\Phi_2(\Phi_1)\cdot\mathcal{J}\Phi_1$.*

REMARK 2. *If Φ is a regular transformation of E onto E_1, then the inverse Φ^{-1} of Φ is a regular transformation of E_1 onto E.*

(In Remark 1, and throughout the paper, we adhere to Menger's convention whereby the juxtaposition fg of two functions f and g denotes the result of substituting g in f; the dot is never omitted in the symbol $f\cdot g$ for the product of the two functions.)

A function Φ of E into R^n is said to be *i-simple* if there exists a real-valued function ϕ of E such that

$$\Phi(x) = (x_1, \cdots, x_{i-1}, \phi(x), x_{i+1}, \cdots, x_n)$$

for all points $x = (x_1, \cdots, x_n)$ of E.

For each integer $n \geqq 1$ we define the *standard n-cell*, S^n, to be the set of points $(x_1, \cdots, x_n)$ in R^n which satisfy the inequalities

$$0 \leqq x_1 \leqq 1, 0 \leqq x_2 \leqq 1, \cdots, 0 \leqq x_n \leqq 1, 0 \leqq x_1 + \cdots + x_n \leqq 1.$$

A set of points E in R^n is said to be a *regular n-cell* if there exists a regular transformation of S^n onto E. The following results are easily verified.

REMARK 3. *Let E be a regular n-cell. Then E is a compact and connected set. Further, for every positive real number η there exists an interval sum* which contains all the points on the boundary of E and has volume less than η.*

From the second part of Remark 3 it follows by standard arguments that if f is a real-valued function continuous on a regular n-cell E then f is integrable over E; the value of its integral is denoted by $I(f; E)$. In particular, the function u which takes the value 1 everywhere is integrable over E; hence E has a well-defined *volume* $v(E) = I(u; E)$.

In order to give an adequate account of the effect of a "change of variable" in integrals over regular n-cells, it is essential to introduce the notion of an orientation. This idea is—informally, at least—quite familiar for regular 1-cells. The direction from left to right along the real line is conventionally called the positive direction; and so we say intuitively that a regular 1-cell is "positively oriented" if we traverse it from left to right, or if we attach to it an arrow point-

* By an interval sum we mean a finite union of closed intervals of R^n.

ing from left to right, and we say it is "negatively oriented" if we traverse it from right to left or attach to it an arrow pointing from right to left. In the same way there are familiar conventions for "orienting" regular 2-cells; we say that a 2-cell is "positively oriented" if we attach to it a counterclockwise-turning arrow and "negatively oriented" if we attach to it a clockwise-turning arrow. Now in these examples, whether we proceed by attaching an arrow to the cell E or by some other method, what we are in fact doing is attaching a sign to the cell E, or—what amounts to the same thing—we are attaching to E one of the numbers -1 or $+1$. Thus we are led to make the following definition: a *regular oriented n-cell* (E, ω) is a pair consisting of a regular n-cell E and a number ω which is either -1 or $+1$; the number ω is called the *orientation* of (E, ω). Let Φ be a regular transformation of E; then we define the image of (E, ω) under Φ to be the regular oriented n-cell $(\Phi(E), \omega_\Phi)$ where

$$\omega_\Phi = \begin{cases} +\omega & \text{if } \mathfrak{J}\Phi \text{ is everywhere positive in } E, \\ -\omega & \text{if } \mathfrak{J}\Phi \text{ is everywhere negative in } E. \end{cases}$$

($\mathfrak{J}\Phi$ is of course of constant sign on E since E is connected and $\mathfrak{J}\Phi$ is continuous and nonvanishing on E.) If f is a real-valued function continuous on E, we define the integral of f over the oriented n-cell (E, ω) to be

$$I(f; E, \omega) = I(\omega \cdot f; E).$$

We may remark in passing that the traditional notation (and Menger's) for integrals over regular 1-cells, *i.e.* over bounded closed intervals of the real line, blurs the distinction between integration over a 1-cell *simpliciter* and over an *oriented* 1-cell. The fundamental notion is that of integration *over* an interval E, but if a and b are the end-points of the interval then to write $\int_a^b f(x)dx$, or even $\int_a^b f$, and to talk of the integral *from a to b*, is to deal not with the integral over the unoriented interval E but with the integral over the oriented interval (E, ω), where $\omega = +1$ or -1 according as $a < b$ or $a > b$.

Before proceeding to our main theorem we show how a general regular transformation can be "factorized" into simple transformations.

LEMMA. *Let E be a connected set of points in R^n. Let $\Phi = (\phi_1, \cdots, \phi_n)$ be a regular transformation of E onto a set E' in R^n such that none of the first-order partial derivatives $\mathfrak{D}_1\phi_1, \cdots, \mathfrak{D}_n\phi_n$ ever vanishes in E. Then there exist a sequence of sets $E = E_0, E_1, E_2, \cdots, E_n = E'$, and a sequence of regular transformations Φ_i of E_{i-1} onto E_i $(i = 1, \cdots, n)$ such that each Φ_i is i-simple and $\Phi = \Phi_n\Phi_{n-1} \cdots \Phi_1$.*

We shall give the proof only in the case $n = 2$; the proof for the general case, which proceeds by induction, is a little more complicated.

Since $\mathfrak{D}_1\phi_1$ never vanishes in the connected set E, it follows that $\mathfrak{D}_1\phi_1$ is of constant sign in E. Let Φ_1 be the function of E onto $E_1 = \Phi_1(E)$ given by

$$\Phi_1(x, y) = (\phi_1(x, y), y)$$

for all points (x, y) of E. Then Φ_1 is certainly 1-simple; its coordinate functions are continuous and have continuous first-order partial derivatives in E; and its Jacobian vanishes nowhere in E, since clearly $\mathfrak{J}\Phi_1 = \mathfrak{D}_1\phi_1$. Further, Φ_1 is a (1, 1) mapping of E onto E_1; for if $\Phi_1(x, y) = \Phi_1(x', y')$ we have at once $y = y'$ and $\phi_1(x, y) = \phi_1(x', y')$, whence $\phi_1(x, y) = \phi_1(x', y)$. Now consider the function ψ defined by setting $\psi(t) = \phi_1(t, y)$ for all appropriate real numbers t; since, by definition, $\mathfrak{D}_1\phi_1(t, y) = \mathfrak{D}\psi(t)$, it follows that $\mathfrak{D}\psi$ is of constant sign and hence that ψ is a (1, 1) function. Hence $x = x'$. This discussion shows that Φ_1 is a 1-simple regular transformation of E onto E_1.

Now set $\Phi_2 = \Phi\Phi_1^{-1}$; by Remarks 1 and 2, Φ_2 is a regular transformation of E_1 onto E'. To show that Φ_2 is 2-simple, let $(z, y) = (\phi_1(x, y), y)$ be any point of E_1; then $\Phi_2(z, y) = \Phi(x, y) = (\phi_1(x, y), \phi_2(x, y)) = (z, \phi_2(x, y))$.

This completes the proof in the case $n = 2$.

We shall use this "factorization" lemma in the proof of our central result, which we now state as follows.

THEOREM. *Let (E, ω) be a regular oriented n-cell; let Φ be a regular transformation of E onto E'. Then for every real-valued function f continuous on E' we have*

$$I(f; E', \omega_\Phi) = I(f\Phi \cdot \mathfrak{J}\Phi; E, \omega). \tag{1}$$

We call (1) *the transformation rule for the situation* $(E; \Phi)$.

From this formulation of the result it is easy to deduce the two methods by which it can be applied in the practical evaluation of integrals. Reading (1) from left to right and from right to left respectively, we see that

1. to find the integral of a continuous function f over a regular n-cell E' we may try to find a regular n-cell E and a regular transformation Φ of E onto E' such that the integration of $f\Phi \cdot \mathfrak{J}\Phi$ over E can be conveniently carried out;

2. to find the integral over a regular n-cell E of a function g which can be expressed in the form $g = f\Phi \cdot \mathfrak{J}\Phi$, where Φ is a regular transformation of E and f is a continuous function of $\Phi(E)$, we merely integrate f over $\Phi(E)$.

We consider first the case where E and E' are bounded closed intervals of the real line and ϕ is a function of E onto E' with continuous nonvanishing derivative. The proof of (1) here is a very familiar one, slightly modified to take account of the orientation. We remark first that if F is any antiderivative of f in E', then $F\phi$ is an antiderivative of $f\phi \cdot \mathfrak{D}\phi$ in E.

Now suppose $\mathfrak{D}\phi$ is everywhere positive in E; then ϕ is an increasing function of E. Hence, if E is the interval $[a, b]$, then E' is the interval $[\phi(a), \phi(b)]$. According to the Fundamental Theorem of the Calculus, it follows that

$$I(f; E') = F(\phi(b)) - F(\phi(a)) = F\phi(b) - F\phi(a).$$

But, also by the Fundamental Theorem, we have

$$F\phi(b) - F\phi(a) = I(f\phi \cdot \mathfrak{D}\phi; E).$$

Since in the present case $\omega_\phi = \omega$, we have the desired result

$$I(f\phi \cdot \mathfrak{D}\phi;\, E,\, \omega) = I(f;\, E',\, \omega_\phi). \tag{2}$$

If, on the other hand, $\mathfrak{D}\phi$ is everywhere negative in E, then ϕ is decreasing and hence $E' = [\phi(b),\, \phi(a)]$. So in this case

$$I(f;\, E') = F\phi(a) - F\phi(b) = -\, I(f\phi \cdot \mathfrak{D}\phi;\, E).$$

Here, however, $\omega_\phi = -\omega$, so our final result is (2) as before.

To complete the proof of our theorem we make use of two reduction steps.

First we remark that if the transformation Φ can be carried out in two stages then it suffices to prove the theorem for each of the stages. To make this statement more explicit, let $\Phi = \Phi_2\Phi_1$, where Φ_1 is a regular transformation of E onto some regular n-cell E_1 and Φ_2 is a regular transformation of E_1 onto E'. Suppose we can prove the transformation rule for the situations $(E;\, \Phi_1)$ and $(E_1;\, \Phi_2)$. Then the rule for the situation $(E;\, \Phi)$ will follow at once by simple computations using Remark 1.

Next suppose that we can prove the transformation rule in the situation $(R;\, \Phi)$ for every regular n-cell R contained in E such that $\Phi(R)$ is a bounded closed *interval.* We claim that we can deduce the transformation rule for the situation $(E;\, \Phi)$.

To establish this result we notice first that we can certainly deduce the transformation rule for every situation $(R;\, \Phi)$ in which R is a regular n-cell contained in E and $\Phi(R) = R'$ is an interval sum. Now let S be an interval sum contained in E; let R' be an interval sum contained in E' and containing $\Phi(S)$; and let $R = \Phi^{-1}(R')$. Then

$$\begin{aligned} \left| I(f\Phi \cdot \mathfrak{J}\Phi;\, E, \omega) - I(f;\, E', \omega_\Phi) \right| \leqq{} & \left| I(f\Phi \cdot \mathfrak{J}\Phi;\, E,\, \omega) - I(f\Phi \cdot \mathfrak{J}\Phi;\, R,\, \omega) \right| \\ & + \left| I(f;\, R',\, \omega_\Phi) - I(f;\, E',\, \omega_\Phi) \right| . \end{aligned}$$

Both terms on the right-hand side can be made arbitrarily small by choosing S and R' such that $v(E) - v(S)$ and $v(E') - v(R')$ are sufficiently small.

We now apply our lemma and the two reduction steps to show that we need prove our theorem only in the special case for which Φ is a simple transformation and $E' = \Phi(E)$ is a bounded closed interval.

To see this, let E be any regular n-cell, Φ an arbitrary regular transformation. Let x be any point of E. Since the Jacobian

$$\mathfrak{J}\Phi = \sum \pm\, \mathfrak{D}_{i_1}\phi_1 \cdot \mathfrak{D}_{i_2}\phi_2 \cdot \cdots \cdot \mathfrak{D}_{i_n}\phi_n \tag{3}$$

(where the summation runs over all permutations $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$) is nonzero at x, it follows that at least one of the terms in the summation is nonzero at x. Then there exists a permutation $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$ such that

$$\mathfrak{D}_{i_1}\phi_1,\ \mathfrak{D}_{i_2}\phi_2,\ \cdots,\ \mathfrak{D}_{i_n}\phi_n$$

do not vanish at x; hence there is a neighborhood of x in which they do not vanish. Applying the Heine-Borel Theorem to the covering of E provided by

the neighborhoods which correspond in this way to every point of E, we deduce easily that E can be expressed as the union of a finite number of regular n-cells E_k, having only boundary points in common and such that for every k there exists a permutation of $(1, \cdots, n)$ for which the corresponding term in the summation (3) is nonvanishing on E_k. For each E_k it follows then by a suitable modification of the indexing in our lemma that there exist sequences of regular n-cells

$$E_k = E_{k,0},\ E_{k,1},\ E_{k,2},\ \cdots,\ E_{k,n} = \Phi(E_k)$$

and of *simple* regular transformations $\Phi_{k,r}$ of $E_{k,r-1}$ onto $E_{k,r}$ such that $\Phi_{k,n}\Phi_{k,n-1}\cdots\Phi_{k,1}$ is the restriction of Φ to E_k.

If now we can prove our transformation rule in every situation $(R;\Phi_{k,r})$ where R is a subset of $E_{k,r-1}$ for which $\Phi_{k,r}(R)$ is a bounded closed interval, then, from our second reduction step the transformation rule for every situation $(E_{k,r-1};\Phi_{k,r})$ follows at once. Using the first reduction step we can deduce the rule for every situation $(E_k;\Phi)$, and so finally, by addition, for the situation $(E;\Phi)$.

We conclude the proof of our theorem, then, by proving the transformation rule for a situation $(E;\Phi)$ where $E'=\Phi(E)$ is a bounded closed interval of R^n and Φ is a simple transformation. Suppose, then, that E' is defined by the inequalities $a_i \leqq x_i \leqq b_i$ $(i=1, \cdots, n)$ and that Φ is n-simple, *i.e.* that there exists a real-valued function ϕ of E such that $\Phi(x)=(x_1, \cdots, x_{n-1}, \phi(x))$ for all points $x=(x_1, \cdots, x_n)$ of E.

Let $\overline{E}$ be the bounded closed interval of R^{n-1} defined by the inequalities $a_i \leqq y_i \leqq b_i$ $(i=1, \cdots, n-1)$. For every point $y=(y_1, \cdots, y_{n-1})$ of $\overline{E}$ consider the set of real numbers t such that $(y, t)=(y_1, \cdots, y_{n-1}, t)$ belongs to E; these points t form an interval E_y of the real line such that $\phi(y, E_y)=[a_n, b_n]$.

Let ϕ_y be the function of E_y onto $[a_n, b_n]$ defined by setting $\phi_y(t)=\phi(y, t)$. Then $\mathfrak{D}\phi_y(t)=\mathfrak{D}_n\phi(y, t)=\mathcal{J}\Phi(y, t)$; hence $\mathfrak{D}\phi_y$ is continuous and of constant sign in E_y. It follows from the special case of our theorem for intervals of the real line that for every real-valued function f continuous on E we have

$$I(f_y\phi_y\cdot\mathfrak{D}\phi_y;\ E_y,\ \omega) = I(f_y;\ [a_n, b_n],\ \omega_\Phi), \tag{4}$$

where f_y is defined in the same way as ϕ_y above.

Now define the functions F and G of $\overline{E}$ by setting for every point y of $\overline{E}$

$$F(y) = \text{left-hand side of (4)},\quad G(y) = \text{right-hand side of (4)}.$$

Since it is easily seen that $f_y\phi_y\cdot\mathfrak{D}\phi_y=(f\Phi\cdot\mathcal{J}\Phi)_y$, it follows from standard results on the reduction of multiple integrals to repeated integrals that

$$I(F;\overline{E}) = I(f\Phi\cdot\mathcal{J}\Phi;\ E,\ \omega).$$

Similarly we have $I(G;\overline{E})=I(f;\ E',\ \omega_\Phi)$. Since, by (4), $F=G$, our desired result follows and our theorem is completely proved.

We remark that many of the situations $(E;\ \Phi)$ with which we have to deal in the practical evaluation of multiple integrals do not satisfy the condition that Φ is regular on E. We may mention as an example the polar-coordinate transformation Π in the real plane given by

$$\Pi(r,\theta) = (r\cos\theta,\ r\sin\theta),$$

which is not regular in any regular 2-cell including the origin. Only slight modifications are necessary to deal with situations, such as that of the example, in which the lack of regularity is not too unpleasant. If E is a regular n-cell in R^n, we say that a function Φ of E onto a regular n-cell E' in R^n is *almost regular* if for every positive real number η there exists a regular n-cell E_η contained in E such that Φ is a regular transformation of E_η and such that $v(E)-v(E_\eta)$ and $v(E')-v(\Phi(E_\eta))$ are both less than η. It is then a simple matter to prove the following result.

THEOREM. *Let $(E,\ \omega)$ be a regular oriented n-cell; let Φ be an almost regular transformation of E onto a regular n-cell E'. Then for every real-valued function f continuous on E' we have*

$$I(f;\ E',\ \omega_\Phi) = I(f\Phi\cdot\mathcal{J}\Phi;\ E,\ \omega).$$

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GROUPOIDS WITH ADDITIVE ENDOMORPHISMS

I. M. H. ETHERINGTON, University of Edinburgh

The main purpose of this note is to prove two results (Theorems 1(ii) and 3) concerning the logarithmetic [**3**, **6**, **7**] of a groupoid with additive endomorphisms. These results apply in particular to groupoids with the property $xy\cdot zw = xz\cdot yw$, since these have additive endomorphisms [**8**]. Theorem 1 as applied to quasigroups with this property was given by Murdoch [**10**]. Theorem 4 states a general result on logarithmetics, and Theorem 5 a condition under which a logarithmetic has, with respect to addition, additive endomorphisms.

1. Endomorphisms of groupoids. Let G be a groupoid, *i.e.*, a set of elements closed with respect to a binary operation, which I shall write as multiplication. If

$$\alpha\colon x \longrightarrow x^\alpha$$

is a mapping of G into itself, and β also is such a mapping, we define the mappings $\alpha+\beta$, $\alpha\beta$ by the rules

$$x^{\alpha+\beta} = x^\alpha x^\beta, \qquad x^{\alpha\beta} = (x^\alpha)^\beta.$$

This addition is in general nonassociative, like the multiplication in G, but the multiplication of mappings is associative.

If γ also is such a mapping, then of the two distributive laws

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta, \qquad (\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

the first is always valid, for we have

$$x^{\gamma(\alpha+\beta)} = x^{\gamma\alpha}x^{\gamma\beta} = x^{\gamma\alpha+\gamma\beta} \qquad (\text{all } x \text{ of } G).$$

If γ is an endomorphism of G, *i.e.* such that $(xy)^\gamma = x^\gamma y^\gamma$ for all x, y of G, then the second distributive law also holds, for

$$x^{(\alpha+\beta)\gamma} = (x^\alpha x^\beta)^\gamma = x^{\alpha\gamma}x^{\beta\gamma} = x^{\alpha\gamma+\beta\gamma} \qquad (\text{all } x \text{ of } G).$$

The set E_G of endomorphisms of G, together with the operation of multiplication of endomorphisms, forms the endomorphism semigroup $E^\times(G)$. If G has the property that the sum of two endomorphisms is always an endomorphism, we shall say that G has *additive endomorphisms.* In this case E_G with addition forms an additive groupoid $E^+(G)$, while E_G with both operations forms the endomorphism ringoid $E(G)$, in which the associative law of multiplication and both distributive laws hold.

If a groupoid has a set of generators $a, b, \cdots$ (as every finite groupoid must have), then an endomorphism α is fully determined when we know the images of the generators; for every element x can be expressed as a finite "word" in the generators, whence we can calculate its image x^α. In other words, any mapping of the generators onto elements of G can be extended in at most one way to an endomorphism of G.

A groupoid with one generator will be called *cyclic.*

2. Entropic groupoids. By the *entropic* law of multiplication I mean the identity*

$$xy \cdot zw = xz \cdot yw. \tag{1}$$

The word *entropic*, introduced in [5], refers by etymology to "inner turning" and seems preferable to *symmetric* which was used with the same meaning by Frink in a recent note† [8]. An objection to use of the word *symmetric* is that this conflicts with its established meaning when applied to groups and semigroups; also that *totally symmetric* has been used by Bruck [2] very appropriately for the property that $xy = z$ implies the five other equations derived by permuting x, y, z. This was in the context of quasigroups, and Bruck followed Murdoch [10, 11] in calling entropic quasigroups *abelian* on account of analogies with abelian groups. Unfortunately *abelian* has also been used by some writers

* The generalization for n-ary operations [5] is $\prod_i \prod_j x_{ij} = \prod_j \prod_i x_{ij}$ $(i, j = 1, \cdots, n)$.

† Frink uses the additive notation for groupoids and the opposite order for products of endomorphisms.

on quasigroups [*e.g.* **9**] to mean commutative. I propose to use *entropic* for all groupoids having the property (1). (Stein [**12**] uses the term *medial.*)

Frink observes that an entropic groupoid G has additive endomorphisms. For if α, β are endomorphisms, we have, using the entropic property,

$$(xy)^{\alpha+\beta} = (xy)^{\alpha}(xy)^{\beta} = x^{\alpha}y^{\alpha}\cdot x^{\beta}y^{\beta} = x^{\alpha}x^{\beta}\cdot y^{\alpha}y^{\beta} = x^{\alpha+\beta}y^{\alpha+\beta},$$

so that $\alpha+\beta$ is an endomorphism. But Frink goes further and [**8**, pp. 700, 701, 703] implies, almost says, that the converse is true, that the sum of two endomorphisms is always an endomorphism *only* if G is entropic. This is not so. I give below the multiplication tables of three groupoids, having respectively 2, 6 and 7 elements, which all have additive endomorphisms but are *not* entropic.

(2) Ex. 1. $G = \{a, b\}$. $a^2 = b^2 = ab = b,$ $ba = a.$

Ex. 2. $G = \{a, b, c, d, e, f\}$.

All products involving e or $f = f$.

	a	b	c	d
a	b	c	d	e
b	c	d	e	f
c	d	f	f	f
d	e	f	f	f

Ex. 3. $G = \{a, b, c, d, e, f, g\}$.

All products involving e, f or $g = g$.

	a	b	c	d
a	b	d	g	g
b	c	g	e	g
c	g	g	g	g
d	g	f	g	g

These groupoids are not entropic because in each case $aa\cdot ba \neq ab\cdot aa$. They are cyclic, the element a being in case the generator. It is readily verified that Example 1 has two endomorphisms, each determined by a mapping of a:

$$\alpha\colon a \to a,\ b \to b, \qquad \beta\colon a \to b,\ b \to b,$$

with the closed addition table, isomorphic to (2):

$$\alpha + \alpha = \beta + \beta = \alpha + \beta = \beta, \qquad \beta + \alpha = \alpha.$$

Similar results will be found for Examples 2 and 3; in each case $E^{+}(G) \simeq G$ $(+ \to \times)$.

(Example 2 is derived from an example in [**7**, p. 158]. Example 3 was communicated to me by Mr. H. Minc.)

3. Logarithmetics. Let powers of elements of G be indicated by indices, beginning with $x^1 = x$, $x^2 = xx$, and let addition and multiplication of indices be defined as for mappings:

$$x^{A+B} = x^A x^B, \qquad x^{AB} = (x^A)^B.$$

Thus for example

$$x^{2+1} = x^2x, \qquad x^{(1+2)2} = (xx^2)^2, \qquad x^{2(1+2)} = x^2(x^2)^2 = x^{2+2.2}. \tag{3}$$

Given the groupoid G, we equate two indices, $A = B$, if $x^A = x^B$ for all x of G. The *logarithmetic* $L(G)$ then consists of the classes of equated indices, with the operations $+$ and $\times$. A class of equated indices will be called an *index number*, or *index* for short. Just as with additive endomorphisms, we can also consider $L^+(G)$ and $L^\times(G)$ separately, each having the indicated operation.*

In general, addition of indices is nonassociative, but multiplication is associative, and the first distributive law

$$A(B + C) = AB + AC \tag{4}$$

holds, as illustrated in (3). Further identities will hold in particular logarithmetics; for example, G is *palintropic* [**4**] if $x^{AB} = x^{BA}$ for all x of G and all indices A, B, so that

$$AB = BA \tag{5}$$

in $L(G)$, and then the second distributive law holds:

$$(B + C)A = BA + CA. \tag{6}$$

Conversely, (6) implies (5) [**5**, Theorem 3]. G is *power entropic* if $L^+(G)$ is entropic, *i.e.* if indices satisfy $(A+B)+(C+D) = (A+C)+(B+D)$; this implies, but is not implied by, the palintropic property of G [**5**, **7**]. G may be called *trivially palintropic* if all products of six elements are the same. For this implies validity of

$$x^{(1+2)2} = x^{2(1+2)}, \qquad x^{(2+1)2} = x^{2(2+1)},$$

and of all more complicated cases of $x^{AB} = x^{BA}$, while the simpler cases where A and B are 1 or 2 hold automatically. The groupoids of Examples 1, 2, 3 are trivially palintropic but not power entropic.

4. We shall use what may be called *nonassociative induction*: If a (multiplicative) groupoid G has generators $a, b, \cdots$, then in order to prove a theorem Θ_x for all x of G, it is enough to verify $\Theta_a, \Theta_b, \cdots$ and to prove that Θ_x, Θ_y together imply Θ_{xy} for all x, y of G.

THEOREM 1. *If a groupoid G has additive endomorphisms, then* (i) *powers are endomorphisms of G,* (ii) *G is palintropic.*

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* In speaking of a *power* x^A we shall have in mind the corresponding mapping $x \to x^A$ of G into itself. Thus two powers which effect the same mapping are identified. The set of such mappings with the operation $+$ forms the free algebra $F_1(G)$ [**1**, p. viii]. $L^+(G)$ is isomorphic to $F_1(G)$, of which it is an abstract model. $F_1(G)$ consists of mappings of G, while $L^+(G)$ consists of classes of indices; hence two different groupoids may have the *same* logarithmetic when it can only be said that their F_1's are isomorphic. If further operations besides multiplication are defined in G, the logarithmetic remains as defined while the free algebra becomes more complicated: see [**7**] for further remarks in this connection.

(i) means that if P is any index $(xy)^P = x^P y^P$. The proof is by nonassociative induction on the index. Indices form a cyclic additive groupoid with generator 1. The result is obvious when $P=1$; and if two powers $x \to x^A$, $x \to x^B$ are endomorphisms of G, then so is the power $x \to x^{A+B}$, since this is the sum of two endomorphisms. This proves (i).

It follows that the indices of powers obey the second distributive law (6), because endomorphisms obey it. This is equivalent to the palintropic property (5). (The proof of this equivalence given in [5] was again by nonassociative induction.)

I do not suppose that the converse of (i) holds. Consideration of nonabelian groups shows that the converse of (ii) is false.

THEOREM 2. *If G is a cyclic groupoid with generator a, then its additive logarithmetic $L^+(G)$ is homomorphic to G, addition corresponding to multiplication and* 1 *corresponding to a.*

For any element x of G is expressible as a power of the generator, $x = a^P$, not necessarily with uniquely determined index number. Thus the correspondence $P \to a^P$ maps $L^+(G)$ on all G, with $1 \to a$. It is a homomorphism because

$$P + Q \to a^{P+Q} = a^P a^Q,$$

i.e., the image of the sum of two indices is the product of their images.

THEOREM 3. *If G is a cyclic groupoid with additive endomorphisms, then its logarithmetic $L(G)$ and endomorphism ringoid $E(G)$ are isomorphic, and additively are isomorphic to G.*

For any mapping of the generator onto an element of G, say $a \to a' = a^P$, extends in *at most* one way to an endomorphism of G (see Section 1), and in *at least* one way to a power $x \to x^P$, and hence by Theorem 1(i) in one and only one way to each, powers and endomorphisms coinciding. $L(G) \simeq E(G)$ follows. The fact that a mapping of a extends in only one way to a power means that each possible image a' is expressible in only one way as a power a^P. The correspondence in Theorem 2 is therefore an isomorphism. Thus $L^+(G) \simeq E^+(G) \simeq G$ with $+ \to + \to \times$.

Example. For the cyclic group C_n, L and E are both isomorphic to the ring of residues mod n.

THEOREM 4. *For any groupoid G, $L^\times(G)$ is anti-isomorphic to the endomorphism semigroup of $L^+(G)$.*

Proof. Any fixed index number A determines the mapping

$$\alpha_A : P \to AP \tag{7}$$

of $L^+(G)$ into itself, in which $1 \to A$. The distributive law (4) shows that α_A is an endomorphism of $L^+(G)$. Since $L^+(G)$ is cyclic, this is the unique endomorphism in which the generator 1 is mapped on A; hence all endomorphisms of $L^+(G)$

are accounted for thus. The correspondence $A \leftrightarrow \alpha_A$ is therefore one-one both ways between $L^{\times}(G)$ and $E^{\times}L^{+}(G)$; and it is an anti-isomorphism because $\alpha_A\alpha_B$ is the mapping $P \rightarrow BAP$, *i.e.*, α_{BA}.

THEOREM 5. *If G is palintropic, $L^{+}(G)$ has additive endomorphisms.*

For the endomorphisms of $L^{+}(G)$ are of the form (7), and by (6) $\alpha_A + \alpha_B$ is the mapping $P \rightarrow AP + BP = (A+B)P$, which is the endomorphism α_{A+B}. (From Theorems 1 and 5, if G has additive endomorphisms, so has $L^{+}(G)$.)

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THE CONCURRENCY OF PERPENDICULARS

PAUL A. CLEMENT, State College of Washington

Introduction. In the determination of possible concurrency of cevians of a triangle, Ceva's theorem ([1] p. 50) plays an important and useful role. We establish here a theorem which is a kind of "companion" to Ceva's in that it plays a similar role in the determination of concurrency of perpendiculars to the sides of a triangle.

Definitions. The symbol (K, L, M) will indicate three points K, L, and M, on the sides (possibly produced) BC, CA and AB respectively of a triangle ABC. We consider directed segments with $\overline{AM} = -\overline{MA}$, so that for example $\overline{AM} + \overline{MB} = \overline{AB}$ for all positions of M on the line $\overline{AB}$.

The points (K, L, M) will be said to satisfy condition C if the cevians AK, BL, and CM are concurrent, and then the point of concurrency will be called the C-point of (K, L, M).

The points (K, L, M) will be said to satisfy condition P if the perpendiculars to the sides of the triangle erected at K, L, and M are concurrent, and then the point of concurrency will be called the P-point of (K, L, M).

Thus, if A', B', and C' are the midpoints of the sides opposite A, B, and C, (A', B', C') satisfy condition C, and the C-point here is the centroid of the triangle. We note that the orthocenter is both a C-point and P-point.

The "Companion" Theorem. Ceva's theorem and its converse may now be stated in the following way: The points (K, L, M) satisfy condition C if and only if $\overline{AM}\cdot\overline{BK}\cdot\overline{CL}=\overline{MB}\cdot\overline{KC}\cdot\overline{LA}$. We shall prove the following

THEOREM. *The points (K, L, M) satisfy condition P if and only if*

$$\overline{AM}^2 + \overline{BK}^2 + \overline{CL}^2 = \overline{MB}^2 + \overline{KC}^2 + \overline{LA}^2. \tag{1}$$

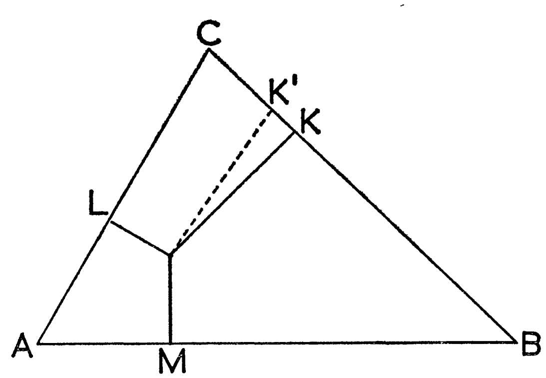

FIG. 1

Proof. (*Necessity*). From P (see Fig. 1) drop perpendiculars to the sides and join P to A and B. In right triangles PAM and PBM we have that

$$\overline{AM}^2 = \overline{AP}^2 - \overline{PM}^2, \text{ and } \overline{MB}^2 = \overline{BP}^2 - \overline{PM}^2.$$

By subtraction, we get that

$$\overline{AM}^2 - \overline{MB}^2 = \overline{AP}^2 - \overline{BP}^2. \tag{2c}$$

Similar relations, (2a) and (2b), are obtained in the triangles BCP and CAP, and the addition of these three, (2a), (2b), and (2c), results in (1).

(*Sufficiency*). Suppose (1) to hold, that perpendiculars at M and L intersect at P, and that the perpendicular from P to BC intersects at K'. Then, by the necessity proof, we have that

$$\overline{AM}^2 + \overline{BK'}^2 + \overline{CL}^2 = \overline{MB}^2 + \overline{K'C}^2 + \overline{LA}^2. \tag{3}$$

Now subtract (3) from (1) to get that

$$\overline{BK}^2 - \overline{BK'}^2 = \overline{KC}^2 - \overline{K'C}^2, \text{ or } \overline{BK}^2 - \overline{KC}^2 = \overline{BK'}^2 - \overline{K'C}^2;$$

and it is easily seen that this holds only if $K = K'$.

COROLLARY. *Each of the following four relations is equivalent to* (1), (*a, b, and c denote the length of the sides opposite A, B, and C*).

$$c\cdot\overline{AM} + a\cdot\overline{BK} + b\cdot\overline{CL} = c\cdot\overline{MB} + a\cdot\overline{KC} + b\cdot\overline{LA}, \tag{4}$$

$$2(c\cdot\overline{AM} + a\cdot\overline{BK} + b\cdot\overline{CL}) = c^2 + a^2 + b^2, \tag{5}$$

$$\overline{AM}\cdot\sin C + \overline{BK}\cdot\sin A + \overline{CL}\cdot\sin B = \overline{MB}\cdot\sin C + \overline{KC}\cdot\sin A + \overline{LA}\cdot\sin B, \tag{6}$$

and

$$\pm c\cdot\overline{MM_1} \pm a\cdot\overline{KK_1} \pm b\cdot\overline{LL_1} = 0, \tag{7}$$

where (K_1, L_1, M_1) *are the isotomic points of* (K, L, M) *and* (7) *holds for some choice of signs.*

Proof. We obtain (4) by transposing in (1) and factoring pairs; (5) comes from (4) when an expression equal to the left member of (4) is added to both sides of (4); the Sine Law substituted in (4) produces (5); (7) is obtained by transposing in (4) and identifying segments.

We remark that (4) seems to be the most useful in this set of equivalent relations. Using (4), one easily deduces the concurrency of the altitudes, the perpendicular bisectors of the sides, *etc.*

COROLLARY. *If* (K, L, M) *satisfy condition P, then so also do* (K_1, L_1, M_1), *where K_1, L_1, and M_1 are the isotomic points of K, L, and M.*

Proof. This is obvious either from (7) or (4). Two P-points related as in this corollary will be called *isotomic mates* (cf. isotomic conjugate points ([1] p. 169). Thus, perpendiculars erected at the "internal" touch points of the excircles of a triangle are concurrent, and the corresponding P-point is the isotomic mate of I, the incenter of the triangle.

Points on the Circumcircle. We may now conveniently restate two theorems ([1] p. 152) on the Simson line.

THEOREM. *The points* (K, L, M) *determine a point on the circumcircle of the triangle ABC* (*which is their P-point*) *if and only if both* (K, L, M) *satisfy condition P and* (K, L, M) *are collinear.*

Proof. See reference given; however, it is an interesting exercise to prove this theorem using segment lengths in (4) and in $\overline{AM}\cdot\overline{BK}\cdot\overline{CL} = (-1)\ \overline{MB}\cdot\overline{KC}\cdot\overline{LA}$, the latter being a necessary and sufficient condition that (K, L, M) be collinear.

A Property of the Nine-Point Circle. The following theorem generalizes a property of the nine-point circle and also presents a kind of converse to another theorem ([1] p. 170, prob. 9).

THEOREM. *Let (K, L, M) be noncollinear and satisfying condition P. Then the circumscribing circle of (K, L, M) intersects the sides of the triangle ABC in (K', L', M') which also satisfy condition P; further, the corresponding P-points, P and P', are isogonal conjugates and R, the center of the circle, is the midpoint of $\overline{PP'}$.*

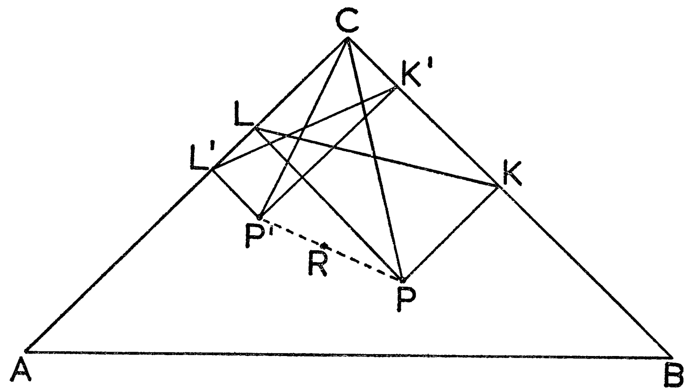

FIG. 2

Proof. The circle circumscribing (K, L, M) intersects the sides of the triangle (Fig. 2) with chords $\overline{KK'}$, $\overline{LL'}$, and $\overline{MM'}$; let K_1, L_1, and M_1 be the midpoints of these chords. Obviously (K_1, L_1, M_1) satisfy condition P, the P-point being R, the center of the circle. Thus, we have that

$$c\cdot\overline{AM} + a\cdot\overline{BK} + b\cdot\overline{CL} = c\cdot\overline{MB} + a\cdot\overline{KC} + b\cdot\overline{LA}, \tag{8}$$

and

$$c\cdot\overline{AM}_1 + a\cdot\overline{BK}_1 + b\cdot\overline{CL}_1 = c\cdot\overline{M_1B} + a\cdot\overline{K_1C} + b\cdot\overline{L_1A}. \tag{9}$$

Subtract (8) from (9) to get that

$$c\cdot\overline{MM}_1 + a\cdot\overline{KK}_1 + b\cdot\overline{LL}_1 = c\cdot\overline{M_1M} + a\cdot\overline{K_1K} + b\cdot\overline{L_1L};$$

and since (K_1, L_1, M_1) are midpoints, it follows that

$$c\cdot\overline{M_1M'} + a\cdot\overline{K_1K'} + b\cdot\overline{L_1L'} = c\cdot\overline{M'M}_1 + a\cdot\overline{K'K}_1 + b\cdot\overline{L'L}_1. \tag{10}$$

On adding (9) and (10), we find that

$$c\cdot\overline{AM'} + a\cdot\overline{BK'} + b\cdot\overline{CL'} = c\cdot\overline{M'B} + a\cdot\overline{K'C} + b\cdot\overline{L'A},$$

which is the condition that (K', L', M') satisfy condition P. It is obvious that

R is the midpoint of $\overline{P}\overline{P}'$. Finally,

$$
\begin{aligned}
\angle L'CP' + \angle P'CP &= \angle LKP, \text{ since } CLPK \text{ is inscribable,}\\
&= 90^\circ - \angle K'KL\\
&= 90^\circ - \angle LL'K', \text{ since } KK'LL' \text{ is inscribable,}\\
&= \angle K'L'P'\\
&= \angle KCP + \angle P'CP, \text{ since } CL'P'K' \text{ is inscribable.}
\end{aligned}
$$

Thus $\angle L'CP' = \angle KCP$, and similarly at the other vertices, so that P and P' are isogonal conjugates.

It is evident that our circumscribing circle is the nine-point circle of the triangle ABC if (K, L, M) are the midpoints of the sides, in which case P and P' become 0, the circumcenter, and H, the orthocenter, respectively.

Reference

1. D. R. Davis, Modern College Geometry, 1949.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

NOTES ON MATRIX THEORY—XIV: ON THE JACOBI RELATION FOR THE BRACKET SYMBOL

RICHARD BELLMAN, The RAND Corporation

1. Introduction. Let $[A, B]$ denote the commutator expression $AB-BA$ where A and B are square matrices. The object of this note is to derive, rather than verify, the classical three-term relation

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{1}$$

2. Preliminaries. Let e^A denote the matrix exponential, and write

$$e^A e^B = e^{A+B+f(A,B)}, \tag{1}$$

where

$$f(A, B) = [A, B]/2 + g(A, B) + h(A, B), \tag{2}$$

where $g(A, B)$ is a homogeneous polynomial of degree 3 in A and B satisfying the relation

$$g(A, B) = g(B, A), \tag{3}$$

and $h(A, B)$ is a sum of homogeneous polynomials in A and B beginning with one of degree 4.

3. Associativity. Let us now employ the associative property of matrices. We have

$$e^A(e^Be^C) = (e^Ae^B)e^C. \tag{1}$$

Hence

$$e^Ae^{B+C+f(B,C)} = e^{A+B+f(A,B)}e^C. \tag{2}$$

Using (2.1) again, this yields the basic functional equation

$$f(B, C) + f(A, B + C + f(B, C)) = f(A, B) + f(A + B + f(A, B), C). \tag{3}$$

4. Derivation of Jacobi relation. Let us now use the relation of (2.2) in (3.3) We obtain

$$\begin{aligned} &f(B, C) + f(A, B + C + f(B, C)) \\ &\quad = [B, C]/2 + g(B, C) + 1/2[A, B + C + [B, C]/2] \\ &\qquad + g(A, B + C) + \text{terms of degree four or more.} \end{aligned} \tag{1}$$

Hence to terms of degree four,

$$\begin{aligned} &f(B, C) + f(A, B + C + f(B, C)) \\ &\quad = [B, C]/2 + [A, B + C]/2 + [A, [B, C]]/4 + g(B, C) + g(A, B + C), \\ &f(A, B) + f(A + B + f(A, B), C) \\ &\quad = [A, B]/2 + [A + B, C]/2 + [[A, B], C]/4 + g(A, B) + g(A + B, C). \end{aligned} \tag{2}$$

Using the linearity of the bracket symbol, $[X+Y, Z]=[X, Z]+[Y, Z]$, this relation reduces to

$$\begin{aligned} &\frac{[A, [B, C]]}{4} + g(B, C) + g(A, B + C) \\ &\qquad = \frac{[[A, B], C]}{4} + g(A, B) + g(A + B, C). \end{aligned} \tag{3}$$

Let us now permute the symbols cyclically, replacing A by B, B by C, C by A, and then A by C, B by A, C by B, and then add.

Cancelling the common terms, we obtain

$$\begin{aligned} &(1/4)\{[A, [B, C]] + [B, [C, A]] + [C, [A, B]]\} \\ &\qquad + \{g(A, B + C) + g(B, C + A) + g(C, A + B)\} \\ &\quad = (1/4)\{[[A, B], C] + [[B, C], A] + [[C, A], B]\} \\ &\qquad + \{g(A + B, C) + g(B + C, A) + g(C + A, B)\}. \end{aligned} \tag{4}$$

Referring to (1.3), we see that the g-terms cancel. Using the result that $[A, B] = -[B, A]$, we obtain (1.1).

A TEMPERATURE FUNCTION WHICH VANISHES INITIALLY

P. C. ROSENBLOOM, University of Minnesota and D. V. WIDDER, Harvard University

In the theory of heat conduction it is well known that the temperature of an infinite bar is not uniquely determined by its temperature at any given instant. The function

$$u(x, t) = xt^{-3/2}e^{-x^2/4t}$$

satisfies the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1}$$

for $t>0$, and $u(x, 0+)=0$ for $-\infty<x<\infty$. This example, usually cited to substantiate the failure of uniqueness, is unsatisfactory for some purposes since it is discontinuous at the origin. The purpose of the present note is to exhibit a solution of equation (1) which is of class C^∞ over the whole plane, vanishes for $t=0$, but which is not identically zero. Various other authors, including E. Hille [1], L. Hörmander [2], S. Täcklind [3], and A. Tychonov [4], have shown the existence of such solutions. The advantage of the present example is that it can be set down explicitly and that its properties can be derived in a very elementary way.

Set

$$g(x, t) = e^x \cos(x + 2t) + e^{-x} \cos(x - 2t),$$
$$a(y) = e^{-y^{4/3}}y \cos(\sqrt{3}y^{4/3}).$$

The required function is

$$u(x, t) = \int_0^\infty a(y)g(xy, ty^2)dy. \tag{2}$$

It is clear that $g(xy, ty^2)$ satisfies (1) over the whole plane for each value of the parameter y. It will be evident that $u(x, t)$ satisfies (1) and is of class C^∞ if differentiation under the integral sign is valid. But

$$\left|\frac{\partial^{2n} g(x, t)}{\partial x^{2n}}\right| = \left|\frac{\partial^n g(x, t)}{\partial t^n}\right| \leqq 2^{n+1} \cosh x. \tag{3}$$

Now differentiate the integral (2) n times with respect to t or $2n$ times with respect to x under the integral sign. The step will be valid if the resulting integral is dominated, say for $|x| \leqq R$, $-\infty<t<\infty$, by another convergent one in which the integrand is independent of x and t. But (3) gives the dominant integral

$$2^{n+1}\int_0^\infty e^{-y^{4/3}}y^{2n+1} \cosh Ry\, dy,$$

obviously convergent for every $R>0$.

We now show that

$$u(x, 0) = 2\int_0^\infty a(y) \cosh xy \cos xy \, dy = 0, \qquad -\infty < x < \infty. \tag{4}$$

By adding the four series

$$e^{\pm xy \pm ixy} = \sum_0^\infty (\pm 1 \pm i)^n \frac{(xy)^n}{n!} \ll \sum_0^\infty \frac{2^n |xy|^n}{n!},$$

we have

$$\cosh xy \cos xy = \sum_0^\infty \frac{a_n (xy)^{4n}}{4n!} \ll \sum_0^\infty \frac{2^n |xy|^n}{n!} \tag{5}$$

for suitable, but irrelevant, constants a_n. Now substitute the series (5) in the integral (4) and integrate term by term, a valid step since

$$\int_0^\infty e^{-y^{4/3}} y e^{2|xy|} dy < \infty, \qquad -\infty < x < \infty.$$

But now (4) is established since it is well known that

$$\int_0^\infty a(y) y^{4n} dy = 0, \qquad n = 0, 1, 2, \cdots. \tag{6}$$

For completeness we add the proof. The integral (6) is the real part of

$$\int_0^\infty e^{-y^{4/3} z} y^{4n+1} dy = (3/4)\, \Gamma[3(2n+1)/2] z^{-3(2n+1)/2}, \tag{7}$$

where $z = 1 - i\sqrt{3}$. But

$$\arg z^{-3(2n+1)/2} = \frac{3(2n+1)}{2} \cdot \frac{\pi}{3} = (2n+1)\frac{\pi}{2} \cdot$$

That is, the integral (7) is purely imaginary, and (6) is established. Finally,

$$u(0, t) = 2\int_0^\infty a(y) \cos 2\, t y^2 dy \tag{8}$$

is not identically zero by the uniqueness theorem for Fourier-cosine transforms of functions in the class L. We observe that $a(y)$ might be replaced by many others which are known to make the moments (6) all vanish.

From the general theory of Hörmander [2] one could also exhibit the closely-related solution

$$u(x, t) = \int_{a-i\infty}^{a+i\infty} e^{st + x\sqrt{s} - s^{2/3}} ds, \qquad a > 0,$$

where the integration is along the line Re $s=a$ of the complex s-plane. This solution has the additional feature that it vanishes identically for negative t. Equation (8) shows that this is not true of our example.

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ON CIRCULAR PERMUTATIONS

L. E. Clarke, University College of Ghana and James Singer, Brooklyn College

Let a set of n objects contain r_1 elements a_1, r_2 elements $a_2, \cdots, r_p$ elements a_p. The problem is to determine P, the number of circular permutations of the n objects. This problem has been discussed in the note *A general formula for circular permutations* (this Monthly, vol. 64, 1957, pp. 347–348), but the statement and proof of the result (Theorem 2 of the note) contain an error. For example, if $n=12$, $r_1=r_2=6$, the formula in the paper cited does not give an integral value for P; our result gives $P=80$. The error arises from improper linking of linear permutations. We obtain the circular permutations by linking d similar linear permutations, where d is any divisor of h, the greatest common divisor of $r_1, \cdots, r_p$.

Theorem. *With the above notation*

$$P = \sum_{d|h} \frac{\phi(d)F_d}{n},$$

where

$$F_d = \frac{(n/d)!}{(r_1/d)! \cdots (r_p/d)!},$$

$\phi(d)$ is Euler's function, and the summation is over all divisors d of h.

Proof. Let a linear permutation of the n objects be called "*d-fold*" if it consists of d repetitions of a linear permutation of n/d of the objects; let it be called "*precisely d-fold*" if it is d-fold, but is not d_1-fold for any $d_1>d$. It is clear that if a linear permutation is d-fold or precisely d-fold then d divides h.

The number of d-fold linear permutations is F_d; let the number of precisely d-fold linear permutations be denoted by X_d.

There is an n/d: 1 correspondence between precisely d-fold linear permuta-

tions and those circular permutations obtained by linking d (and not more than d) identical linear permutations. Therefore

$$P = \sum_{d|h} \frac{d}{n} X_d. \tag{1}$$

Further, if a d-fold linear permutation is precisely d_1-fold then d divides d_1; conversely, if d divides d_1, then a precisely d_1-fold linear permutation is also d-fold. It follows that

$$F_d = \sum_{k|h/d} X_{kd} \qquad (d \mid h). \tag{2}$$

It is necessary to solve the equations (2) for X_d $(d| h)$ and substitute in the equation (1). This will be done by means of the Möbius inversion formula and by use of the substitutions

$$G_d = F_{h/d}, \qquad Y_d = X_{h/d} \qquad (d \mid h). \tag{3}$$

By (2) and (3)

$$G_d = \sum_{k|d} Y_{d/k} = \sum_{k|d} Y_k. \tag{4}$$

The Möbius inversion formula applied to (4) gives

$$Y_d = \sum_{k|d} \mu\left(\frac{d}{k}\right) G_k. \tag{5}$$

By (1) and (3)

$$\begin{aligned} P &= \sum_{d|h} \frac{h}{dn} Y_d \\ &= \sum_{d|h} \frac{h}{dn} \left\{ \sum_{k|d} \mu\left(\frac{d}{k}\right) G_k \right\} && \text{(by (5))} \\ &= \sum_{k|d|h} \frac{h}{dn} \mu\left(\frac{d}{k}\right) G_k \\ &= \sum_{k|h} \frac{1}{n} G_k \left\{ \frac{h}{k} \sum_{l|h/k} \frac{\mu(l)}{l} \right\} && \text{(on putting } d = kl) \\ &= \sum_{k|h} \frac{1}{n} G_k \phi\left(\frac{h}{k}\right) = \sum_{k|h} \frac{1}{n} F_k \phi(k) && \text{(by (3)).} \end{aligned}$$

This result, though certainly not original, does not seem to be widely known. It is a particular case of a very general theorem of Pólya's (Acta Mathematica, vol. 68, 1937, pp. 145–254; Pólya's "Hauptsatz" is stated in Section 16).

A PROBLEM IN GRAPH THEORY

DONALD J. NEWMAN, AVCO Research and Advanced Development Division

Our purpose is to give a simple constructive proof of the

THEOREM. *Any graph with $2n$ vertices each of order not less than n must contain a $2n$-gon.*

A colorful interpretation of this problem is the following:

Given $2n$ people, each one a friend of at least n others (friend is reflexive), there is a way of seating them around a table such that *only* friends sit next to one another.

It is easily seen that the theorem fails if the vertices need only be of order $n-1$, for then they may form two disconnected graphs of n vertices each.

We turn now to the proof. It will be seen that despite its "*Reductio ad absurdum*" form it actually gives a construction for determining the $2n$-gon. (Seating arrangement.) This construction, furthermore, seems to be fairly economical. We shall prefer the language of the "Seating" formulation.

Proof. The trick is to introduce m new people of the Dale Carnegie type, that is, people of whom everyone is a friend. It may or may not be that with this new blood the seating is possible. It is clear, however, that if $m=2n$ it will be possible by the simple expedient of alternation. Let k, then, be the minimum m for which the seating is possible. We must prove $k=0$, so assume $k>0$, and suppose the seating arrangement to be

$$APB \cdots A$$

where P is a Dale Carnegier, and the two A's above refer to the same person. If in the sequence following the B there ever occurred an $A'B'$, (A' means friend of A, *etc.*), then forming

$$A[B \cdots A']B' \cdots A$$

and reversing those in the bracket we obtain

$$AA' \cdots BB' \cdots A,$$

an acceptable seating arrangement! [It is understood that if $B'=A$ the reversal is unnecessary.]

But, without P there can be no acceptable seating arrangement.

$\therefore$ $A'B'$ never occurs past B.

$\therefore$ $A'B'$ never occurs at all.

$\therefore$ All A' are followed by non-B',

hence number of non-B''s $\geqq$ number of A''s $\geqq n+k$. Also, number of B''s $\geqq n+k$.

Adding these two inequalities gives

$$\text{total number of people} \geqq 2n+2k,$$

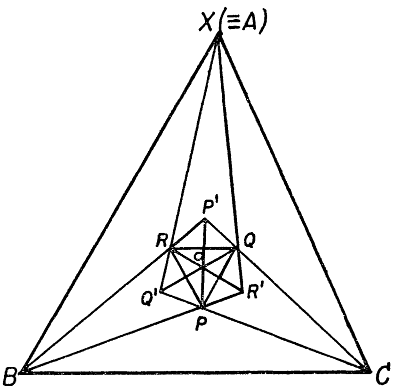
or $2n+k \geqq 2n+2k$, a contradiction since $k>0$.

AN ELEMENTARY PROOF OF MORLEY'S THEOREM

K. Venkatachaliengar, Mysore University

Morley's theorem states that *the points of intersection of the adjacent trisectors of the interior angles of a triangle meet at the vertices of an equilateral triangle.*

This theorem has been proved in many ways (*cf.* [1]). We prove the result here in a simple way, using only the elementary theorems of congruence-geometry.



ABC is a triangle. First of all draw the triangle $P'BC$ with angles $P'\hat{B}C = 2/3(A\hat{B}C)$, $P'\hat{C}A = 2/3(A\hat{C}B)$ and let P be the incentre of this latter triangle (*i.e.* $P'BC$). Determine the points Q and R on CP', BP' respectively, such that $P'\hat{P}Q = P'\hat{P}R = \pi/6$. It follows that PQR is an equilateral triangle. Let O be the centre of the triangle PQR. Produce QO and RO to meet CP produced and BP produced in Q' and R' respectively, and let $Q'R$ and QR' meet in X. We shall show that X and A are identical and that the equilateral triangle PQR is the "Morley Triangle" of ABC.

From the figure we see that

$$\begin{aligned} X\hat{R}B &= P'\hat{R}Q' = P'\hat{R}Q + Q\hat{R}Q' \\ &= \frac{\pi}{2} - R\hat{P}'O + Q\hat{P}Q' = \frac{\pi}{2} - R\hat{P}'O + Q\hat{P}P' + P'\hat{P}Q' \\ &= \frac{\pi}{2} - R\hat{P}'O + \frac{\pi}{6} + (P'\hat{C}P + P\hat{P}'C) = \frac{2\pi}{3} + \frac{C}{3}, \end{aligned}$$

and

$$X\hat{R}'B = 2R\hat{R}'P = 2\left(\pi - \frac{\pi}{6} - R\hat{P}R'\right)$$

$$= 2\left[\left(\frac{5\pi}{6}\right) - \left(\frac{\pi}{6} + P'\hat{P}R'\right)\right]$$

$$= \frac{4\pi}{3} - 2\left(\frac{B}{3} + B\hat{P}'P\right)$$

$$= \frac{4\pi}{3} - \frac{2B}{3} - \left(\pi - \frac{2B}{3} - \frac{2C}{3}\right)$$

$$= \frac{\pi}{3} + \frac{2C}{3}$$

so that

$$X\hat{R}B = \tfrac{1}{2}\pi + \tfrac{1}{2}(X\hat{R}'B). \tag{1}$$

Since RR' bisects $X\hat{R}'B$, we obtain from (1) that R is the incentre of the triangle $XR'B$. Therefore $X\hat{B}R = 1/3(A\hat{B}C)$ and $B\hat{X}R = R\hat{X}R'$ and X is on the side AB. We may see in a similar way that Q is the incentre of the triangle $XQ'C$ and therefore $X\hat{C}Q = 1/3(A\hat{C}B)$ and $C\hat{X}Q = Q\hat{X}R$ and X is on AC. Consequently, since X is on both AB and AC, $X \equiv A$. It follows that AR, AQ are the trisectors of $B\hat{A}C$ and that PQR is the "Morley Triangle" of ABC.

We see further that the triangle $P'RQ$, $Q'PR$, $R'QP$ are all isosceles with their equal angles given by $1/3(B+C)$, $1/3(C+A)$, $1/3(A+B)$ respectively, if A, B, C denote the angles of the original triangle ABC. Therefore $A\hat{R}Q = 1/3(\pi+B)$ and we may now determine the side of the Morley triangle on employing the relations

$$a \operatorname{cosec} A = b \operatorname{cosec} B = c \operatorname{cosec} C = 2\rho = \text{circular diameter}$$

for a triangle. We check the well-known formula

$$PQ = QR = RP = 8\rho \sin\frac{A}{3} \sin\frac{B}{3} \sin\frac{C}{3} \cdot$$

Reference

1. Clarence Lubin, A proof of Morley's theorem, this MONTHLY, vol. 62, 1955, pp. 110–112. (Also the references contained therein.)

A WARNING ABOUT TRANSLATING AXIOMS*

H. Hiż, Pennsylvania State University

In a recent paper [1] Halmos used conjunction (A) and negation (N) as primitives of the sentential calculus and considered disjunction and implication between s and t as abbreviations for $NANsNt$ and $NAsNt$ respectively. As

* This paper was written while the author was engaged in a research project sponsored by the National Science Foundation.

axioms for the (A, N) system, he took the standard [2] set based on disjunction and negation as primitives and considered them as abbreviations for

(1) $NANANsNsNs$,
(2) $NAsNNANsNt$,
(3) $NANANsNtNNANtNs$,
(4) $NANAsNtNNANANuNsNNANuNt$.

The rule of inference (without abbreviations) states that if s and $NAsNt$ are tautologies, then t is a tautology.

However, this set of axioms is not complete. To see this consider the following three-valued interpretation of A and N where 1 is the only designated value:

A	1	2	3	N
*1	1	1	3	3
2	2	2	3	1
3	3	3	3	1

The rule of inference leads from expressions assigned the designated value to expressions assigned the designated value. The axioms (1)–(4) take the designated value always. But $NANss$ (which is a tautology) assumes the value 3 when one uses 2 for s. Thus, $NANss$ does not follow from the axioms. Similarly, *e.g.*, $NANsAst$, $NAAstNNNAsNNt$ are not provable. Note that in checking the fact that (1)–(4) assume always 1 and that the rule leads from $s = NAsNt = 1$ to $t = 1$, one never uses the equality $A12 = 1$ (this results from the circumstance that a nonnegated variable never occurs in the axioms as the second argument of A), which is essentially involved in the quoted examples of unprovable tautologies. Still, addition of, *e.g.*, $NANAstNNAtNNs$ would leave the quoted tautologies unprovable. For complete axiomatisations of (A, N) consult [3] and [4]. Note also that $NAAstNNNAsNNt$ can be considered as a part of the definition of conjunction by implication (C) and negation: $Ast = NCsNt$ (write it as an implication and translate into (A, N) language). A translation of a complete set of axioms to another set of primitives would be complete only if from the resulting axioms the definitions of the first set of primitives followed.

This mistake is of no consequence for the totality of Halmos' paper where the considerations about the axiomatic setup of the sentential calculus were only parenthetical. Halmos himself raised (in a conversation) the problem of completeness of the translated axiomatisation; the possibility that the answer might be negative was first signaled by J. B. Rosser (in a letter to Halmos).

References

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2. D. Hilbert and W. Ackermann, Principles of Mathematical Logic, New York, 1950, p. 27.
3. Bolesław Sobociński, Axiomatisation of a conjunctive-negative calculus of propositions. J. Computing Systems, vol. 1, 1954, pp. 229–242. (This is a translation of a paper from Collectanea Logica, vol. 1, 1939.)
4. J. Barkley Rosser, Logic for Mathematicians, New York, 1953, p. 55.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

ON SOME THEOREMS ON PERMUTATIONS

W. A. AL-SALAM, Duke University

Recently Mullin [1] discussed the function

$$\Phi(n) = \sum_{r=0}^{n} {}_nP_r$$

where n is a nonnegative integer and ${}_nP_r = n!/(n-r)!$.

It might be of interest to consider the more general function

$$\Phi_k(n) = \sum_{k=0}^{n} \{{}_nP_r\}^k, \ k \text{ a positive integer.}$$

We have the following theorems:

THEOREM 1. *A canonic generator for* Φ_k *is*

$$\Phi_k(n+1) = (n+1)^k\Phi_k(n) + 1.$$

Proof. We have

$$\Phi_k(n+1) = \sum_{r=0}^{n+1} \frac{((n+1)!)^k}{(r!)^k} = (n+1)^k \sum_{r=0}^{n+1} \left(\frac{n!}{r!}\right)^k = (n+1)^k\Phi_k(n) + 1.$$

THEOREM 2. *An asymptotic expansion for* $\Phi_k(n)$ *is*

$$\Phi_k(n) \sim (n!)^k \, {}_0F_{k-1}(1, 1, \cdots, 1; 1)$$

where, in the notation of the hypergeometric functions,

$${}_0F_{k-1}(\alpha_1, \cdots, \alpha_{k-1}; z) = \sum_{r=0}^{\infty} \frac{z^r}{r!(\alpha_1)_r \cdots (\alpha_{k-1})_r},$$

$$(\alpha)_r = \alpha(\alpha+1) \cdots (\alpha+r-1) \qquad (r \geqq 1),$$

$$(\alpha)_o = 1.$$

${}_oF_k(\alpha_1, \cdots, \alpha_k; z)$ *converges for all* z.

Proof.

$$\Phi_k(n) = (n!)^k\left\{ \sum_{r=0}^{\infty}\left(\frac{1}{r!}\right)^k - \sum_{r=n+1}^{\infty}\left(\frac{1}{r!}\right)^k\right\}$$

$$= (n!)^k \, {}_oF_{k-1}(1, 1, \cdots, 1; 1) + o((n!)^k).$$

THEOREM 3.

$$\Phi_k(n) \sim (2\pi)^{k/2} n^{k(n+1/2)} e^{-kn} {}_0F_{k-1}(1, 1, \cdots, 1; 1).$$

Proof. As in [1] this theorem follows from Sterling's formula and Theorem 2.

Now let $J_0(x)$ be the Bessel function of order zero. It is given by

$$J_o(x) = {}_oF_1\left(1; -\frac{x^2}{4}\right).$$

In particular we find from the tables that $J_o(2i) = 2.280$.

We have the following corollary to Theorems 2 and 3:

COROLLARY.

$$\Phi_2(n) = \sum_{r=0}^{n} \{{}_nP_r\}^2 \sim 2.280(n!)^2$$
$$\sim 4.560\pi \cdot n^{2n+1} \cdot e^{-2n}.$$

In this connection we also note that

$$\sum_{r=0}^{n} {}_nP_r \, {}_nC_r = L_n(-1),$$

where $L_n(x)$ is the Laguerre polynomial of order zero [2].

References

1. A. A. Mullin, Three theorems on permutations, this MONTHLY, vol. 64, 1957, pp. 669–670.
2. D. V. Widder, Advanced Calculus, New York, 1947, p. 383.

AN ANALYTIC APPROACH TO TRIGONOMETRIC FUNCTIONS

D. A. KEARNS, Merrimack College

Consideration of the elementary functions from purely analytic points of view can provide excellent instructional material. Unfortunately, although such developments occur here and there in mathematical literature, they may be inaccessible to many students and to some teachers, either because they are written in languages other than English or are found in texts of a fairly advanced nature.

In this note we deal with the sine and cosine functions, deriving their major properties from the equation for simple harmonic motion and without the introduction of infinite series. The few properties of linear differential equations needed can be made at least plausible even to students having only an elementary background in calculus.

We begin by considering the differential equation

$$\frac{d^2y}{dx^2} + y = 0. \tag{1}$$

By a solution of this equation we shall mean a continuous function, $y(x)$, which satisfies (1) and which has a continuous first derivative. The fundamental existence and uniqueness theorems of the theory of ordinary differential equations apply to (1). In particular, *there exists one and only one solution*, $y(x)$, *of* (1) *such that* $y(x_0)=y_0$ *and* $y'(x_0)=y_0'$, *where* x_0, y_0, y_0' *are arbitrary constants.* Geometrically, this says that there is one and only one curve passing through (x_0, y_0) with a slope of y_0' at this point, the equation of which satisfies (1).

Two solutions of (1), $y_1(x)$ and $y_2(x)$, are called linearly independent if the identity $c_1y_1(x)+c_2y_2(x)\equiv 0$ implies $c_1=c_2=0$.

The only other result we use from the theory of differential equations is the fact that any solution of (1) can be expressed as a linear combination of two linearly independent solutions. That is, if $y(x)$ is a solution of (1) and $y_1(x)$ and $y_2(x)$ are linearly independent solutions, then there are constants c_1 and c_2 such that $y(x)=c_1y_1(x)+c_2y_2(x)$.

Let us then consider two solutions of (1), $y=s(x)$ and $y=c(x)$ such that

$$s(0) = 0, \qquad s'(0) = 1; \tag{2}$$

and

$$c(0) = 1, \qquad c'(0) = 0. \tag{3}$$

THEOREM 1. *$s(x)$ and $c(x)$ are linearly independent.*

Proof. Suppose there are constants, c_1 and c_2, such that $c_1s(x)+c_2c(x)\equiv 0$. Then $c_1s'(x)+c_2c'(x)\equiv 0$.

In particular these identities hold when $x=0$, so that $c_1s(0)+c_2c(0)=0$ and $c_1s'(0)+c_2c'(0)=0$.

These equations together with (2) and (3) imply $c_1=c_2=0$.

THEOREM 2. $s'(x)=c(x)$.

Proof. Since $s(x)$ is a solution of (1), we must have $s''(x)+s(x)=0$. Just as this equation defines $s''(x)$ as a continuous function, the equation $s'''(x)+s'(x)=0$ shows that $s'''(x)$ exists and is continuous. Letting $y=s'(x)$, the last equation becomes equation (1). Furthermore, $y(0)=s'(0)=1$ by (2). Since $y'(x)=s''(x)=-s(x)$, it follows that $y'(0)=0$. Thus $y=s'(x)$ is a solution of (1) satisfying equations (3). But there is only one such function because of the uniqueness asserted by the theorem quoted above. Since $c(x)$ is this function, $s'(x)\equiv c(x)$.

THEOREM 3. $c'(x)=-s(x)$.

Proof. $c'(x)=s''(x)=-s(x)$ by Theorem 2 and equation (1).

THEOREM 4. $s^2(x)+c^2(x)\equiv 1$.

Proof. Since $s(x)$ satisfies (1), $s''(x)+s(x)=0$.
Multiplying this by $s'(x)$ we obtain, $s'(x)s''(x)+s'(x)s(x)=0$. Integrating,

it follows that $(s'(x))^2+(s(x))^2=\text{constant}$, or $c^2(x)+s^2(x)=\text{constant}$. But at $x=0$, because of (2) and (3), $c^2(0)+s^2(0)=1$. Therefore, $s^2(x)+c^2(x)\equiv 1$.

THEOREM 5. *For arbitrary* a, $s(x+a)=s(x)c(a)+s(a)c(x)$.

Proof. Since $s(x)$ is a solution of (1), so is $s(x+a)$. Therefore $s(x+a)$ can be expressed as a linear combination of the linearly independent solutions $s(x)$ and $c(x)$. That is, there are constants, c_1 and c_2, such that

$$s(x+a) = c_1 s(x) + c_2 c(x). \tag{4}$$

At $x=0$, this becomes $s(a)=c_1 s(0)+c_2 c(0)=c_2$, where we have used (2) and (3). Differentiating (4), we obtain

$$c(x+a) = c_1 c(x) - c_2 s(x). \tag{5}$$

Again letting $x=0$ and using (2) and (3), $c(a)=c_1 c(0)-c_2 s(0)=c_1$. Substituting these values for c_1 and c_2 in (4) yields the desired identity.

THEOREM 6. *For arbitrary* a, $c(x+a)=c(x)c(a)-s(x)s(a)$.

Proof. Substitute the c_1 and c_2 obtained in the proof of Theorem 5 in (5).

THEOREM 7. $s(-x)=-s(x)$ *and* $c(-x)=c(x)$.

Proof. Letting $a=-x$ in the identities of Theorems 5 and 6, we obtain two linear equations in $s(-x)$ and $c(-x)$:

$$s(x)c(-x) + c(x)s(-x) = 0,\ c(x)c(-x) - s(x)s(-x) = 1.$$

Since the determinant of this system is $-s^2(x)-c^2(x)=-1\neq 0$, we may solve the equations for $s(-x)$ and $c(-x)$ to obtain the theorem.

THEOREM 8. $s(x)$ *has at least one relative maximum point.*

Proof. Since $s(0)=0$, $s'(0)=1$, and $s(x)$ and $s'(x)$ are continuous functions, there is a neighborhood to the right of $x=0$ in which $s(x)$ is positive and monotonically increasing. As x becomes positively infinite, $s(x)$ must behave in one of the following mutually-exclusive ways:

(i) $s(x)$ becomes positively infinite monotonically;
(ii) $s(x)$ approaches a horizontal asymptote monotonically;
(iii) $s(x)$ has a relative maximum point for some positive value of x.

We show that (i) and (ii) are impossible.

Elimination of the first case follows immediately from Theorem 4. Since $s^2(x)+c^2(x)\equiv 1$, $s(x)$ is bounded.

To eliminate the possibility of (ii), suppose it to be true. It would then follow that

$$\lim_{x\to\infty} s'(x) = \lim_{x\to\infty} c(x) = 0,$$

which implies

$$\lim_{x\to\infty} c(x + a) = 0 \text{ for arbitrary } a,$$

and

$$\lim_{x\to\infty} s(x) = \lim_{x\to\infty} \sqrt{1 - c^2(x)} = 1.$$

But then,

$$\lim_{x\to\infty} c(x + a) = \lim_{x\to\infty} (c(x)c(a) - s(x)s(a)) = - s(a).$$

Since $s(x)$ is not identically zero, this is a contradiction.

Therefore, (iii) must be the case.

Remark. Since $c(x)$ is continuous and not identically zero, there must be a smallest positive value of x for which $s'(x) = c(x) = 0$. We shall designate this value by $\frac{1}{2}p$ and note that $c(\frac{1}{2}p) = 0$ and $s(\frac{1}{2}p) = 1$.

THEOREM 9. *$s(x)$ and $c(x)$ are periodic with period $2p$. That is, for any x, $s(x+2p) = s(x)$ and $c(x+2p) = c(x)$.*

Proof. $s(x + \frac{1}{2}p) = s(x)c(\frac{1}{2}p) + s(\frac{1}{2}p)c(x) = c(x)$, and $c(x + \frac{1}{2}p) = c(x)c(\frac{1}{2}p) - s(x)s(\frac{1}{2}p) = -s(x)$. Therefore,

$$s(x + p) = c(x + \tfrac{1}{2}p) = - s(x), \tag{6}$$

and $c(x+p) = -s(x+\frac{1}{2}p) = -c(x)$. It follows that $s(x+2p) = -s(x+p) = s(x)$ and $c(x+2p) = -c(x+p) = c(x)$.

THEOREM 10. *$2p$ is the smallest period of $s(x)$ and $c(x)$.*

Proof. By (6) and Theorem 7,

$$s(\tfrac{1}{2}p - x) = - s(x - \tfrac{1}{2}p) = s(x + \tfrac{1}{2}p),$$

so that $s(x)$ is symmetric with respect to the line $x = \frac{1}{2}p$. Now suppose there exists a P smaller than $2p$ but greater than zero such that $s(x+P) \equiv s(x)$. Then $s(P) = s(0) = 0$. Because of the symmetry proven above, p is the smallest positive zero of $s(x)$. Therefore $P = p$. But p is not a period of $s(x)$, and hence neither is P.

THEOREM 11. $p = \pi$.

Proof. The curve $u = c(x)$, $v = s(x)$, $0 \leqq x \leqq 2p$, represents a unit circle in the u, v-plane the circumference of which is given by

$$s = \int_0^{2p} \sqrt{(s'(x))^2 + (c'(x))^2}dx = \int_0^{2p} dx = 2p.$$

Hence, $2p = 2\pi$.

Remarks. This last proof points directly to a geometric interpretation of the functions $u=c(x)$ and $v=s(x)$. The curve $u^2+v^2=1$ is a unit circle. Any point on this circle has an abscissa, $c(x)$, and an ordinate, $s(x)$. The angle which the radius vector to any point (u, v) makes with the positive u-axis is x since the subtended arc is given by

$$\int_0^x (s^2(t) + c^2(t))^{1/2}\, dt = \int_0^x dt = x.$$

The infinite series expansions of $s(x)$ and $c(x)$ can be found in the usual way. That is, we try to find functions $y=\sum a_n x^n$ which satisfy (1), and (2) or (3). The resulting series and their derivatives converge, and by the uniqueness of solutions of these differential systems, are identical to $s(x)$ and $c(x)$ respectively.

Reference

W. F. Osgood, Lehrbuch der Funktionentheorie, I, Leipzig and Berlin, 1912.

THE INDEPENDENCE OF THE ASSOCIATIVE LAW

L. O. KATTSOFF, Harpur College

Independence proofs are not indispensable, to judge by their absence in various algebras. Quite often textbooks on abstract algebras or discussions of geometric postulates pay lip-service to the concept and then add that they will not show the independence of all the postulates. Historically, of course, independence proofs, or attempts at them, have been quite fruitful in revealing varieties of algebras and diversities of geometric systems.

The existence of nonassociative algebras is quite intriguing. An easy way to show that existence is to exhibit the independence of the associative law from the remaining postulates for the group. This is not too difficult if we also give up the commutative law. However, to show independence in the case of an Abelian group seems to be more difficult. I offer the following as demonstration of the independence of the associative law in an Abelian group.

Postulates: Given the set G containing elements $a, b, c, \cdots$, and a binary operation $\circ$ on G:

1. For every pair of elements a and b, $a \circ b$ is in G.
2. For any three elements a, b, and c, $(a \circ b) \circ c = a \circ (b \circ c)$.
3. There exists an element i in G such that $a \circ i = i \circ a = a$.
4. For every element a there exists an element a^{-1} in G, such that $a \circ a^{-1} = a^{-1} \circ a = i$.
5. For any two elements a and $b(a, b \neq i)$, $a \circ b = b \circ a$; if $a=b=i$, $i \circ i = i$.

To show the independence of Postulate 2, the associative law, and hence to show the consistency of the remaining set with the denial of Postulate 2, we construct nonassociative algebras in this fashion.

A) Consider the infinite set G:

This set is nonassociative for $n>2$.

Set C is intrinsically interesting because it is a finite set.

Both B) and C) differ from A) in that, in each of these, each element is its own inverse, while in the first example each element (except i) has a distinct inverse.

The following postulates for a nonassociative algebra suggest themselves:

(i) 1 is a number.

(ii) To every pair of numbers a and b in G, there is a number $a \circ b$ in G.

(iii) There are no numbers a and b such that $a \circ b = 1$.

(iv) If the numbers a, b, c, d are such that $a \circ b = c \circ d$, then $\max(a, b) = \max(c, d)$.

(v) If a set of numbers contains 1 and if whenever it contains a and b it contains $a \circ b$, then it contains all numbers.

ON THE DEFINITION OF ln a

S. Leader, Rutgers University

In defining the natural logarithm by the definite integral

$$\ln a = \int_1^a x^{-1} dx \tag{1}$$

for $a>0$, the logarithmic properties of $\ln a$ can be derived quite simply by first proving

$$\ln a = \lim_{h\to 0} \frac{a^h - 1}{h}\cdot \tag{2}$$

To prove (2) note that for $b>1$ and $1/b \leqq x \leqq b$ we have $|x^{h-1}-x^{-1}| = x^{-1}|x^h - 1| \leqq b(b^{|h|}-1)$ for all h. Hence, $\lim_{h\to 0} x^{h-1} = x^{-1}$ uniformly in $[1/b, b]$. This implies

$$\int_1^a x^{-1}dx = \lim_{h\to 0}\int_1^a x^{h-1}dx. \tag{3}$$

Now for $h \neq 0$,

$$\int_1^a x^{h-1}dx = \frac{x^h}{h}\Big]_1^a = \frac{a^h - 1}{h}\cdot \tag{4}$$

So (2) follows from (1), (3), and (4). The logarithmic properties of $\ln a$ can be derived from (2) in the following way.

$$\ln ab = \lim_{h\to 0}\frac{(ab)^h - 1}{h} = \lim_{h\to 0}\frac{a^h b^h - b^h + b^h - 1}{h}$$

$$= \lim_{h\to 0}\left(\frac{a^h - 1}{h}\right)b^h + \left(\frac{b^h - 1}{h}\right)$$

$$= \left(\lim_{h\to 0}\frac{a^h - 1}{h}\right)\left(\lim_{h\to 0} b^h\right) + \left(\lim_{h\to 0}\frac{b^h - 1}{h}\right)$$

$$= \ln a + \ln b.$$

$$\ln a^c = \lim_{h\to 0}\frac{a^{ch} - 1}{h} = \lim_{h\to 0} c\left(\frac{a^{ch} - 1}{ch}\right) = c\lim_{p\to 0}\frac{a^p - 1}{p}$$

$$= c \ln a, \text{ where } p = ch \text{ and } c \neq 0.$$

Thus, defining e by $\ln e = 1$, we have $\ln e^x = x \ln e = x$. So $\ln a$ is the logarithm of a to the base e.

To obtain the differentiation formula for a^x multiply (2) by a^x:

$$a^x \ln a = a^x \lim_{h\to 0}\frac{a^h - 1}{h} = \lim_{h\to 0}\frac{a^x(a^h - 1)}{h} = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} = \frac{d}{dx}a^x.$$

[For related ideas see *A note on the base of natural logarithms* by E. Leach, this MONTHLY, vol. 60, 1953, p. 622.]

NOTE ON A PAPER BY R. STEINBERG

HERBERT J. CURTIS, University of Illinois

In his Classroom Note *Note on linear differential equations* (January, 1957) R. Steinberg suggests a method for finding particular solutions of certain linear differential equations.

It should be noted that this method fails to yield a particular solution of such a simple equation as $(D-1)z = e^x$, nor indeed of any equation $f(D)z = u$, if the polynomial f is a divisor of g. In this case $k = 1$, $h = f$, and the method requires that we know a particular solution of $h(D)v = u$. But this equation is equivalent to $f(D)z = u$.

Moreover, *only* if f and g are relatively prime will the method yield solutions. Unless f and g are relatively prime, h and g have a common factor of degree greater than zero. To find a solution of $h(D)v = u$ we apply the method again and split h into two factors, h_1 and k_1, with k_1 relatively prime to g. Then h_1 and g have a common factor of degree greater than zero. Ultimately we are faced with the necessity of obtaining a particular solution of an equation $h_n(D)v = u$ with h_n a factor of g. But such a solution the method will not give.

The important part of Steinberg's theorem is the statement that there exists a solution of $f(D)z = u$ which is also a solution of $g(D)h(D)z = 0$. This is essentially the basis of the method of undetermined coefficients given, for example, in Rainville's *Elementary Differential Equations*, page 108 and following.

PARTICULAR INTEGRALS FOR NONHOMOGENEOUS, LINEAR, ORDINARY DIFFERENTIAL EQUATIONS

G. E. LATTA, Stanford University

In standard references which treat elementary integration techniques for ordinary linear differential equations, the methods given for obtaining particular integrals include variation of parameters, and special procedures for constant coefficient equations. The role of the adjoint equation in this context seems to be almost completely overlooked and, indeed, the adjoint equation is seldom used in elementary courses on this subject. However, a straightforward application of the adjoint equation leads to a well-motivated construction of a particular integral, and the method has certain advantages.

In the following, let

$$Ly \equiv y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = r(x) \tag{1}$$

be a given linear equation. The adjoint expression corresponding to Ly is

$$\bar{L}v = (-1)^n v^{(n)} + (-1)^{n-1}(p_1(x)v)^{(n-1)} + \cdots - (p_{n-1}(x)v)' + p_n(x)v \tag{2}$$

and we may introduce the adjoint equation by trying to construct an integrating factor for the expression Ly; *i.e.*, $v(x)Ly$ is to be a total derivative. Thus

$$vLy = \frac{d}{dx}\{P(y, v)\} + y\bar{L}v \tag{3}$$

is Lagrange's identity, $\bar{L}v$ the adjoint of Ly, and $P(y, v)$ is a bilinear expression, linear in $y, y', \cdots, y^{(n-1)}$, and $v, v', \cdots, v^{(n-1)}$, called the bilinear concomitant. In this way, v is an integrating factor for Ly provided $\bar{L}v=0$. Since the adjoint of $\bar{L}v$ is Ly, we see that, reciprocally, y is an integrating factor of $\bar{L}v$ if $Ly=0$.

An immediate consequence of (3) is that if we know all the solutions of $Ly=0$, then the solutions of $\bar{L}v=0$ can be obtained by purely algebraic means. To see this, let $y_1, \cdots, y_n$ be a set of linearly independent solutions of $Ly=0$. Then (3) becomes

$$vLy_i = \frac{d}{dx}\{P(y_i, v)\} + y_i\bar{L}v = 0 \qquad i = 1, \cdots, n,$$

since $Ly_i=0$. To solve $\bar{L}v=0$, we have

$$\frac{d}{dx}\{P(y_i, v)\} = 0, \quad \text{or} \quad P\{y_i, v\} = c_i \qquad i = 1, \cdots, n. \tag{4}$$

Equations (4) are a set of n linear algebraic equations in $v, v', \cdots, v^{(n-1)}$. An elementary calculation shows that the determinant of the coefficients of $v, v', \cdots, v^{(n-1)}$ in these equations is

$$\pm\begin{vmatrix} y_1 & y_1' \cdots y^{(n-1)} \\ \cdot & \\ \cdot & \\ \cdot & \\ y_n & y_n' \cdots y_n^{(n-1)} \end{vmatrix} = \pm W(y_1, \cdots, y_n),$$

the Wronskian of $y_1, \cdots, y_n$, and is thus nonzero. Using Cramer's rule, we find $v(x)$ as a quotient of determinants, involving n arbitrary constants $c_1, \cdots, c_n$, and is thus the general solution of $\bar{L}v=0$.

To find a particular integral of $Ly=r(x)$, we assume that we know the solutions $y_1, \cdots, y_n$ of the homogeneous equation, or, equivalently, $v_1, \cdots, v_n$ of $\bar{L}v=0$. Then (3) becomes

$$v_iLy = \frac{d}{dx}\{P(y, v_i)\} + y\bar{L}v_i \qquad i = 1, \cdots, n, \qquad \frac{d}{dx}\{P(y, v_i)\} = v_ir(x)$$

and integrating,

$$P(y, v_i) = \int^x v_i(t)r(t)dt + c_i. \tag{5}$$

Proceeding as above, we can solve for y as the quotient of two determinants, and may take each $c_i=0$ if we merely desire a particular integral. If we wish to solve the initial value problem, given $y, y', \cdots y^{(n-1)}$ at x_0, then (5) becomes

$$P(y, v_i) = \int_{x_0}^x v_i(t)r(t)dt + c_i$$

and the c_i are given immediately in terms of the initial values. For example $Ly=y''+y=\sec x$ and Ly is self adjoint,

$$\sin x(y'' + y) = \frac{d}{dx}(y' \sin x - y \cos x) = \frac{\sin x}{\cos x},$$

$$\cos x(y'' + y) = \frac{d}{dx}(y' \cos x + y \sin x) = 1,$$

$$y' \sin x - y \cos x = -\log \cos x,$$

$$y' \cos x + y \sin x = x.$$

Thus $y=x \sin x+\cos x \log \cos x$ is a particular integral.

SELF-INTERSECTIONS OF SPECIAL CURVES

W. FUNKENBUSCH, Michigan College of Mining and Technology

The curves we consider are defined by $x=f(t)$, where $f(h+k)=f(h-k)$, h being a constant, and y being an nth degree polynomial in t designated by $\phi(t)$. The plotting of these curves from their parametric equations may be expedited by the following direct method for finding points of self-intersection.

Let us define $\psi(t)=\phi(t+h)$. In order that the two t values, $t=h\pm k$, produce the same y in addition to producing the same x, it is required that $\psi(k)=\psi(-k)$. This means that the desired values of k are the nonzero zeros of $\theta(k)$, where $\theta(k)$ contains the odd degree terms of $\psi(k)$. Obviously $\theta(0)=0$, and therefore if $n<11$, we need to solve no higher than a fourth degree equation in k^2.

Example. Find any self-intersection points on the curve $x=t^2-4t$, $y=\phi(t)=t^3-2t^2-7t$.

Solution. We have $h=2$, $t=2\pm k$, $\psi(t)=t^3+4t^2-3t-14$, $\theta(k)=k^3-3k$, and $k=0$ or $\pm\sqrt{3}$. Hence $t=2\pm\sqrt{3}$, gives the point $(-1, -2)$ as a point of self-intersection.

Other problems with solutions are as follows:

$$\begin{aligned} &x = \frac{1}{1+t^2}, \qquad y = t^3 - t - 6, \\ &t = \pm 1, \text{ gives } \left(\frac{1}{2}, -6\right). \end{aligned} \tag{1}$$

$$\begin{aligned} &x = \frac{e^t + e^{-t}}{2}, \qquad y = t^3 - t - 6, \\ &t = \pm 1, \text{ gives } \left(\frac{e^2+1}{2e}, \frac{e^2+1}{2e}\right). \end{aligned} \tag{2}$$

$$\begin{aligned} &x = t^2 - 2t, \qquad y = t^5 - 3t^4 - 5t^3 + 15t^2 + 4t - 12 \\ &t = -1 \text{ or } 3 \text{ gives } (3, 0) \text{ and } t = 1 \pm \sqrt{3} \text{ gives } (2, -6). \end{aligned} \tag{3}$$

$$\begin{aligned} &x = \cos t, \qquad y = t^3 - 3t^2 - 3t \\ &t = \pm\sqrt{3} \text{ gives } (\cos\sqrt{3}, -9). \end{aligned} \tag{4}$$

MATHEMATICAL EDUCATION NOTES

Mathematical Education Notes, a new department of the MONTHLY will appear for the first time in the November, 1958, issue. Editors of the department are John R. Mayor, American Association for the Advancement of Science and the University of Maryland, and John A. Brown, University of Delaware. Contributions should be addressed to Mr. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

The national interest in the improvement of science education, perhaps greatest in mathematics as one of the sciences, is increasing. Already much work has been done to improve mathematics education, particularly in grades 9 to 12 and in the first two years of college. But a very great deal remains to be

done. A considerable share of the work, on which improvement can be based, must be carried by members of the Mathematical Association of America. It is hoped that this department may strengthen this effort in providing information and points of view on curriculum in mathematics and the other sciences, experimental studies in progress, educational uses of films and television, educational activities of professional societies, and other reports of importance to mathematics education.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1331. *Proposed by P. L. Duren, Massachusetts Institute of Technology*

Show that, for all $a\geqq 0$ and $b\geqq 1$,

$$ab \leqq e^a + b(\ln b - 1),$$

with equality if and only if $b=e^a$.

E 1332. *Proposed by P. L. Chessin, University of Maryland*

If A, B, C are the angles of a triangle and x is such that

$$\cos(x+A)\cos(x+B)\cos(x+C)+\cos^3 x = 0,$$

then

$$\tan x = \cot A + \cot B + \cot C, \tag{1}$$

$$\sec^2 x = \csc^2 A + \csc^2 B + \csc^2 C. \tag{2}$$

E 1333. *Proposed by V. F. Ivanoff, San Carlos, California*

If $s_k=1^k+2^k+\cdots+n^k$, show that

$$\begin{vmatrix} s_1 & 1 & 0 & 0 & \cdot & 0 \\ s_2 & s_1 & 2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_{n-2} & \cdot & \cdot & \cdot & n \\ s_n & s_{n-1} & \cdot & \cdot & \cdot & s_1 \end{vmatrix} = (n!)^2.$$

E 1334. *Proposed by Burton Randol, The Rice Institute*

If $f(z)$ is any simply periodic entire function, show that there exists a (finite) z_0 such that $f(z_0)=z_0$.

E 1335. *Proposed by C. N. Campopiano, Polytechnic Institute of Brooklyn*

Let $f(z)$ be a complex function of the complex variable z defined for all finite z. Suppose there is a constant k, $0<k<1$, such that

$$|f(z)-f(w)| \leqq k|z-w|$$

for all z, w. Show that there is a unique solution to the equation

$$z=f(z)+a,$$

where a is an arbitrary constant.

SOLUTIONS

A Triangle with Angles in Geometric Progression

E 1301 [1958, 122]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the angles of a triangle ABC are consecutive terms of a geometric progression of common ratio 3, then

$$\cos B \cos C+\cos C \cos A+\cos A \cos B=-1/4.$$

Solution by M. T. Salhab, Carnegie Institute of Technology. We may take $A=3B=9C$, $C=\pi/13$. Using $\cos x \cos y=[\cos (x+y)+\cos (x-y)]/2$, we have to show that

$$\sum_{k=1}^{6} \cos 2kC=-1/2.$$

Now using

$$\sum_{k=1}^{n} \cos 2k\theta=\frac{\sin (2n+1)\theta}{2 \sin \theta}-\frac{1}{2}$$

with $n=6$, $\theta=\pi/13$, the desired result follows immediately.

Also solved by A. N. Aheart, R. G. Albert, J. B. Baird III, Edward Barbeau, B. H. Bissinger, L. R. Bragg, Julian Braun, D. R. Brillinger, Thédose Brown, C. S. Carlson, P. L. Chessin, A. G. Clark, J. W. Clawson, Mary Dean Clement, A. E. Danese, J. E. Darraugh, E. L. Ellis and D. L. Muench (jointly), G. W. Erwin, Jr., P. A. Fillmore and E. H. Kanning III (jointly), Leona Freeman, Jose Gallego-Diaz, H. S. Gaples, Jr., L. L. Garner, Michael Goldberg, S. H. Greene, J. W. Haake, A. R. Hyde, M. S. Klamkin, J. D. E. Konhauser, Morton Kupperman, Leon Lefton, Joe Lipman, R. L. London, D. C. B. Marsh, C. T. Molloy, Jr., J. B. Muskat, E. F. Myers, F. D. Parker, L. A. Ringenberg, Jeff Ritterman, D. A. Robinson, Azriel Rosenfeld, Nathan Schwid, T. H. Slook, R. H. Sprague, W. B. Stovall, Jr., Harry Weingarten, Ralph Wiggins, Samuel Wolf, Dale Woods, Roscoe Woods, J. W. Young, David Zeitlin, the proposer, and two anonymous solvers.

Garner proposed the allied problem: If the angles A, B, C, D of a (reëntrant) quadrilateral $ABCD$ are consecutive terms of a geometric progression of common ratio 3, then

$$\cos A \cos B + \cos B \cos C + \cos C \cos D + \cos D \cos A = 1/2.$$

Locus of a Centroid

E 1302 [1958, 122]. *Proposed by M. S. Klamkin, AVCO, Research and Development, Lawrence, Mass.*

A square is divided into two parts by an arbitrary diameter through its center. Determine the locus of the centroid of one of the equal areas.

Solution by B. H. Bissinger, Lebanon Valley College. Consider the square with vertices at $(\pm 1, \pm 1)$. Taking moments about the coordinate axes we find the centroid of the region below the diameter of slope m, $|m| \leqq 1$, has coordinates $x=m/3$, $y=-1/2+m^2/6$. Considerations of symmetry prove the closed path of the centroid for one complete revolution of the diameter consists of four parabolic sections whose equations are

$$4y^2 = (3x^2 - 1)^2, \qquad x^2 \leqq 1/9,$$
$$4x^2 = (3y^2 - 1)^2, \qquad 1/9 \leqq x^2 \leqq 1/4.$$

It is interesting to note that the direction of the path of the centroid is parallel to the parametric diameter and therefore that the derivative exists at the four points $(\pm 1/3, \pm 1/3)$ where the parabolas are pieced together.

Also solved by R. G. Albert, E. F. Allen, Ferrel Atkins, Edward Barbeau, Merrill Barnebey, Julian Braun, D. R. Brillinger, E. W. Brown, P. L. Chessin, J. W. Clawson, David Fink, Jose Gallego-Diaz, Michael Goldberg, R. E. Graf, S. H. Greene, J. W. Haake, J. D. Haggard and R. G. Smith (jointly), Corinne Hattan, A. R. Hyde, J. D. E. Konhauser, Morton Kupperman, Peter Landweber, J. H. Leshler and Dale Woods (jointly), Joe Lipman, E. W. Marchand, D. C. B. Marsh, J. B. Muskat, C. S. Ogilvy, P. D. Parker, M. J. Pascual, W. D. Peeples, Jr., L. A. Ringenberg, Jeff Ritterman, D. A. Robinson, Azriel Rosenfeld, Nathan Schwid, Donato Teodoro, C. W. Trigg, S. I. Vrooman, Herbert Wolf, David Zeitlin, and one anonymous solver.

Pascual considered the analogous problem in three space, where the square is replaced by a cube and the diametral line by a diametral plane. Ogilvy proposed the allied problem: What is the locus of the centroid if the arbitrary diameter is fixed and the square is rotated?

An Unusual Differential Equation

E 1303 [1958, 122]. *Proposed by M. J. Hellman, Rutgers University*

Solve the differential equation $dy/dx=\mathcal{R}(z^n)/\mathcal{I}(z^n)$, where $z=x+iy$, n is a positive integer, $\mathcal{R}(z^n)$ denotes the real part of z^n, and $\mathcal{I}(z^n)$ denotes the imaginary part of z^n.

I. *Solution by J. D. E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pa.* In polar coordinates the differential equation becomes $dr/d\theta = r \tan (n+1)\theta$, the solution of which may be written

$$r^{n+1} \cos (n + 1)\theta = \mathcal{R}(z^{n+1}) = \text{constant}.$$

II. *Solution by Chih-yi Wang, University of Minnesota.* Since $x=(z+\bar{z})/2$, $y=(z-\bar{z})/2i$, where $\bar{z}$ is the complex conjugate of z, the given differential equa-

tion is reduced at once to $z^n dz + \overline{z^n dz} = 0$. Hence the general solution is $\mathcal{R}(z^{n+1}) = \text{constant}$.

III. *Solution by L. R. Bragg, Duke University.* Noting that $\mathcal{I}(z^n) = -\mathcal{R}(iz^n)$, the given equation can be written $\mathcal{R}[z^n] + \mathcal{R}[iz^n dy/dx] = 0$, or

$$\mathcal{R}[z^n(1 + idy/dx)] = 0.$$

This has the solution $\mathcal{R}[z^{n+1}] = \text{constant}$.

Also solved by J. W. Adams, D. S. Adorno, A. N. Aheart, R. G. Albert, Philip Bacon, J. L. Baker, B. H. Bissinger, Louis Brand, Julian Braun, D. A. Breault, D. R. Brillinger, E. W. Brown, C. N. Campopiano, W. B. Carver, P. L. Chessin, A. G. Clark, A. E. Danese, E. S. Eby and H. F. Illson (jointly), E. L. Ellis and D. L. Muench (jointly), Jose Gallego-Diaz, Michael Goldberg, R. N. Gordon and R. H. Sprague (jointly), S. H. Greene, Corinne Hattan, A. R. Hyde, D. A. Kearns, M. S. Klamkin and D. J. Newman (jointly), Donald Knuth, Morton Kupperman, C. E. Langenhop, L. E. Larson, Joe Lipman, R. L. London, S. Marein-Efrón, D. C. B. Marsh, C. S. Ogilvy, M. J. Pascual, J. L. Pietenpol, B. E. Rhoades, D. A. Robinson, Azriel Rosenfeld, Nathan Schwid, P. W. Shaw, T. H. Slook, O. E. Stanaitis, Donato Teodoro, S. I. Vrooman, R. J. Wagner, Dale Woods, J. W. Young, David Zeitlin, and the proposer.

Alternative expressions for the solution are

$$x\mathcal{R}(z^n) - y\mathcal{I}(z^n) = C,$$

$$\sum_{k=0}^{[(n+1)/2]} (-1)^k \binom{n+1}{2k} x^{n-2k+1} y^{2k} = C.$$

Klamkin and Newman gave the generalization: If

$$dy/dx = \mathcal{R}[f(z)(a - ib)]/\mathcal{I}[f(z)(a - ib)],$$

then

$$y = a\mathcal{R}\int f(z)dz + b\mathcal{I}\int f(z)dz.$$

The given problem is the case where $f(z) = z^n$, $a = 1$, $b = 0$.

A Property of the Morley Configuration

E 1304 [1958, 122]. *Proposed by W. B. Carver, Cornell University*

Let A_1, A_2, A_3 be the vertices of any triangle and let the arc of the circumcircle from A_i to A_j be trisected by the points T_{ij} and T_{ji}, T_{ij} adjacent to A_i and T_{ji} adjacent to A_j, $i, j = 1, 2, 3$. Further, let A_1T_{32} and A_3T_{12} intersect in P_2, A_1T_{23} and A_2T_{13} intersect in P_3. Show that $T_{31}T_{21}$ is parallel to P_2P_3.

Solution by Jose Gallego-Diaz, Vanderbilt University. Let A_3T_{21} and A_2T_{31} intersect in P_1. We know (see, *e.g.*, H. D. Grossman, *The Morley triangle: a new geometric proof*, this MONTHLY [1943, 552]) that $P_1P_2P_3$ is the Morley triangle of $A_1A_2A_3$ and that $\sphericalangle P_3P_2A_1 = \pi/3 + A_3/3$. But, if Q is the intersection of $T_{21}T_{31}$ with A_1T_{32}, we also have $\sphericalangle T_{21}QA_1 = 2A_3/3 + A_1/3 + A_2/3 = \pi/3 + A_3/3$. It follows that $T_{31}T_{21}$ is parallel to P_2P_3.

Also solved by R. G. Albert, Leon Bankoff, J. W. Clawson, A. R. Hyde, J. D. E. Konhauser, D. C. B. Marsh, and Sister M. Stephanie.

Bankoff gave the reference, V. Thébault, *The triangle*, Scripta Mathematica, vol. 22, 1956, p. 27.

Existence and Construction of a Certain Matrix

E 1305 [1958, 122]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Given a positive integer n and a symmetric 3×3 matrix $A=(a_{ij})$ with nonnegative integral elements, when does there exist an $n\times 3$ matrix B such that $B^*B=A$ (here B^* denotes the transpose of B) and every element of B is either 0 or 1? Describe a method for finding all solutions B.

Solution by the proposer. First observe that if B is a solution, then any matrix obtained from B by a permutation of the rows is also a solution. It is therefore natural to consider two solutions as equivalent if they differ only by a permutation of the rows.

If every element of an $n\times 3$ matrix B is 0 or 1, then there are eight possible types of row vectors in B, namely:

$$\begin{array}{cccc} (1,1,1), & (1,1,0), & (1,0,1), & (0,1,1), \\ (1,0,0), & (0,1,0), & (0,0,1), & (0,0,0). \end{array}$$

Let the numbers of row vectors in B of each of these eight types be denoted by b_{123}, b_{12}, b_{13}, b_{23}, b_1, b_2, b_3, b respectively. These eight numbers are nonnegative integers with sum equal to n, and they determine B up to a permutation of the rows. If $B^*B=A$, then we must have

$$b_{123} \geqq 0, \tag{1}$$

$$b_{12} = a_{12} - b_{123} \geqq 0, \tag{2}$$

$$b_{13} = a_{13} - b_{123} \geqq 0, \tag{3}$$

$$b_{23} = a_{23} - b_{123} \geqq 0, \tag{4}$$

$$b_1 = a_{11} - (a_{12} + a_{13}) + b_{123} \geqq 0, \tag{5}$$

$$b_2 = a_{22} - (a_{12} + a_{23}) + b_{123} \geqq 0, \tag{6}$$

$$b_3 = a_{33} - (a_{13} + a_{23}) + b_{123} \geqq 0, \tag{7}$$

$$b = n - (a_{11} + a_{22} + a_{33}) + (a_{12} + a_{13} + a_{23}) - b_{123} \geqq 0. \tag{8}$$

This can easily be seen if we regard B as the incidence matrix of three subsets S_1, S_2, S_3 of the set $\{1, \cdots, n\}$. (Then a_{ij} is the number of elements of $S_i \cap S_j$; b_1 is the number of those elements which belong to S_1 and only to S_1; b_{12} is the number of those elements which belong to S_1, S_2, but not to S_3; *etc.*)

Let p denote the least of the four integers

$$a_{12},\ a_{13},\ a_{23},\ n - (a_{11} + a_{22} + a_{33}) + (a_{12} + a_{13} + a_{23}),$$

and let q denote the greatest of the four integers

$$0,\ (a_{12} + a_{13}) - a_{11},\ (a_{12} + a_{23}) - a_{22},\ (a_{13} + a_{23}) - a_{33}.$$

Then inequalities (1)–(8) imply

$$p \geqq q. \tag{9}$$

Therefore inequality (9) is a necessary condition for B to exist.

This necessary condition (9) is also sufficient. In fact, when (9) is fulfilled, we can choose an integer b_{123} satisfying all inequalities (1)–(8). There are $p-q+1$ possible choices for b_{123}, since it can be taken to be any one of the integers $q, q+1, \cdots, p-1, p$. After having chosen b_{123}, the nonnegative integers b_{12}, b_{13}, b_{23}, b_1, b_2, b_3, b are determined by the equations (2)–(8), and we have

$$b_{123} + b_{12} + b_{13} + b_{23} + b_1 + b_2 + b_3 + b = n.$$

Then any $n\times 3$ matrix B with these preassigned numbers $b_{123}, b_{12}, \cdots, b_3, b$ of rows of each of the eight types will satisfy $B^*B=A$. Since there are $p-q+1$ possible choices for b_{123}, the total number of nonequivalent solutions B is $p-q+1$.

Remark 1. The necessary and sufficient condition (9) can be written as follows:

$$a_{ii} - a_{ij} \geqq 0, \qquad (i \neq j), \tag{9.1}$$

$$n - (a_{ii} + a_{jj}) + a_{ij} \geqq 0, \qquad (i \neq j), \tag{9.2}$$

$$a_{ii} - (a_{ij} + a_{ik}) + a_{jk} \geqq 0, \qquad (i, j, k \text{ all distinct}), \tag{9.3}$$

$$n - (a_{11} + a_{22} + a_{33}) + (a_{12} + a_{13} + a_{23}) \geqq 0. \tag{9.4}$$

Remark 2. The same problem for a symmetric 4×4 matrix A is surprisingly difficult. No necessary and sufficient condition is known.

Also solved by D. A. Breault, D. R. Brillinger, A. S. Davis, S. H. Greene, J. W. Haake, Joe Lipman, D. C. B. Marsh, and D. A. Robinson.

The following examples were given by Marsh. Consider the three matrices

$$A_1 = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

If $A=A_1$, there is no solution; if $A=A_2$, there is essentially only one solution provided $n>6$; if $A=A_3$, there are two and only two nonequivalent solutions provided $n>4$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4804. *Proposed by I. S. Gál, Cornell University*

A theorem of Baire states that every lower semi-continuous function on a metric space is the pointwise limit of an increasing sequence of continuous functions. Show by an example that the theorem cannot be extended to arbitrary uniform spaces.

4805. *Proposed by R. R. Goldberg, Pittsburgh, Penna.*

The following relation is found to be true for $n=1, 2, \cdots, 6$.

$$n^2 \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(n+k-1)!}{k!k!} \sum_{j=1}^{n+k-1} \frac{1}{j} = 1.$$

Is it true for all positive integers n?

4806. *Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Let $0=a_0<a_1<a_2< \cdots$ be integers. Prove that $\sum_{n=0}^{\infty} z^{a_n}$ has no zeros in $|z|<(\sqrt{5}-1)/2$. Also, this is the best possible constant.

4807. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the integral solutions of the equation

$$(m-1)x^2 + my^2 = (m+1)z^2.$$

4808. *Proposed by J. M. Gandhi, Jain Engineering College, Panchkoola, India*

Consider the expression $S_2(n, i) = \sum a_1^2 a_2^2 \cdots a_i^2$, where the sum is taken over all possible integral choices of the a's such that $1 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_i \leqq n$. Prove that

$$\begin{aligned} 2(2^{2n}-1)B_{2n} = (-1)^{n-1}\{&(n-1)!^2 - (n-2)!^2 S_2(n-2, 1) \\ &+ (n-3)!^2 S_2(n-3, 2) - \cdots + (-1)^n 1!^2 S_2(1, n)\}, \end{aligned}$$

where the B_{2n} are the well-known Bernoulli numbers.

4809. *Proposed by Sylvan H. Greene, General Electric Co., Philadelphia*

Let n be an odd, square-free integer greater than 3; let A_n be the set of all

$a \pmod{n}$ such that $(a/n)=+1$, and B_n that of all $b \pmod{b}$ such that (b/n) $=-1$, where (c/n) is the Jacobi symbol. Prove that in a complete system of residues (mod n), $\sum a \equiv \sum b \equiv 0 \pmod{n}$. The summations are taken over all elements of A_n and B_n respectively.

SOLUTIONS

Power Sums of Roots

4758 [1957, 676]. *Proposed by Leonard Carlitz, Duke University*

Let $x_1, \cdots, x_n$ denote the roots of

$$(x-a+\sqrt{1-2ax+x^2})^n+(x-a-\sqrt{1-2ax+x^2})^n = 2(a^2-1)^{n/2}.$$

Evaluate the power sums $\sigma_k = x_1^k + \cdots + x_n^k$ $(0 \leqq k < n)$.

Solution by W. A. Al-Salam, Duke University. Consider the polynomial

$$P(x) = (x-a+\sqrt{1-2ax+x^2})^n+(x-a-\sqrt{1-2ax+x^2})^n - 2(a^2-1)^{n/2}.$$

On one hand we have, for large values of x,

$$\frac{P'(x)}{P(x)} = \sum_{k=1}^{n} \frac{1}{x-x_k} = \sum_{r=0}^{\infty} \frac{\sigma_r}{x^{r+1}}. \tag{1}$$

On the other hand from the definition of $P(x)$

$$\begin{aligned}
&\frac{P'(x)}{P(x)} \\
&= \frac{n}{\sqrt{1-2ax+x^2}} \cdot \frac{(x-a+\sqrt{1-2ax+x^2})^n-(x-a-\sqrt{1-2ax+x^2})^n}{P(x)} \\
&= \frac{n}{\sqrt{1-2ax+x^2}} - \frac{2n}{\sqrt{1-2ax+x^2}} \\
&\quad \cdot \frac{(x-a-\sqrt{1-2ax+x^2})^n-(a^2-1)^{n/2}}{P(x)}.
\end{aligned}$$

We note that the first term in the right hand side of (2) is the generating function for $P_r(a)$, the Legendre polynomials, and for large values of x gives

$$\frac{n}{\sqrt{1-2ax+x^2}} = n \sum_{r=0}^{\infty} \frac{P_r(a)}{x^{r+1}}.$$

Now

$$\frac{(x-a+\sqrt{1-2ax+x^2})^n-(a^2-1)^{n/2}}{(x-a-\sqrt{1-2ax+x^2})^n-(a^2-1)^{n/2}} = -\left(\frac{x-a+\sqrt{1-2ax+x^2}}{x-a-\sqrt{1-2ax+x^2}}\right)^{n/2}$$

$$= -\left(1 + \frac{2}{a^2 - 1}\cdot(x^2 - 2ax + 1) + \frac{2(x - a)}{a^2 - 1}\sqrt{1 - 2ax + x^2}\right)^{n/2}.$$

Thus for large values of x we can see by easy calculation that the second term in the right hand side of (2) has a development in which all the powers of $1/x$ are $\geqq n$. Hence, comparing (1) and (2), we get $\sigma_r = nP_r(a)$, $(0 \leqq r < n)$.

Also solved by Emil Grosswald, Peter Henrici, M. S. Klamkin, D. C. B. Marsh, F. D. Parker, Blagovest Sendov, G. Szegö, Chih-yi Wang, and the proposer.

Nonuniform Convergence

4760 [1957, 676]. *Proposed by G. U. Brauer, University of Minnesota*

Let $f(x)$ be a real function such that $f(0)=0$, $f(1)\neq 0$, $\lim_{n\to\infty} f(n)=0$, where n takes on positive integral values. Construct a sequence of integers $\{a_n\}$, $a_n \to \infty$ as $n\to\infty$, and a compact set C such that $f(a_n x)\to 0$ nonuniformly for x in C.

Solution by Hans Schneider, The Queen's University, Belfast, Ireland. Let C be the compact set consisting of 0 and all rationals of the form $1/n$, where n is a positive integer. Set $a_n = n!$. For each nonzero x in C it is clear that $a_n x$ is an integer for all sufficiently large n and that $\lim_{n\to\infty} a_n x = \infty$. Hence $\lim_{n\to\infty} f(a_n x) = 0$ for all x in C.

Now let $0 < \epsilon < |f(1)|$. Then, for each positive integer n, we have the inequality $|f(a_n x)| = |f(1)| > \epsilon$ if $x = 1/n!$, and therefore the convergence is nonuniform on C.

Also solved by Montford Plebstnoch, R. L. Plunket, Blagovest Sendov, and the proposer.

A Summation Related to No. 4592

4762 [1957, 676]. *Proposed by Samuel Beatty, University of Toronto*

Define γ_i by the relations

$$\sum_{r=1}^{n} \frac{\log^i r}{r} = \int_1^n \frac{\log^i x}{x}\,dx + \gamma_i + o(1).$$

Show that

$$\sum_{r=1}^{\infty} (-1)^{r+1}\frac{\log^p r}{r} = \frac{\lambda^{p+1}}{p+1} - \left[\lambda^p\gamma + \binom{p}{1}\lambda^{p-1}\gamma_1 + \cdots + \binom{p}{p-1}\lambda\gamma_{p-1}\right],$$

$(p=2, 3, \cdots)$, where $\lambda = \log 2$ and γ is Euler's constant. This is a generalization of no. 4592 [1954, 350].

Editorial Note. It was overlooked that a solution of the current problem appears, culminating in equation (8), in *The Power Series Coefficients of* $\zeta(s)$, by Briggs and Chowla, this MONTHLY, vol. 62, 1955, pp. 323–325. This paper is referred to in the solution of Problem 4592 which appears in the October, 1955 issue and, as a matter of fact, it was this problem which instigated the paper.

Solutions submitted by W. A. Al-Salam, W. E. Briggs, R. G. Buschman, L. Carlitz, Lawrence Glasser, Emil Grosswald, M. S. Klamkin, Bryant Tuckerman and Tien Chi Chen, Chih-yi Wang, David Zeitlin, and the proposer.

A Minimax Problem

4763 [1957, 746]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

For n positive numbers x_i, let $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ denote respectively the least and the greatest of the $n+1$ quantities

$$(1)\qquad x_1, \frac{1}{x_1}+x_2, \cdots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\cdot$$

Prove that

$$(2)\qquad \operatorname*{Max}_{x_i>0} f(x_1, \cdots, x_n) = \operatorname*{Min}_{x_i>0} g(x_1, \cdots, x_n) = 2\cos\frac{\pi}{n+2}\cdot$$

Solution by D. S. Greenstein, University of Michigan. Assume that u is a positive real such that

$$(3)\qquad u = x_1 = \frac{1}{x_1}+x_2 = \cdots = \frac{1}{x_{n-1}}+x_n = \frac{1}{x_n}\cdot$$

Then $u=\max f=\min g$ for all choices of x_i. For, to increase f we would have to increase x_1, which would necessitate increasing x_2, *etc.*, finally necessitating an increase of x_n, which would decrease $1/x_n$ and decrease f. A similar argument applies to decreasing g.

From (3) we see that $x_1 > \cdots > x_n$. Further, $x_1<2$ since $x_1 \geqq 2$ implies $x_n>1$ contrary to $x_n=1/x_1$. We may therefore put $u=2\cos\theta$. For (3) to hold, we must have $x_1=u$, $x_2=u-1/u$, $x_3=u-1/(u-1/u)$, *etc.*, which may be rewritten $x_k=Q_k(u)/Q_{k-1}(u)$, $(k=1, \cdots, n)$, where the Q_k are defined by

$$(4)\qquad Q_0 = 1,\ Q_1 = u,\ Q_{k+1} = uQ_k - Q_{k-1}.$$

Since $1/x_n=u$, we must have $u=Q_{n-1}(u)/Q_n(u)$, which is equivalent to $Q_{n+1}(u)=0$. From $u=2\cos\theta$ it is easily verified that the unique solution of (4) is given by $Q_k=\sin(k+1)\theta/\sin\theta$. Hence θ is a multiple of $\pi/(n+2)$. But $Q_1, \cdots, Q_n$ all positive require $\theta=\pi/(n+2)$. Finally $u=2\cos(\pi/(n+2))$ as required.

Also solved by Ward Cheney and Allan Goldstein, Emil Grosswald, D. C. B. Marsh, M. F. Neuts, Roly Rasmussen, G. Szegö, R. S. Varga, Chih-yi Wang, and the proposer.

A Limit of a Sum

4764 [1957, 737]. *Proposed by A. E. Currier, U. S. Naval Academy*

Prove

$$\lim_{n\to\infty} \sum_{j=0}^{n} \binom{2n-j}{n}(-4)^j \Big/ \binom{2n}{n} = \frac{1}{3}\cdot$$

Solution by D. G. Cantor, University of California, Los Angeles. Let

$$f_n(x) = \binom{2n}{n}^{-1} \sum_{j=0}^{n} \binom{2n-j}{n} x^j.$$

By direct substitution one may verify that

$$2(x-1)f_n(x) = x - 2 + x^2(n/(2n-1))\cdot f_{n-1}(x),$$

and by induction show that, for $x \neq 1$,

$$f_n(x) = \frac{x-2}{2x-2} \sum_{k=1}^{n} \prod_{i=k}^{n-1} \left(\frac{x^2}{4(x-1)} \cdot \frac{2i+2}{2i+1} \right) + \prod_{i=0}^{n-1} \left(\frac{x^2}{4(x-1)} \cdot \frac{2i+2}{2i+1} \right).$$

Clearly the sequence $\{f_n(x)\}$ converges when $|x^2/4(x-1)| < 1$, *i.e.*, when $-2-2\sqrt{2} < x < -2+2\sqrt{2}$, by comparison with the geometric series. Then $f(x) = \lim_{n\to\infty} f_n(x)$ exists and

$$2(x-1)f(x) = x - 2 + x^2(1/2)f(x),$$

whence $f(x) = 2/(2-x)$, and $f(-4) = 1/3$ as required.

Also solved by M. T. Bird, D. C. B. Marsh, Yoshio Matsuoka, G. Szegö, and the proposer.

Editorial Note. By a somewhat extended analysis, Szegö shows that, for $n \to \infty$,

$$f_n(x) \to (1 - x/2)^{-1}, \qquad -2 - 2\sqrt{2} < x < 2;$$

$$f_n(x) \cong \frac{x}{x-1} \cdot \left(\frac{x^2}{4(x-1)} \right)^n \cdot \sqrt{\pi n}, \qquad x \geqq 2 \text{ or } x \leqq -2 - 2\sqrt{2}.$$

Continuous Increasing Function

4765 [1957, 746]. *Proposed by J. L. Massera, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

Let $f(x)$ be a real function defined for $x \geqq 0$. If (i) f has a finite upper bound in any finite interval, and (ii) there are two positive numbers h, k such that $x' - x'' \geqq h$ implies $f(x') - f(x'') \geqq k$, then there is an increasing function $g(x)$ having as many continuous derivatives as we please, such that $g(x-h) < f(x) < g(x)$ for all sufficiently large x.

Discussion by J. L. Pietenpol, Columbia University. A stronger hypothesis is called for. As it stands, the theorem is false, as an example will show: Let $f(x) = 3[x] - x$, $h = k = 1$, where $[x]$ is the greatest integer function. The conditions of the problem are met. We now seek an increasing function $g(x)$ such that $g(x-1) < f(x) < g(x)$, which is equivalent to $f(x) < g(x) < f(x+1)$, for sufficiently large x. But it is clear that any $g(x)$ which satisfies this condition is necessarily discontinuous at all integral values of x.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Consiglio Nazionale dell Ricerche Monografie Matematiche. Vol. 4. *Calcolo dell Matrici.* By Salvatore Cherubino. Edizioni Cremonese, Rome, 1957. vi+322 pp.

The reviewer read this book with a great deal of pleasure because it is different from most of the books on matrices which are currently appearing. While the concept of vector space is introduced, it is not used very much. The reduction to canonical form under similarity transformations is limited to the field of the characteristic roots. A paragraph entitled "the concepts of abstract algebra" consists essentially of the derivation of the first and second regular representations of a finite linear associative algebra. The concepts of ring and field are introduced, but most of the book is concerned with matrices with either real or complex elements.

There is really a lot of factual material in this book, much of it not readily accessible in textbooks. The chapters dealing with the characteristic roots are especially complete, treating approximation to the dominant root, Hurwitz' stability theorem, maxima and minima of quadratic forms, Borchardt's theorem, bounds for the roots, *etc.* But the author's principal interest seems to be in the differentiation and integration of matrices whose elements are functions of one or more independent variables. Even the Cayley-Hamilton theorem is proved by a power series expansion.

Coverage and references are international, with Italian authors getting only a shade the best of it. Classical theorems seem to be attributed to the proper authors. There are many minor turns and points of view which the reviewer has not encountered elsewhere. The paper and printing are excellent, and so is the cover.

C. C. MACDUFFEE
University of Wisconsin

Flachenverbiegung im Grossen. By N. W. Efimow. Akademie-Verlag, Berlin, 1957. xi+233 pp. DM 35.50 (about $8.50).

The book consists of two parts: the first, ending at page 112, is a translation of Efimow's report (Uspehi 1948) on *Qualitative questions in the theory of deformations of surfaces*; the second, pp. 113–230, by E. Rembs and K. P. Grotemeyer, consists of notes referring to topics of the first part and containing mostly results discovered after 1948, frequently answers to problems raised in the first part.

Efimow's report is a very clear exposition which can be read and enjoyed

also by the nonspecialist. The notes, while individually well written, are by their very nature incoherent and cannot be read as a book. However, they provide a great deal of valuable information to the specialist.

In both parts the longer proofs are only briefly sketched. This permits the authors to include a wealth of material. Consequently we must limit ourselves here to indicating a few of the main themes.

(1) The existence and uniqueness of the realizations as convex surfaces of metrics with nonnegative curvature. The classical methods of Weyl, Lewy, Cohn-Vossen, Herglotz are discussed, but the emphasis, at least for realization problems, lies on the modern methods developed by A. D. Alexandrov and Pogorelov.

(2) Deformations in the large of convex surfaces mainly by means of Alexandrov's Gluing Theorem. Part 2 gives many explicit deformations of special surfaces.

(3) Deformations in the small of, not necessarily convex, smooth surfaces, in particular Efimow's results on nondeformable surfaces with flat points.

(4) A general theory of infinitesimal deformations of the first and higher orders. The rigidity of special surfaces like surfaces with positive curvature, closed or with boundaries and sometimes boundary conditions, of tori, and (in a modified sense) of convex polyhedra. The second part emphasizes, in particular, the integral formulae entering this theory.

A very extensive list of references and a subject index conclude the book.

H. Busemann
University of Southern California

Plane Trigonometry. By Frank A. Rickey and J. P. Cole. Holt, New York, 1958. x+260 pp. $3.25.

This book gives an excellent analytic treatment of its subject. At the same time it presents with sufficient detail and applications the solution of right and general triangles. It treats the usual topics,—trigonometric functions of angles measured in degrees and radians, evaluation of trigonometric functions of special angles, use of tables of trigonometric functions, graphs of trigonometric functions, solution of triangles, trigonometric identities, solution of equations, inverse trigonometric functions,—and includes graphs in polar coordinates and a chapter on complex numbers. The appendix discusses accuracy of approximate numbers and contains a square- and cube-root table and four-place tables of natural functions and of logarithms of numbers and trigonometric functions. Answers to the odd-numbered problems are given.

Throughout the book the authors stress derivation of formulas rather than formula memorization. They succeed in making the general proofs of theorems clear and accurate; the addition formulas are proved by direct application of the distance formula in cartesian geometry. Trigonometric graphs are introduced early; possibly they are overworked as a visual aid in solving trigonometric equations. Exercises are carefully selected to take care of necessary drill and to

train the student in analytic methods. As a quite natural part of the subject of plane trigonometry the authors present concepts and topics useful in more advanced mathematics, for example, trigonometric functions of a number, principal values of inverse trigonometric functions, the function $(\sin x)/x$ when x is small, polar coordinates, complex numbers.

This trigonometry is a welcome addition to the textbooks which emphasize an analytical approach to the subject.

HELEN G. RUSSELL
Wellesley College

Calculus With Analytic Geometry. By R. E. Johnson and F. L. Kiokemeister. Allyn and Bacon, Boston, 1957. xi+650 pp. $7.95.

It is common for "*A*-students" in a "standard" calculus course to remark to the effect: "I can work all of the problems but I don't understand it!" This book goes a long way toward all that a textbook can do now for such students. (It is necessarily a compromise between the here-is-how-you-do-it-and-never-mind-why type of text and a course in real variables.) It is an impressive job of exposition with a fair amount of modern terminology and a considerable amount of mathematical care. It has plenty to challenge bright students and would not need to be fatal to average students.

Miscellaneous notes for flavor: Functions are named by single letters—f is the function whose value at x is $f(x)$. The phrase "if and only if" is used without blush. Absolute value and inequality symbols are used freely. Limits are hit heavily. The greatest integer function and other "unusual" functions are used to give interesting examples and problems and to show that not all functions are everywhere continuous. The tangent to a curve is treated with more than usual rigor. Physicists will like the distinction between "speed" and "velocity." The notion of open and closed intervals is used throughout. A region bounded by a curve is *defined* to have an area given by an integral and the common "proof" by witchcraft is omitted. (The additivity of area and its lineage from polygons is effectively hinted at.) L.U.B. and G.L.B. are introduced on page 197 in preparation for defining an upper integral and a lower integral for a function over a given domain.

A striking feature of the book is the frequent correspondence between concise, technical language and an easygoing interpretation of it. After three equivalent definitions of continuity (in hope that one will strike home) and three pages of examples and explanation we find the familiar: "Roughly speaking if a function f is continuous in an interval $[a, b]$, the graph of f between $x=a$ and $x=b$ has no jumps in it. This piece of the graph of f may be sketched without lifting the pencil from the paper" (p. 76).

Although this book contains a rather complete introduction, it is a rare student who will be able to cover the introduction at the implied pace. Either the first ninety pages will have to be taken slowly or students with strong high school backgrounds will be required. However, as implied by the title, analytic

geometry is not a prerequisite. Enough analytic geometry is contained in the first ninety pages to make differentiation of polynomials understandable. After polynomial differentiation and its applications are well established, this knowledge is turned back onto an efficient treatment of "Conic Sections and Other Algebraic Curves." Analytic geometry is therefore effectively intermixed with calculus to make both subjects more understandable and interesting.

There are some flaws. For example, the authors are not sure whether to call angles or line segments of the same measure congruent or equal; they try it both ways. Rarely, nonsense creeps in: "A collection of points satisfying some geometric condition is called a locus" (p. 27). Then certainly every set of points is a locus and the new term is superfluous. The treatment of coordinates on a line on page 11 is confusing. Differentials are as slippery as ever. This list could be much extended. Toward the end of the book it runs downhill toward the usual agglomeration of mysterious topics.

A principal objection is that it is not reformed enough. The word "set" is painfully avoided until page 196. The language and intuition of set theory should be used throughout. Functions should be sets of ordered pairs of numbers (with the single-valued condition). Sequences should be functions over the positive integers. $\sum$-notation should be introduced earlier. And so on. The only trouble is that by the time all such changes were made it would be an extremely demanding task to prepare a text as useful in the classroom as this one.

D. A. PAGE
University of Illinois

The Numerical Solution of Two-Point Boundary Problems in Ordinary Differential Equations. By L. Fox. Oxford University Press, 1957. xi+371 pp. $9.60.

This book deals with differential equations having the extra conditions imposed for more than one value of the independent variable. It serves a very useful purpose, since most numerical techniques are set for one point initial conditions.

A great deal of the work being done in this country with numerical methods is, however, done in and around large computers. One must be warned, therefore, that while almost every technique introduced by the author can be adapted to a large computer, there is often a difficult choice as to which technique to use. Again, this is not the fault of the author, for this book is aimed at the competent computer, not the competent machine programmer. Let the latter be warned by the quotation from Chapter 6, "The choice of method depends solely on the structure of the equation and computational convenience."

The first two chapters are devoted to notation and derivation of formulae. Chapter three supplies the necessary algebraic techniques for solving the simultaneous equations that will continually appear. Succeeding chapters then treat second-order, first-order, and higher-order problems. It might be thought that first-order problems hardly belong, but their solution is peculiarly amenable to second-order techniques by introducing an artificial boundary value.

Eigenvalue problems are well cared for, starting with a useful description of methods of finding eigenvalues. Cases are included where one of the boundary values is infinite. A chapter describing initial value methods points out the differences from other methods.

All too frequently books on numerical methods do not emphasize accuracy enough, but in this case an entire chapter studies the subject thoroughly. If one tries to sum up quite briefly, here is a book which covers its problem in very great detail, with a tremendous number of examples worked out, so that any reasonably interested person may learn a great deal of useful information. It is not a reference book in that without knowing most of the material it is not possible to make a reasonable choice of method of attack. It belongs in the library of anyone seriously interested in such problems, and it should be available to any large-machine computing group.

R. G. SELFRIDGE
U. S. Naval Ordnance Test Station

Queues, Inventories and Maintenance. By Philip M. Morse. Wiley, New York, 1957. ix+202 pages. \$6.50.

This volume is the first in a series of monographs published by Operations Research Society and the editors are certainly to be congratulated for having made an ideal initial choice.

The author, one of the pioneers in the field of Operations Research, has taken a problem area of great analytic interest and practical significance, the problem of queuing, and discussed various aspects of it in a very lucid fashion.

What is particularly attractive about this book is the clear and leisurely fashion in which concepts are introduced, problems are posed and equations are treated. The author is clearly not ashamed to use the English language rather than symbols to make his points, an attitude which, unfortunately, is not shared by a large number of writers of current research articles and books.

The general problem is that of servicing objects which arrive at a facility, wait until attended to, and then depart. The question is that of determining the departure rate from the arrival rate, the servicing rate and the structure of the facility. Examples of systems of this nature are furnished by telephone exchanges, department stores, airfields, hospital clinics, production lines, and automobile traffic.

The author explains how to consider these problems in terms of a probabilistic model, and how this model may be used to treat a number of interesting situations, corresponding to different assumptions concerning arrival time and the structure of the facility.

The last two chapters are devoted to some applications of these techniques to inventory control and maintenance of equipment. Here it might be well to add to the references given by the author those treating these questions by means of a different technique, that of dynamic programming; *cf.* Arrow, Harris

and Marschak, *Optimal Inventory Policy*, Econometrica, 1951; Dvoretzky, Kiefer and Wolfowitz, *The Inventory Problem*, I, II, Econometrica, 1952; and Bellman, *Dynamic Programming*, Princeton University Press, 1957, where other references may be found.

The author appends a sizable number of references he has found useful in writing the book, and refers to other sources which contain extensive bibliographies of these topics.

This volume is highly recommended either as textbook or as collateral reading in a course on Operations Research, Mathematical Economics, or Applied Probability Theory. Even if the reader is not interested in queues, *per se*, he will find both the detailed discussion of model-building and the mathematical analysis extremely valuable.

RICHARD BELLMAN
The RAND Corporation

Integrated Algebra and Trigonometry. By Robert C. Fisher and Allen D. Ziebur. Prentice-Hall, New Jersey, 1958. 427 pp. $7.95.

It is the novel way of treating many familiar topics which makes this work a stimulating one to read. The concept which unites the many topics in this book is that of function, a relationship between two sets of numbers. More emphasis is placed on inequalities and absolute values than is customary in the usual text on algebra. Trigonometric functions are introduced via a trigonometric point on a unit circle (essentially radian measure) and all the properties of these functions, such as periodicity, variation, addition formulae and evenness or oddness are developed without any reference to the traditional definitions using right triangles.

The factoring of quadratic trinomials is relegated to a position after the solution of quadratic equations, and hence the relation between the factors and the roots of the associated quadratic equation is apparent. Systems of simultaneous linear equations are solved by reducing them to a canonical triangular system. With this approach, the concepts of the matrices associated with such systems and their properties are easily presented. Determinants then appear as numbers associated with square matrices.

Throughout the text the material is excellently presented, and the aim of the authors seems to be to stress ideas rather than the mechanical aspects of the subject. As a result, computational problems and long lists of drill exercises are omitted, and the few tables are abbreviated and poorly labeled. Much of the work is rigorously presented, yet intuitional proofs are given when expedient.

This is a text which should prove stimulating to capable students interested in mathematics and to teachers who are willing to deviate from the traditional methods used in teaching trigonometry and algebra.

R. G. SANGER
Kansas State College

Raum und Gegenraum. Einführung in die neuere Geometrie. By Louis Locher-Ernst. Philosophisch-Anthroposophischer Verlag am Goetheanum, Dornach, Switzerland, 1957. $6.50.

The center of the author's attention is occupied by the principle of duality. The usual way in which this basic idea is presented to novices in the study of projective geometry is fragmentary and unconvincing. The author undertakes not only to remedy this situation, but to work out a complete dual for metrical geometry. He goes about his task with skill and imagination, backed by a consumate art of exposition, aided by an abundance of excellent drawings. The results obtained are applied to some extent to the study of plane curves.

But the real aim of this discussion, in fact the "raison d'être" of the book as a whole, transcends the field of geometry. The principle of duality, which emerged from projective geometry during the first half of the 19th century, put alongside the accustomed point-space a new "plane-space" or "counter-space" ("Gegenraum") which, in the hands of the geometers, became a powerful tool. Furthermore, the principle of duality exerted a great influence upon mathematical philosophy in general, and upon the axiomatic method in particular. To a geometer of the stature of Michel Chasles (1793–1880), this did not exhaust the potentialities of this principle. He perceived that plane-space was destined to play an important role in the study of the physical sciences, and others [*Aperçu historique sur l'origine et le développement des méthodes en géométrie*, first ed., 1837, p. 408]. Towards the end of the first quarter of the present century this idea found adherents in some quarters (see pp. 207 ff). Locher-Ernst's elaboration of the "plane-space" is intended to facilitate the use of that space, rather than the point-space, as a frame of reference by the physicists, biologists, *etc.*

Such a suggestion could not be made at a more appropriate time than the present, when mathematical thinking is predominantly oriented toward applications. Nevertheless, to think of space in terms of planes remains a big order. Be that as it may, students and especially teachers of projective geometry will read Locher-Ernst's work with much profit. The book contains more than two hundred first-rate figures, an interesting preface, a brief table of contents, and no index of any kind.

NATHAN ALTSHILLER COURT
University of Oklahoma

Integral Equations. By S. G. Mikhlin. Pergamon Press, New York, Los Angeles, 1957. xii+338 pp. $12.50.

This book is very elegantly written, and the translator has done an excellent job of retaining its simplicity of style. Part I, of two parts, deals with methods of solution of integral equations, and introduces the reader gradually and smoothly to all the notions required for a complete treatment of the subject at an elementary level. Each topic is illustrated by example, with a practical

view in mind of actually obtaining numerical estimates and solutions. Many classical proofs of some complexity are given in much simplified form.

Even though it is necessary to use the notions of the Lebesgue integral, and mean convergence in many proofs, the presentation is simple enough that a reader without these mathematical disciplines can still follow the proofs, and need only accept the existence of certain integrals. Apart from this, the book is admirably suited as a text in a first course in integral equations, and the wealth of examples and applications (mostly in two dimensions) is sufficient to appeal to nearly all applied interests.

Part I includes the Fredholm theory for the equation of the second kind, including systems, for Volterra, degenerate, and weakly singular kernels; the Hilbert-Schmidt theory of symmetric kernels, and selected examples of singular equations with Hilbert and Cauchy type kernels. Part II gives applications of these integral equations to many problems in two and three dimensions, including Dirichlet's problem, Neumann's problem, and related examples, the biharmonic equation, and numerous problems from hydrodynamics, elasticity theory, and conformal mapping.

GORDON LATTA
Stanford University

Elements of Mathematics. By J. Houston Banks. Allyn and Bacon, Boston, 1956. x+422 pp. $5.75.

This book could be used as a text or reference in a variety of courses. No prerequisite other than elementary arithmetic is necessary, but some previous experience with algebra and geometry would be very desirable. The chapters are, with exceptions noted by the author, independent, and starred sections can be omitted without affecting continuity. Topics currently much discussed, particularly in the fields of elementary and secondary education, are numerous. For examples: the number system, mathematical proof, groups, fields, sets, Boolean algebra, functions and statistics. Some further idea of the wide range of this text may be obtained by noting a few selected topics: bases other than ten, algorithms, indirect proof, Zeno's paradoxes, transfinite numbers and systems of measurement. The logical development of the arithmetic of the natural numbers should be of interest and value to elementary teachers. The chapters on algebra and mathematical proof should be of particular interest to secondary teachers. The text is clearly written but may be somewhat concise for use as a text for prospective and in-service teachers—a use for which it seems otherwise well suited. At times the limited coverage of some topics should be sufficiently stimulating to prompt the student to seek further information. This is not an easy book for backward students but a careful presentation "designed to make more meaingful the fundamental concepts and techniques of mathematics." It serves this purpose well.

JAMES L. SIMPSON
Montana State College

Lie Groups. By P. M. Cohn. Cambridge University Press, 1957, vi+164 pp. \$4.00 (Cambridge Tracts in Mathematics and Mathematical Physics, No. 46).

This monograph is intended, as the author mentioned in the preface, to develop the beginning of the theory of Lie groups, especially the fundamental theorems of Lie relating the group to its Lie algebra. The approach is somewhat standard. Nevertheless the author presents the material in a very clear and self-contained manner. Only the most elementary knowledge of groups, vector spaces and general topology is assumed. Among the propositions and definitions, there are illuminating remarks giving the motivation of the concepts and theorems. The organization is good, and the style is so lucid that it can serve as an excellent introduction to the theory of Lie groups.

Chapter I introduces the concepts of analytic manifolds, infinitesimal transformations and differential forms. Chapter II opens with some elementary properties of topological groups; Lie groups and their analytic subgroups are then defined. In Chapter III, the Lie algebra of a Lie group is introduced by means of right infinitesimal translations. One-parameter subgroups and exponential mapping are discussed. Chapter IV deals with the exterior algebra of differential forms. By applying it to Lie groups, the equations of Maurer-Cartan are established.

Chapter V is concerned with the converse of the fundamental theorems of Lie. The author gives, in detail, the proof of the 1-1 correspondence between local Lie groups and their Lie algebras. Chapter VI covers canonical charts (coordinate systems), continuous homomorphisms, quotient Lie groups, the relation between analytic subgroups and Lie subalgebras, and the adjoint representation. In Chapter VII, the author constructs the universal covering group of a Lie group and proves the uniqueness theorem. The book concludes with an appendix on completely integrable system of total differential equations (Frobenius Theorem).

The classification problem and global properties of the underlying space of Lie groups are not touched.

H. C. WANG
Northwestern University

The Analysis of Multiple Time-Series. By M. H. Quenouille. Hafner, New York, 1957. 105 pp. \$4.75.

In a series of papers published in the 1920's, G. U. Yule discussed the difficulties encountered when regression or correlation analysis is applied to time-series and the impossibility of interpreting the correlations between time-series without reference to the internal structure of the series. This led to a more intensive study of the manner in which time-series may be generated. Led by Yule, statisticians began to pay more attention to the problems of recognizing, testing, and estimating the structure of time-series.

Because of the complex analysis involved, much of the work has dealt with the analysis of individual time-series although the need for studying multiple time-series is quite apparent, particularly in the field of economics. Numerous papers on various aspects of the analysis of multiple time-series have appeared in the literature from time to time but there remain numerous gaps both in theory and applications.

The author states that his purpose in writing this monograph is to fill some of the gaps and to present a unified account of the methods available for the analysis of multiple time-series.

The manner in which the author has woven together the various aspects of the theory is indicated by the chapter headings: (1) Introduction; (2) Specification; (3) Identification; (4) Preliminary Investigation; (5) Practical Complications; (6) Estimation; (7) Significance and Goodness-of-fit Tests; (8) Practical Examples.

The author has done a remarkably good job on a difficult task. Here is a summary of one approach to the problems of analyzing multiple time-series. The work is concise, at times almost too much so, and is generously illustrated with numerical examples. Shortcomings and gaps remaining in the theory are pointed out rather than suppressed. The printing is good and, except for a few lost decimal points, the book is relatively free of errors.

PAUL M. HUMMEL
University of Alabama

Ordinary Difference-Differential Equations. By Edmund Pinney. University of California Press, 1958. 262 pp. $5.00.

In the last fifty years there has been a respectable number of mathematical articles treating integro- and difference-differential equations, but until the present volume appeared there has been no book in this field. The author's purpose is twofold: (i) to develop the general theory of a wide class of difference-differential equations and (ii) to give techniques for solving them. Integral transforms (with known inverses) play an important role in the theory which covers such types as first and second order mixed difference-differential equations with constant coefficients and one lag, and various mixed and nonmixed equations. Systems of mixed integro-differential and difference-differential equations with constant coefficients are also treated. A special feature of the book is the first practical treatment of a number of nonlinear equations, including Minorsky's equation. Applications are in many fields, including biology, economics, engineering and physics, and the book will be of great interest to all areas of applied mathematics in which the meaningful models involve a time lag. There are fourteen pages of bibliography.

CLETUS O. OAKLEY
Haverford College

BRIEF MENTION

101 *Puzzles in Thought and Logic.* By C. R. Wylie, Jr. Dover, New York, 1957. $1.00.

This collection of, for the most part, new logical puzzles will delight mathematicians everywhere. Certainly this low-cost volume belongs on the shelf of every mathematics library, both public and private. In fact, your reviewer feels it would be a welcome addition in a great many doctors' waiting rooms.

Calculus of Variations and Its Applications. Edited by Lawrence M. Graves. McGraw-Hill, New York, 1958. 153 pp. $7.50.

The papers of the Eighth symposium in Applied Mathematics, sponsored by the American Mathematical Society and the Office of Ordnance Research on April 12–13 at the University of Chicago, comprise the main meat of this up-to-date volume. The main topics include elasticity, plasticity, hydrodynamics, and dynamic programming.

The contributors include Eric Reissner, D. C. Drucker, Joseph B. Keller, J. B. Diaz, J. L. Synge, M. M. Schiffer, Richard Bellman, S. Chandrasekhar, and E. H. Rothe. In addition to the nine addresses, two brief notes by P. G. Hodge, Jr., and H. F. Weinberger are included. The individual papers either have been, or probably will be, reviewed in *Mathematical Reviews.*

Table for the Solution of Cubic Equations. By Herbert E. Salzer, Charles H. Richards and Isabelle Arsham. McGraw-Hill, New York, 1958. 161 pp. $7.50.

This tabular presentation will assist in obtaining rapid solutions of cubic equations using a desk calculator.

Analytic Geometry Problems. By C. O. Oakley. College Outline Series, Barnes and Noble, 1958. $1.95.

This excellent and extensive revision of Oakley's earlier work will be welcomed by mathematics students and teachers everywhere. The carefully-drawn figures, numerous illustrative examples, and excellent problems with answers, should make this small book a valuable adjunct for any textbook. Indeed, it could well be used as a text in itself.

Magic House of Numbers. Irving Adler. New American Library, New York, 1958. 123 pp. $.35.

This is an interesting collection of puzzles and tricks, available at your corner drugstore.

Education in Mathematics for the Slow Learner. By Mary Potter and Virgil Mallory. National Council of Teachers of Mathematics, Washington, D. C., 1958. 36 pp. $.75.

Mathematics Clubs in High Schools. By Walter H. Carnahan. National Council of Teachers of Mathematics, Washington, D. C., 1958. 32 pp. $.75.

This brief pamphlet contains the ideas of several teachers who have been connected with successful mathematics clubs. The wide spread interest in mathematics clubs may be judged by the phenomenal growth of the high school and junior high school mathematics club, Mu Alpha Theta, which is sponsored by the Mathematical Association of America. In one year's time this club has grown to more than two hundred eighty chapters in forty-six of the forty-nine states.

How to Use Your Library in Mathematics. By Allene Archer. National Council of Teachers of Mathematics, Washington, D. C., 1958. 6 pp. $.40

The Golden Number. By Borissavlievitch. Alec Tiranti, London. Philosophical Library, New York, 1958. 91 pp. $4.75.

More aesthetics than mathematics, but still interesting. Concerned primarily with the geometry of architecture.

History of Mathematics. Vols. I and II. By D. E. Smith. Dover, New York, 1958. Vol. I, 596 pp. $2.75, Vol. II, 725 pp. $2.75.

It is good to welcome back into print, in a relatively inexpensive edition, the well-known historical work of D. E. Smith.

Recreational Mathematics. By William L. Schaaf. National Council of Teachers of Mathematics, Washington, D. C., 1958. 151 pp. $1.20.

This valuable source book is a guide to the literature, prepared in a convenient form so that one may quickly learn where to look for information on the subject of interest. It certainly belongs in every mathematical library.

A Treatise on Plane and Advanced Trigonometry. By E. W. Hobson, Dover, New York, 1958. 383 pp. $1.95.

This reprint of Hobson's *A Treatise on Plane Trigonometry* should be required reading for all writers and teachers of elementary trigonometry.

Solid Geometry. By J. S. Hails and E. J. Hopkins. Oxford University Press, New York, 1957. viii+196 pp. $2.20.

This is a standard solid-geometry text for use in sixth-form courses in English schools. There is no hint of the combined analytic and synthetic approach which is currently being used in many of the more advanced American high schools.

The Taylor Series. By P. Dienes. Dover, New York, 1958. 552 pp. $2.75.

This reissue of Dienes' text, originally published by the Oxford Press, contains an investigation of analytic functions by means of Taylor Series, a useful adjunct for anyone studying complex variable.

Introduction to the Theory of Numbers. By Leonard E. Dickson. Dover, New York, 1958. 183 pp. $1.65.

A welcome republication of the original University of Chicago Press book. This reviewer had forgotten that Dickson's little book contains Siegel's proof on Thue's theorem on the rational approximations to a root of an algebraic equation.

Three Dimensional Dynamics. By C. E. Easthope. Academic Press, New York, 1958. viii+277 pp. $7.80.

This text, intended for undergraduate honor students, presents a vectorial treatment of solid dynamics without the use of tensors.

An Electronic Computer for Statistical Analysis of Radio Propagation Data. By M. Grønlund and C. O. Lund. Acta Polytechnica, Applied Mathematics and Computing Machinery Series, Vol. 1, No. 3, Copenhagen, 1957. $2.30 (address orders to Box 5073, Stockholm 5, Sweden).

The problems of statistical analysis of propagation data are considered and the equipment used is described in some detail in this Danish monograph.

An Introduction to the Theory of Random Signals and Noise. By Wilbur B. Davenport, Jr. and William L. Root. McGraw-Hill, New York, 1958. ix+393 pp. $10.00.

An introduction to the statistical theory needed to study signals and noises in communication systems by means of random processes. The Karhunen-Loeve expansion is used for random functions over a finite interval. This book provides another indication of the growing importance of advanced statistical procedures in engineering today.

Communication, Organization and Science. By Jerome Rothstein. Falcon's Wing Press, Indian Hills, Colorado, 1958. xcvi+110 pp. $3.50.

A discussion based upon the premise that measurement and communication have the same underlying logical structure. A number of the author's ideas are sufficiently provocative so that this reviewer recommends that anyone interested in information theory read it. The eighty-five page foreword by C. A. Muses is interesting in itself.

The Exterior Ballistics of Rockets. By Leverett Davis, Jr., James W. Follin, Jr. and Leon Blitzer. Van Nostrand, New York, 1958. v+457 pp. $8.50.

This well-motivated mathematical treatment of the exterior ballistics of rockets without moving control surfaces assumes a basic familiarity with analytical mechanics, differential equations, and vector analysis.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Lloyd J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Alaska: Dean William Cashen, Dean of Students and formerly head of department, will return as Professor; Dr. Torcom Chorbajian, Iowa State University, has accepted a position as Assistant Professor.

Brown University: Dr. W. H. Fleming, Purdue University, has been appointed Assistant Professor; Dr. Leon Greenberg, Yale University, has been appointed Instructor; Dr. A. M. Duguid has been appointed Research Associate in the Division of Applied Math.; Professor A. A. Bennett is now Professor Emeritus.

Millsaps College: Professor T. L. Reynolds is on sabbatical leave during the year 1958–59 at the U. S. Naval Ordnance Test Station in China Lake, California; Professor S. R. Knox is Acting Head of the Mathematics Department for the year 1958–59; Professor V. B. Temple, recently retired from Louisiana College, has been appointed Visiting Professor.

Mississippi Southern College: Associate Professor B. O. Van Hook has been promoted to Professor and made Chairman of the Department; Dr. G. W. Nicholson, University of Georgia, has been appointed Professor; Dr. R. W. Bagley, Lockheed, has been appointed Professor; Assistant Professor William Sanders, returning from three years study at the University of Illinois, has been appointed Associate Professor; Assistant Professor Virginia Felder, returning from a year's study at Columbia University, has been appointed Associate Professor; Mr. Gaston Smith, University of Alabama, has been appointed Assistant Professor.

West Virginia University: Mr. Bernard Gilbert, New York University, and Mr. H. W. Gould, University of North Carolina, have been appointed Instructors.

Assistant Professor W. S. Bicknell, University of Wisconsin, has been promoted to Associate Professor.

Assistant Professor C. H. Cunkle, Colorado State University, has been appointed Associate Professor at Dickinson College.

Assistant Professor D. E. Deal, Ball State Teachers College, will be studying at the University of Michigan on a Danforth Teacher Study Grant during 1958–59.

Dr. F. C. DeSua, Bell Telephone Laboratories, Whippany, New Jersey, has been appointed Associate Professor at the College of William and Mary.

Mr. L. T. Gardner, Oakwood School, Poughkeepsie, is now a graduate student at Columbia University and Part II Instructor for the Actuaries' Club of New York.

Dr. R. L. Graves, Standard Oil Co. of Indiana, Whiting, Indiana, has been appointed Assistant Professor of Applied Mathematics in the School of Business, and Associate Director of The Operations Analysis Laboratory at the University of Chicago.

Mr. J. G. Hankins, Jr., University of Oklahoma, has accepted a position as Geophysicist with the U. S. Navy Hydrographic Office, Washington, D. C.

Dr. D. K. Harrison, Brown University, has been appointed Assistant Professor at Haverford College.

Professor M. H. Heins, Brown University, has been appointed Professor at the University of Illinois.

Mr. Bruce Kellogg, Northwestern University, has accepted a position as Mathematician with Combustion Engineering, Windsor, Connecticut.

Dr. R. R. Kemp, Institute for Advanced Study, has been appointed Assistant Professor at Queen's University, Kingston, Ontario, Canada.

Dr. L. A. Kokoris, Washington University, will be a Visiting Associate Professor at the Illinois Institute of Technology during 1958–59.

Assistant Professor L. H. Lange, Valparaiso University, as the recipient of a Science Faculty Fellowship from the National Science Foundation, continues on leave at the University of Notre Dame.

Assistant Professor W. G. Lister, Brown University, has been appointed Professor at State University College on Long Island.

Professor Lee Lorch, Philander Smith College, is a Visiting Lecturer at Wesleyan University during 1958–59.

Mr. M. R. Luttrell, Western Electric Co., Chicago, Illinois, is now at Massachusetts Institute of Technology, Lincoln Laboratory, Lexington, Massachusetts, as a Defense Projects Engineer.

Assistant Professor R. T. J. Mahoney, United States Naval Academy, has been appointed Instructor at Washington University.

Mr. W. T. Neis, Aramco, Saudi Arabia, is now Mathematical Statistician, Bureau of the Census, Suitland, Maryland.

Assistant Professor F. R. Olson, University of Buffalo, has been promoted to Associate Professor.

Dr. G. A. Parkinson, University of Wisconsin-Milwaukee, is now Director, Milwaukee Vocational and Adult Schools.

Dr. R. F. Rinehart, Case Institute of Technology, is now in the Office of Ordnance Research, Duke University.

Dr. W. R. Seugling, Allstates Engineering Co., Trenton, New Jersey, is now Sr. Research Engineer, Rocketdyne Division of North American Aviation, Inc., Canoga Park, California.

Mr. L. C. Teng, Massachusetts Institute of Technology, is now a Research Associate, Engineering Research Institute of the University of Michigan.

Dr. R. L. Wilson, Convair, Ft. Worth, Texas, has been appointed Professor and Chairman of Department of Mathematics, Ohio Wesleyan University.

Assistant Professor J. E. Davis, University of Illinois, died March 1, 1958. He had been a member of the Association for 24 years.

Professor Emeritus E. H. Taylor, Eastern Illinois State College, died June 26, 1958. He was a charter member of the Association.

Mr. Jurio Tsuchiya, Mississippi State College, died June 28, 1958.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1958 Preliminary Actuarial Examination are as follows:

First Prize of $200: Daniel G. Quillen, Harvard University

Additional Prizes of $100 each:

Edward J. Barbeau, Jr.	Toronto University
William H. Blake, Jr.	George Washington University
Theodore M. Jungreis	Rensselaer Polytechnic Institute
David H. Krantz	Yale University
Joe Lipman	Toronto University
Dennis W. Moore	Harvard University
Theodore S. Rosky	State University of Iowa
Lawrence A. Shepp	Brooklyn Polytechnic Institute

The Society of Actuaries has authorized a similar set of nine prizes for the 1959 examinations on Part 2.

The 1959 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service under the direction of a committee of actuaries and mathematicians and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 13, 1959. The closing date for applications is April 1, 1959.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 154 persons have been elected to membership by the Board of Governors on applications duly certified.

ESTHER ADLER, M.A. (Columbia) Lecturer, Yeshiva University

GARY L. ANDERSON, Student, Bemidji State College.

DAVID R. ANDREW, Student, Southwestern Louisiana Institute.

PAUL ARMER, A.B. (U.C.L.A.) Head, Numerical Analysis, RAND Corp., Santa Monica, California.

CHARLES H. ARNOLD, M.A. (Lehigh) Teacher, Chestnuthill High School, Brodheadsville, Pennsylvania.

JAMES L. BAILEY, Ph.D. (Michigan S.U.) Asst. Professor, Case Institute of Technology.

MRS. SARABETH T. BARNES, M.S. (Wisconsin) Instr., University of Minnesota.

LT. DAVID R. BARR, M.S. (Miami, Ohio) United States Air Force, Iowa City, Iowa.

BENJAMIN BAUMSLAG, Student, University of Witwatersrand, South Africa.

NORMAN F. BEACH, A.B. (Princeton) Asst. Supt., Film Emulsion Division, Eastman Kodak Co.

RICHARD W. BEALS, Student, Yale University.

MORTON P. BERENSON, Student, University of Oregon.

DAVY L. BERNARD, Student, Southwestern Louisiana Institute.

RAYMOND C. BOEHNE, Student, University of Detroit.

KAYLAND Z. BRADFORD, Student, University of Oklahoma.

PEDRO BRICENO, Licenciatura (Central U. of Venezuela) Actuary, Ministry of Labor, Caracas, Venezuela.

WILLIAM T. BROOKS, B.S. (Texas Tech) Industrial Engineer, Astrodyne, McGregor, Texas.

DORETTA M. BROWN, Student, Ursinus College

CAPT. EDWIN E. BROWN, B.S. (Florida S.U.) U.S.A.F.; Grad. Student, Stanford University.

JAMES W. BROWN, M.A. (Michigan) Teaching Fellow, University of Michigan.

JOHN A. BROWN, Ph.D. (Wisconsin) Professor, Oneonta State Teachers College, New York.

JAMES P. BURLING, M.A. (N.Y.S.C. for Teachers) Asst. Professor, Oneonta State Teachers College, New York.

LINDA M. BUSWELL, Student, Carleton College.

WILLIAM F. CARPENTER, B.S.E.E. (Florida) Grad. Asst., University of Florida.

JOHN G. COOK, Student, University of British Columbia.

HUGH R. COOMES, Student, University of Kentucky.

WALTER E. CRAIG, Student, St. Martin's College.

HOBART L. CURTIS, Student, San Jose State College.

JOHN O. DANIELSON, M.S. (Wisconsin) Dean of Instruction and Chairman of Dept., Wisconsin State College.

REV. BRENDAN P. DONNELLY, OSB, B.A. (St. Anselm's) Instr., St. Anselm's College.

NETTIE A. DOOLITTLE, Ed.D. (Missouri) Asst. Professor, University of Missouri.

CHARLES DRESCHER, B.S. (Poly. Inst. of Brooklyn) Asso. Equipment Engineer, Western Electric Co., New York City.

FORREST E. DRISTY, B.S. (S. Dakota Sch. of Mines & Tech.) Instr., South Dakota School of Mines and Technology.

DANIEL E. DUPREE, B.S. (Louisiana Poly. Inst.) Grad. Student, Alabama Polytechnic Institute.

ALEXANDER DURLEY, M.A. (Atlanta) Instr., Texas State University.

FRANCES P. DUSHEK, M.A. (Texas) Instr., Del Mar College.

BRUCE ERICKSON, Student, University of British Columbia.

MARIO V. FIONDELLA, M.S. (Florida) Instr., University of Florida.

MARY L. FISHER, M.S. (Illinois) Chairman of Dept., Joliet Township High School and Junior College, Illinois.

JAMES L. FLATT, M.A. (Peabody) Asst. Professor, Clemson College.

NEAL E. FOLAND, M.A. (Missouri) Instr., University of Missouri.

ROBERT M. FOSSUM, Student, St. Olaf College.

JAMES K. FOSTER, Student, University of Minnesota.

KENNETH A. FOWLER, Ph.D. (Michigan) Asst. Professor, San Jose State College.

JOSEPH A. FROMME, Student, Purdue University.

LOUIE R. GIESZL, Student, University of Houston.

JIMMIE D. GILBERT, M.S. (Alabama Poly. Inst.) Instr., Alabama Polytechnic Institute.

MYRNA C. GILES, Student, University of Omaha.

MAURICE E. GILMORE, JR., Student, Georgetown University.

RONALD L. GRAHAM, Student, University of Alaska.

WILLIAM M. GRAVES, Student, Georgia Institute of Technology.

LAWRENCE F. GUSEMAN, JR., Student, A. & M. College of Texas.

JOSEPH M. C. HAMILTON, M.A. (U.C.L.A.) Instr., Los Angeles City College.

PAUL M. HARMS, A.B. (Bethel) Grad. Asst., Iowa State College.

DOUGLAS C. HARVIE, M.S. (Victoria) Lecturer, Victoria University of Wellington, N. Z.

ROBERT J. HENDERSON, Student, University of British Columbia.

JOHN R. F. HEWETT, Student, University of Toronto.

WILLIAM B. HIGGINS, M.A. (Columbia T.C.) Asst. Professor, Ball State Teachers College.

PAUL D. HILL, M.S. (Alabama Poly. Inst.) Grad. Student, Alabama Polytechnic Institute.

C. ROBERT HILLAND, Student, A. & M. College of Texas.

JOHN W. HOGAN, A.B. (Berea) Teaching Asst., University of Wisconsin.

LEONARD I. HOLDER, Ph.D. (Purdue) Asst. Professor, San Jose State College.

JOHN E. HOMER, JR., Student, State University of Iowa.

EDGAR H. HOPPER, M.S. (Tennessee) Instr., Alabama Polytechnic Institute.

RICHARD L. HOTCHKISS, Student, University of Minnesota.

WILLIAM A. HUSTED, Student, Hobart College.

ROBERT J. IRWIN, B.S. (Case Inst.) President, Eddie Painton Associates, Cleveland, Ohio.

RICHARD L. JACOBSEN, Student, St. Olaf College.

WILBUR J. JONSSON, B.S. (Manitoba) Summer Student, Defence Research Board, St. Hubert, Quebec.

LEWIS S. KARSH, Student, Wesleyan University.

RICHARD E. KEEFFE, B.S. (Buffalo) Engineer, Rocketdyne, Canoga Park, California.

DENIS J. KIELY, JR., B.E.E. (Santa Clara) Instr., San Jose State College.

JAMES I. KILLIAN, Student, University of Oklahoma.

RICHARD L. KINKEAD, Student, University of Houston.

CLAUDE A. KIRKPATRICK, Student, University of Oklahoma.

ROBERT E. KIRSAMMER, Student, University of Detroit.

MRS. DOROTHY J. KNIGHT, M.A. (Kent S.U.) Asst. Professor, Muskingum College.

DAVID H. KRANTZ, Student, Universite de Nancy, France.

SAMUEL S. KUTLER, B.A. (St. John's, Annapolis) Asso. Math., Johns Hopkins University.

FRANK W. LANE, M.A., M.S. (St. Bonaventure) Asso. Professor, Ohio Northern University.

FREDERICK L. LAWSON, Student, Southwestern Louisiana Institute.

JOSEPH LEV, Ph.D. (Cornell) Acting Chief, Bureau of Statistical Services, New York State Educational Dept., Albany, New York.

LAWRENCE A. LIDDIARD, Student, University of Minnesota.

JERRY L. LINNSTAEDTER, Student, A. & M. College of Texas.

JAMES N. MARTIN, Student, Texas Christian University.

HARRY T. MATHEWS, Student, Georgia Institute of Technology.

JOSEPH L. MAZANEC, M.S. (Wisconsin) Instr., Los Angeles City College.

ESTELLE MAZZIOTTA, M.A. (U.C.L.A.) Instr., Los Angeles City College.

VAN K. MCCOMBS, B.S. (Mississippi S.C.) Grad. Student, Mississippi State College.

THOMAS S. MCFEE, B.S. (Maryland) Math., David Taylor Model Basin.

JACK L. MCGEE, B.S. (Oklahoma) Jr. Engineer, Temco Aircraft Co., Garland, Texas.

GEORGE J. MICHAELIDES, M.S. (Virginia Poly. Inst.) Instr., Lamar State College of Technology.

CAPT. WALTER R. MILLIKEN, B.S. (U.S. Military Academy) Asst. Professor, U.S.A.F. Academy.

CHESTER L. MIRACLE, M.S. (Alabama Poly. Inst.) Grad. Student, University of Kentucky.

BRUCE J. MOODY, Student, University of Oklahoma.

MRS. DORIS L. MOORE, M.A. (Western Carolina) Grad. Student, University of Virginia.

WILLIAM R. NEAL, Student, Arizona State College.

KEITH H. NICAISE, Student, Mississippi State College.

TERRY P. NORRIS, Student, Arizona State College.

KARL K. NORTON, Student, Yale University

WORTH D. NOWLIN, JR., Student, A. & M. College of Texas.

ROBERT D. OBERG, B.S. (Minnesota, Duluth) Grad. Student, University of Minnesota.

LLOYD D. OLSON, M.Ed. (N. Dakota Agric. Coll.) Instr., North Dakota Agricultural College.

HARLAN S. PAFFORD, M.Ed. (Virginia) Teacher, Marion High School, Virginia.

CHARLES E. PETTYPOOL, JR., M.A. (Ohio S.U.) Instr., Eastern Illinois University.

GEORGE W. PETZNICK, JR., Student, Case Institute of Technology.

ROGER A. PURVES, B.A. (British Columbia) Grad. Student, University of British Columbia.

ARLAN B. RAMSAY, Student, University of Kansas.

JOHN R. REAY, B.A. (Pacific Lutheran) Acting Instr., University of Idaho.

CHAVIS L. RENWICK, JR., M.S. (N. Carolina Coll.) Statesville, North Carolina.

WAYNE H. RICHTER, Student, Swarthmore College.

MRS. AGNES Y. RICKEY, M.S. (Barry) Supervisor, Dade County Board of Public Instruction, Miami, Florida.

GEORGE A. RIETZ, B.S.E.E. (S. Dakota S.C.) Ed. Program Development, General Electric Co., New York City.

REINO R. RIIHONEN, Bon Marche, Spokane, Washington.

SHELDON T. RIO, M.A. (Montana S.U.) Grad. Student, Oregon State College.

THACHER T. ROBINSON, M.A. (Princeton) Chappaqua, New York.

JIMMIE H. ROGERS, B.A. (Oklahoma Baptist) Instr., Connors State Agricultural College.

JOSEPH S. ROMANOW, Student, Tufts University.

JORDAN T. ROSENBAUM, Student, Case Institute of Technology.

GEORGE N. ROUNTHWAITE, Student, San Diego State College.

MRS. ISABELLE P. RUCKER, B.A. (Randolph-Macon) Teacher, Louisa County School Board, Virginia.

DAVID H. SANDERS, Student, Princeton University.

RICHARD F. SHEPARD, A.M. (Columbia) Math. Editor, Henry Holt & Co., New York City.

TADAO SHINGU, B.S. (Kyoto, Japan) Asst. Professor, Shimane Agricultural College, Japan.

PETER SIKORYAK, Student, Hobart College.

FRANCIS E. SIWIEC, Student, Carteret School, W. Orange, New Jersey.

RALSTON J. SMITH, Student, Eastern Kentucky State College.

SAMUEL SOBEL, B.S. (C.C.N.Y.) Reliability Engineer, Sperry Gyroscope Co., New York.

HAROLD S. SPRAKER, M.Ed. (Virginia) Grad. Student, University of Virginia.

HENRY E. STERN, B.E. (Tulane) Asso. Scientist, Lockheed Aircraft Corp., Marietta, Georgia.

JOHN E. STROUT, B.S. (Illinois) Grad. Asst., University of Illinois.

ROBERT L. STURGEON, Student, University of British Columbia.

DONALD H. TARANTO, B.S. (C.C.N.Y.) Applied Science Rep., I.B.M., New York.

MRS. DINA G. S. THOMAS, M.A. (Utah) Instr., University of Colorado.

JOHN A. THORPE, Student, Massachusetts Institute of Technology.

CARL L. TIBERY, B.S. (Bates) Asst. Instr., University of Maryland.

GEORGE G. TOWN, M.S. (Wisconsin) Boeing Research Asso., Oregon State College.

BENJAMIN A. TRIMBLE, Decatur, Georgia.

HWA TSANG, Student, University of Chicago.

OSCAR VALDIVIA G., B.S. (San Marcos, Peru) Asso. Professor, Universidad de Trujillo, Peru.

WILLIAM A. VEECH, Student, Dartmouth College.

GUY E. WEBBER, JR., B.S. (Emory & Henry) Teacher, Edgewood, Maryland.

RUSSELL A. WELKER, M.S. (Illinois) Grad. Asst., University of Illinois.

ALAN J. WESTCOTT, Student, University of California.

BRANDON W. WHEELER, Student, Sacramento State College.

ALTON L. WILKINSON, Student, A. & M. College of Texas.

JAMES P. WILLIAMS, Student, University of Detroit.

FLOYD D. WILDER, Student, Central State College.

ALAN WILSON, Ph.D. (Rice Institute) Houston, Texas.

GERALD WOHL, B.A. (Syracuse) Field Engineer, General Electric Co., San Francisco, California.

JOHN E. WOOD, M.A. (Longwood) Instr., Hampden Sydney College.

ROY E. WORTH, Student, University of Georgia.

JOHN W. WYMAN, Student, Olivet Nazarene College.

EDWARD G. ZDINAK, M.S. (Pittsburgh) Grad. Student, University of Pittsburgh.

THE MAY MEETING OF THE INDIANA SECTION

The thirty-fifth annual meeting of the Indiana Section of The Mathematical Association of America was held at Ball State Teachers College, Muncie, Indiana, on Saturday, May 3, 1958. Professor C. B. Gass of DePauw University, Chairman of the Section, presided at both morning and afternoon sessions. There were 52 in attendance, including 39 members of the Association.

The following officers were elected: Chairman, Professor G. N. Wollan of Purdue University; Vice-Chairman, Professor K. H. Carlson of Valparaiso University; Secretary-Treasurer, Professor C. F. Brumfiel of Ball State Teachers College.

Professor Edwards, chairman of the Committee on Awards, announced that four Association Medals had been awarded this year to high school seniors exhibiting high mathematical achievement in the Indiana Science Talent Search program. The following motions were carried: (1) That the Section adopt the policy of holding a fall meeting in joint session with the Mathematics Division of the Indiana Academy of Science. (2) That the Section sponsor the administration in Indiana of the Annual High School Mathematics Contest which is sponsored nationally by the M.A.A. and the Society of Actuaries. (3) That a committee be appointed by the chairman to work with the Indiana Council of Teachers of Mathematics to lay down a plan for securing support for local institutes on the teaching of mathematics.

Professor R. M. Thrall of the University of Michigan gave the invited hour address on the topic, "Applications of Mathematics in the Social Sciences."

The following short papers were presented:

1. *Some remarks on the teaching of elementary algebra*, by Professor A. M. Yaqub, Purdue University.

In attempting to strengthen our undergraduate mathematics courses, the author proposed to combine *understanding* with techniques in the teaching of elementary algebra. In algebra, as in high school geometry, one would naturally begin with the axioms which the real numbers are supposed to satisfy, and from which the theorems are to be derived. These axioms include the usual axioms for a field. On the basis of these one could derive the familiar laws of signs. By assuming a few additional axioms one could then derive the laws of inequalities and the laws of exponents. This modest start would immediately indicate to the student the spirit of the axiomatic approach, and no doubt show him that algebra is every bit as suitable for axiomatic treatment as geometry. Moreover, this method allows the teaching of algebra as a *science* rather than as a collection of recipes together with some mystical laws.

2. *Dexinal gauges*, by Mr. Aaron Miller, 1415 W. 28th Street, Indianapolis, Indiana.

The stop to which the base 2 must be raised to produce a number k is called the dexinal gauge of k. Two numbers are dexinated by adding their gauges and are sindexinated by subtracting their gauges. A dexratio (dexine of a ratio) is the dexnum (dexine of the numerator) sindexed by the dexdenom (dexine of the denominator). A dexratio is reduced by taking the ratio of the logarithms to the base 2 of the dexnum and dexdenom. The equality of the two dexratios is called a poise and is the analogue of a proportion. The mean poisal is the analogue of the mean proportional. Employing these definitions, the author developed some theorems and presented several interesting applications in numerical computations.

3. *The seventeen ornamental groups*, by Professor H. W. Alexander, Earlham College.

The seventeen ornamental groups were discussed from the standpoint of (a) the classification of an actual design under one of these groups and (b) the representation of the groups by means of matrices. Other designs were examined.

4. *The sum of a particular series and the corresponding integral*, by Professor L. W. Stark, Butler University.

The analytic solution of the heat conduction equation is obtained by use of the Laplace transform and the convolution integral. One term in the solution is the cosine series

$$\sum_{n=1}^{\infty} [(-1)^{n-1} \cos (2n-1)y/(2n-1)^{2p-1}]$$

$p \geq 1$. Beginning with the result given by Bromwich for $p=1$, the method for obtaining the result for $p=2$ was given in detail and results were then stated for $p=3, 4, 5$. It was also established that the summations evaluated at $y=0$ are multiples of corresponding Euler numbers.

5. *A progress report on experimental work at Ball State Teachers College*, by Professor C. F. Brumfiel, Ball State Teachers College.

For the last three years an experimental geometry and algebra program has been tested in the Ball State laboratory school. Tenth grade geometry is treated rigorously in a course based upon a modified version of the Hilbert postulates. The algebra is a mild postulational treatment that covers most of the conventional topics of ninth grade algebra. During the past year teachers from Eastern Indiana schools, enrolled at Ball State in a National Science Foundation In-Service Institute, have taught the geometry experimentally. Under a continuation of this grant the experimental program will be continued and expanded to include the algebra in 1958–59.

6. *It's all in your mind*, by Professors J. E. Forbes and W. R. Fuller of Purdue University, presented by Professor Forbes.

This was a preliminary report on an experiment in commercial television in a series of weekly broadcasts of topics in mathematics. Many interesting points concerning the preparation and presentation of the programs and the methods used to enlist and maintain the interest of listeners were discussed.

7. *A report on the 1957 Summer Institute on Mathematics in the Social Sciences at Stanford University*, by Professors J. C. Polley, Wabash College, G. N. Wollan, Purdue University, and K. H. Carlson, Valparaiso University.

The subject was introduced by Mr. Polley with general remarks on the nature, the staff, and the organization of the institute. He stated that the institute had been sponsored by and financed by a grant from the Social Science Research Council for the purpose of acquainting college teachers of mathematics with the current applications of mathematics in the field of the social sciences. Attending members represented colleges of various types widely distributed over the country.

Drawing on the eight-weeks experience, Mr. Wollan emphasized the values of such institutes. He urged that efforts be made to promote setting some up on the local level.

In conclusion, Mr. Carlson discussed in some detail the conduct of the institute and the material presented by the various members of the staff.

J. C. Polley, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual spring meeting of the Minnesota Section of the Mathematical Association of America was held on May 17, 1958 at St. John's University, Collegeville, Minnesota. Reverend Walbert Kalinowski, O.S.B., of St. John's University presided at the morning session. The section chairman, Professor O. E. Stanaitis of St. Olaf College, presided at the afternoon session. There were 53 persons registered, of whom 39 were members of the Association.

The following officers were elected to serve for the academic year 1958–1959: Chairman, Reverend Walbert Kalinowski, O.S.B., of St. John's University; Secretary, Professor F. L. Wolf of Carleton College; Members of the Executive Committee, Professor O. E. Stanaitis of St. Olaf College, Professor David Lewis of Hamline University and Remington-Rand Univac, and Professor James Serrin of the University of Minnesota.

At the business meeting, Professor J. M. H. Olmsted reported on the High School Mathematics Contest which was sponsored by the section in Minnesota this year. In this, the first year that the contest was given in Minnesota, it was very successful. The section owes many thanks to Professor G. K. Kalisch of the University of Minnesota and to the members of his High School Contest Committee for this success. Professor Leon Green of the University of Minnesota was appointed chairman of the High School Contest Committee for 1958–1959.

Professor F. L. Wolf reported for the Committee on High School-College Relations. Several proposals for the improvement of high school-college relations were made by the committee and motions from the floor were passed instructing the committee to proceed with implementation or further study of these.

The following papers were presented:

1. *Equations with trigonometric values as roots*, by Professor K. W. Wegner, Carleton College.

Sixty-four equations were presented, along with illustrations of their use in the classroom, as the only irreducible polynomial equations with integral coefficients and of degree two through seven whose roots are of the form $\pm\sin y$, $\pm\cos y$, $\pm\tan y$, $\pm\cot y$, $\pm\sec y$, or $\pm\csc y$, where y is a rational number of degrees.

2. *Five mutually-tangent spheres*, by Mr. H. E. Fiala, North Dakota State College, introduced by Professor Ruby M. Grimes.

This paper explains the problem of Five Mutually Externally Tangent Spheres. It gives also the general solutions for the volume of an irregular tetrahedron and the radii of its inscribed and its circumscribed spheres. It then shows how the concept of five tangent spheres can be applied to greatly simplify the solutions to most problems dealing with properties of tetrahedrons. The principle involved is essentially a change of variables to reduce the number of unknowns. It suggests also how the concept of five tangent spheres can be applied in designing an inertial guidance system.

3. *Mathematical machines and mathematicians*, by Professor S. C. Kleene, University of Wisconsin.

The class of all functions which can be computed, and of all relations (predicates) which can be decided by machines, is characterized. On the basis of this characterization examples are given of functions which cannot be computed, and of relations which cannot be decided, by machines. (This paper is an exposition of results originally obtained independently, in different forms, by Alonzo Church and by A. M. Turing in 1936. See the speaker's *Introduction to Metamathematics*, New York, 1952, or his lecture notes *Sets, Logic and Mathematical Foundations*, Williams College, 1956, or Martin Davis *Computability and Unsolvability*, New York, 1958).

4. *The oscillation in sign of solutions of linear, ordinary difference equations with constant coefficients*, by Professor Murray Braden, Macalester College.

In investigating the oscillation in sign of solutions of the infinite strip problem for a general class of partial difference operators, Fulton Koehler and the author found it desirable to prove the theorem: "Every nontrivial, real solution of an ordinary, linear, homogeneous difference equation with real, constant coefficients must oscillate in sign infinitely often as x approaches infinity." In the course of the proof, two lemmas were established concerning trigonometric sums of the form $S(x)=\sum_{j=1}^{n}(c_je^{i\theta_jx}+c_je^{-i\theta_jx})$, where $x=0, 1, 2, \cdots$. These are: (1) $S(x)$ will approximate itself as closely as desired for an infinite number of values of x; (2) $S(x)$ must be positive for at least one value of x, and negative for at least one value of x.

5. *Boolgebra: ands, ors, and nots, among other things*, by Dr. D. M. Brown, Remington Rand UNIVAC.

Boolean Algebra may be considered as a means of "mixing" binary information (signals, statements, *etc.*) by using combinations of three mixing devices called AND, NOT, and OR. The NOT device has one input and one output, while each of the AND and OR devices has two inputs and one output.

There are exactly four devices with one input and one output, and exactly sixteen devices with two inputs and one output. Mechanical equipment implementing each device, using two sizes of balls as the binary inputs and outputs is demonstrated. The Boolean equation for each device is presented, along with a schematic diagram and an output ("truth") table.

F. L. Wolf, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The fourteenth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the University of Montreal, Montreal, Quebec, Canada, on May 10, 1958. The Chairman of the Section, Professor E. E. Haskins of Clarkson College of Technology, presided at the morning session, and the Vice-Chairman, Professor Caroline A. Lester of the New York State College for Teachers at Albany, presided at the afternoon session. There were 91 persons in attendance, including 54 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor Caroline A. Lester, New York State College for Teachers at Albany; Vice-Chairman,

Professor Dis Maly, Rensselaer Polytechnic Institute; Secretary-Treasurer, Professor N. G. Gunderson, University of Rochester. A report from a committee of department chairmen was heard to the effect that the New York State Education Department had been contacted and made aware of the opposition of the Section to the specific requirements added for accreditation in teaching mathematics in New York State and to the added requirements in education.

A Committee on the Strengthening of Mathematics in the Section reported on its activities and studies. One recommendation was that individual colleges and departments of mathematics make known their views on the teacher certification problem. Another was that schools give more attention to the Advanced Placement Program. The Committee warned of the danger in attracting more students in Mathematics without strengthening teaching. It was voted that this Committee be continued for another year.

Professor A. J. Coleman of the University of Toronto reported on the extension of the Mathematics Contest into Canada. This year over 1000 students from 41 schools participated. Professor Nura Turner of the New York State College for Teachers reported on the Contest in New York State (exclusive of the metropolitan area and the western counties). This year, the first for this area, 3729 students from 211 schools participated.

The program was as follows:

1. *The Canadian Mathematical Congress*, by Professor R. L. Jeffery, Queen's University. (By invitation.)

The speaker, who is President of the C. M. C., gave a history of the Congress, and described the organization, activities, and plans of the group.

2. *Training teachers of mathematics*, by Professor A. Wittenberg, Laval University. (By invitation.)

The crucial issue in the teaching of mathematics is the adequate training of the teachers. A thorough mathematical training is necessary, but not enough. A teacher must be emotionally committed to teaching, able to see elementary mathematics in their own right, able to teach them without taking for granted the mathematical way of thinking. The speaker suggests a teacher training program including: 1) a thorough introduction to mathematics and their main applications; 2) a good introduction to those subjects enabling a teacher to see mathematics in context—foundations, philosophy, and history of mathematics; 3) on the basis of such knowledge, a thorough and independent examination of elementary mathematics.

3. *A mathematical analysis of the English verb-phrase*, by Professor Joachim Lambek, McGill University.

The method of syntactic types, due to Ajdukiewicz, Bar-Hillel and the author, is used to analyse the English verb-phrase. After suitable types have been assigned to all the forms of the verbs *must, work, like, have, be, etc*, it becomes possible to decide by a mathematical computation whether any given arrangement of these forms, together with the names *John, Jane, etc.*, constitutes a grammatical sentence, such as *John must have been liking Jane.*

4. *The logic of algebras and the algebra of logic*, by Professor L. O. Kattsoff, Harpur College.

There is need to distinguish between an algebra which can be interpreted in terms of propositions, conjunction, negation and implication, and the modes of reasoning used in demonstration. In a sense, it is misleading to speak of the algebra of logic. What we have is simply an abstract algebra. When this abstract algebra is interpreted as a logic, there is already implied an intuitive logic, presupposed by the interpretation. If not, an infinite regression of rules of logic develop. A similar distinction is needed between finite induction as a property of a well-ordered system, and as a mode of demonstration.

5. *Convex representations of planar graphs*, by Professor W. T. Tutte, University of Toronto.

Let G be a triply connected graph, known to be planar. We consider the problem of constructing a representation of G in the plane such that the edges are represented by straight lines and each residual domain is the interior or exterior of a convex polygon. It is found that the vertices of such a representation can be obtained by solving certain linear equations, with arbitrary positive coefficients, associated with the graph. In the simplest representation of this kind the vertices of one circuit of G are represented by the vertices of a convex polygon, arbitrary except for the number of its vertices, and every other vertex is at the centroid of the vertices to which it is directly joined.

6. *The spherical image*, by Professor C. S. Ogilvy, Hamilton College.

If all the unit normal vectors of a regular region of a surface are moved to the origin, their end points describe the spherical image of the region on the unit sphere. This mapping by parallel normals is well known to be conformal if and only if the region is part of a minimal surface or of a sphere. It can be shown that the mapping of the general (non-minimal) surface of negative Gaussian curvature can be described locally by decomposing it into three linear transformations: a rotation through 90°, a reflection, and a certain (specified) simple shear.

7. *On the classical characterizations of the existence a.e. of a derivative for functions of one real variable*, by Professor A. G. Fadell, University of Buffalo.

The best-known condition for the existence a.e. of a derivative for functions of one variable is bounded variation. This condition, of course, is not necessary. The classical characterization apparently is not well known, as evidenced by its absence from the recent wealth of real-variable texts. In this note we outline the proof of the classical necessary and sufficient condition for measurable functions, namely the existence a.e. of locally bounded difference quotients. The proof utilizes the sufficiency of a Lipschitz condition on an open set, the Lipschitz extension theorem, elementary decomposition theorems, and aspects of Lebesgue density.

8. *The minimax theorem (another simple proof)*, by Professor J. E. L. Peck, McGill University.

Another simple proof of the minimax theorem is given, which applies equally well to the case where only one space of pure strategies is finite. The method uses properties of linear functions on convex sets.

N. G. Gunderson, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The eleventh annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at Oregon State College, Corvallis, Oregon on June 20, 1958 in conjunction with the 547th meeting of the American Mathematical Society and an organizational meeting of the Pacific Northwest Section of the Society for Industrial and Applied Mathematics. Professor Sydney Hacker, Chairman of the Section, presided over the meetings. There were 133 persons in attendance, including 82 members of the Association.

The following officers were elected: Chairman, Professor Kenneth Bush, University of Idaho; Vice-Chairman, Professor Arthur Livingston, University of Washington; Secretary-Treasurer, Professor K. S. Ghent, University of Oregon. Brief reports were given concerning the High School Mathematics Contest for British Columbia, Idaho and Oregon. The contest was not sponsored in Montana by the Section.

The program included a session for contributed papers, an invited address by Professor Arthur Erdelyi, three invited half-hour talks, and four fifteen-minute talks on problems of high school mathematics teaching. Professor Lonseth introduced Professor

Erdelyi; Professor Keeping introduced the three lecturers for the half-hour talks and Professor Vatnsdal presided at the evening session. The abstracts for the papers are given below.

Contributed papers (10 minutes):

1. *Classical orthogonal polynomial transforms*, by Dr. T. P. Higgins, Dalmo Victor Company, Belmont, California.

Integral transforms are often useful in solving boundary-value problems when the nature of the transform is such that it can be applied to replace a particular group of terms of the differential equation with a single simple term. Various authors have considered the particular advantages of this type of integral transform. A generalization of the Jacobi transform can be made using a more general polynomial kernel, and this transform reduces, under special choices for certain of the parameters, to either the Jacobi transform, the Gegenbauer transform, the Legendre transform, the Tchebichef transform, the Laguerre transform, or the Hermite transform. This note defines the classical orthogonal polynomial transform, proves its characteristic property, and gives simple examples of applications of the characteristic property.

2. *A method for solving the quartic equation*, by Dr. C. E. Sealander, Mathematical Research Laboratory, Boeing Scientific Research Laboratories, Seattle, Washington.

Consider the quartic equation $f(x) \equiv x^4+bx^3+cx^2+dx+e=0$. The substitution $x=my+n$ transforms this equation to $(my)^4+(1/6)f'''(n)(my)^3+(1/2)f''(n)(my)^2+f'(n)my+f(n)=0$. This is a reciprocal equation if (1) $f(n)=m^4$ and (2) $m^3f'''(n)=6mf'(n)$. The latter two equations can be solved simultaneously for m and n, and the substitution $y+1/y=z$ for solving reciprocal equations then yields the solution of the quartic. Elimination of m from equations (1) and (2) yields a resolvent cubic of the form (3) $An^3+Bn^2+Cn+D=0$. Three exceptional cases are discussed: (I) The case $A=B=C=D=0$; (II) $n=-b/4$ is a root of equation (3); and (III) $f'(n)=0, f'''(n)\neq 0$.

3. *Incomplete elliptic integrals of the third kind*, by Dr. R. G. Selfridge and Dr. J. E. Maxfield, U. S. Naval Ordnance Test Station, China Lake, California, presented by Dr. Selfridge.

The incomplete elliptic integral of the third kind is given by $\pi(\phi, \alpha^2, k)=\int_0^\phi(1-\alpha \sin^2 \theta)^{-1} \cdot(1-k^2 \sin^2\theta)^{-1/2}d\theta$, and occurs in many aerodynamic and hydrodynamic problems. One of the difficulties associated with tabulating this function is that of checking the results. A simple and effective check is available if an independent computation can yield the complete integral ($\phi=\pi/2$). Previous methods of computation of the complete integral proved difficult to handle in that convergence of the numerical methods proved exceedingly cumbersome. The final method chosen was that of solving the differential equation (with respect to k), that is satisfied by the complete integral. This involved some careful handling because of the singularities of the equation but provided a very effective check on the computations. The resultant table is in press at Dover Publications.

Invited address:

Operational calculus and generalized functions, by Professor Arthur Erdelyi, California Institute of Technology.

Expository account of J. Mikusinski's work (Rachunek operatorow, 1953) which provides a mathematical background both for Heaviside's operational calculus and for delta functions and other "improper functions." This calculus has been applied successfully to ordinary differential equations, integral equations, and hyperbolic and parabolic partial differential equations.

Invited thirty-minute talks:

1. *The problem of optimal control*, by Professor D. W. Bushaw, Washington State College.

The problem arises out of that of designing a control system in which the available control force is bounded and the object is to reduce the difference between the current and desired outputs,

together with all time derivatives of this quantity, to zero as rapidly as possible. It has been conjectured (in the so-called "Bang-Bang Principle") that in any solution of this problem only the extreme values of the control force will be used. The current mathematical status of this principle was discussed, and various solved and unsolved questions connected with it were described.

2. *Variational principles and eigenvalues*, by Professor H. J. Sagan, University of Idaho.

Let L be a self-adjoint linear differential operator and consider the homogeneous boundary value problem $L(u)+\lambda ru=0$ with $u+\alpha(\partial u/\partial n)=0$ on the boundary. In general there are infinitely many values λ_k of λ (Eigenvalues) for which this problem has nontrivial solutions (Eigenfunctions). These Eigenfunctions u_k turn out to be solutions of a variational problem of the type $I(u)/C(u)\to$minimum where I and C are positive definite quadratic functionals and where u has to be orthogonal, with the weight function r, to the preceding $k-1$ Eigenfunctions. The corresponding Eigenvalues λ_k are given by the minima of I/C. This principle—widely referred to as "Rayleigh's Principle"—opens up possibilities for numerical computations of upper and lower bounds of the Eigenvalues as suggested by Ritz and Weinstein.

3. *Approximation by polynomials with integral coefficients*, by Professor Edwin Hewitt and Professor H. S. Zuckerman, University of Washington, presented by Professor Hewitt.

Let X be a compact Hausdorff space, $\mathfrak{C}(X)$ the set of all real-valued continuous functions on X, and $\mathfrak{F}$ a subset of $\mathfrak{C}(X)$ separating points of X. Let $\mathfrak{P}$ be the set of all functions on X that are polynomials with integral coefficients in functions from $\mathfrak{F}$. Let $\mathfrak{B}=\{g: g\in\mathfrak{P},\ 0\leq g(x)<1\}$ for all $x\in X$. Let $J=\bigcap g\in\mathfrak{B}\{x: x\in X,\ g(x)=0\}$. THEOREM. *A function $\phi\in\mathfrak{C}(X)$ is arbitrarily closely uniformly approximable on X by functions g from $\mathfrak{P}$ if and only if there is a $g\in\mathfrak{P}$ such that $g(x)=\phi(x)$ for all $x\in J$. For $X=[-\alpha,\alpha](\alpha>0)$, and $\mathfrak{P}$=all polynomials with integral coefficients, and for certain other cases, the set J is identified explicitly.*

High school Mathematics Teaching (four fifteen-minute invited talks followed by discussion period):

1. *In-service programs for teachers*, by Miss Lesta Hoel, Supervisor of Mathematics, Portland Public Schools.

The emphasis in the in-service training program of mathematics teachers has changed from a consideration of *how* to teach to *what* to teach. The critical analysis of the past program and the investigation of what the program of the future should be has taken care of the "*how.*" Colleges should plan more summer courses designed for teachers to fill in the gaps in their pre-service training. The product which the high school sends to the college is as strong as the teachers which the college trains. Mathematics education must be a cooperative project.

2. *What Portland does for the gifted child*, by Mr. Cecil Jenkins, Grant High School, Portland.

Special courses are offered for "exceptionally endowed" students in one section each of second year algebra (sophomores), plane and solid geometry (juniors), and trigonometry, advanced algebra (seniors). Student-participants in EE classes have high ratings in intelligence, academic ability and emotional maturity. Participants may apply through or be nominated by a teacher, counselor or the EE coordinator and must be approved by a committee. Class size of 10 to 15 may be raised to 25 to 30 next year. Grading is left to the teacher, but EE designation is removed from the transcript of a student receiving C or less in EE class. Extra funds are budgeted for books and instructional materials for EE classes. A teacher, chosen for subject matter competence, serves three years in the program.

3. *Reed College courses for teachers*, by Professor Burrowes Hunt, Reed College.

A brief description was given of the content of recent courses, primarily in-service courses or

summer workshops, for high school teachers. The emphasis has been on subjects directly relevant to high school courses: elementary number theory, rudiments of set theory and modern algebra, the nature of a deductive system, and basic properties of the real and complex number systems.

4. *The Meder Committee and pre-college mathematics*, by Professor J. M. Kingston, University of Washington.

The Commission on Mathematics of the College Entrance Examination Board believes that drastic revision of the content and point of view of high school mathematics is essential. Mathematics should not be presented as an accumulation of rules and tricks but rather the structure and pattern of mathematics should be emphasized. The Commission favors a gradual modification, modernization and improvement of existing curricula rather than any attempt at hasty replacement or improvisation. They have outlined a flexible four-year high school program consisting of two years of Elementary Mathematics, a third year of Intermediate Mathematics and a fourth year of Advanced Mathematics.

K. S. GHENT, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

LLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Marian College, Indianapolis, November 7, 1958.

IOWA, State University of Iowa, Iowa City, October 17, 1958.

KANSAS

KENTUCKY, Centre College of Kentucky, Danville, April, 1959.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 6, 1958.

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University of Agriculture and Applied Science, East Lansing, March 28, 1959.

MINNESOTA, University of Minnesota, Duluth, October 11, 1958.

MISSOURI, Lindenwood College, St. Charles, April 25, 1959.

NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN, College of the Holy Cross, Worcester, Massachusetts, November 29, 1958.

NORTHERN CALIFORNIA, Stanford University, January 17, 1959.

OHIO

OKLAHOMA, Oklahoma City University, October 24, 1958.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.

PHILADELPHIA, Lehigh University, Bethlehem, November 29, 1958.

ROCKY MOUNTAIN, Utah State University of Agriculture and Applied Science, Logan, Spring, 1959.

SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.

SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.

SOUTHWESTERN, Arizona State College, Tempe, Spring, 1959.

TEXAS, University of Texas, Austin, April, 1959.

UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

WISCONSIN, Wisconsin State College, Platteville, May, 1959.

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No. 4. *Projective Geometry,* by J. W. Young. 1930.

No. 5. *History of Mathematics in America before 1900,* by D. E. Smith and Jekuthiel Ginsburg. 1934. (Out of print)

No. 6. *Fourier Series and Orthogonal Polynomials,* by Dunham Jackson. 1941.

No. 7. *Vectors and Matrices,* by C. C. MacDuffee. 1943.

No. 8. *Rings and Ideals,* by N. H. McCoy. 1948.

No. 9. *The Theory of Algebraic Numbers,* by Harry Pollard. 1950.

No. 10. *The Arithmetic Theory of Quadratic Forms,* by B. W. Jones. 1950.

No. 11. *Irrational Numbers,* by Ivan Niven. 1956.

One copy of each monograph may be purchased by members of the Association for $1.75 per copy. Orders should be sent to Harry M. Gehman, Secretary-Treasurer, Mathematical Association of America, University of Buffalo, Buffalo, 14, New York.

Additional copies and copies for non-members of Monographs 1-8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois.

In the case of Monographs 9-11, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, New York.

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PART II

Introduction to Arithmetic Factorization and Congruences from the Standpoint of Abstract Algebra

H. S. VANDIVER and MILO W. WEAVER

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INTRODUCTION TO ARITHMETIC FACTORIZATION AND CONGRUENCES FROM THE STANDPOINT OF ABSTRACT ALGEBRA

By

H. S. VANDIVER, *Professor of Mathematics*

and

MILO W. WEAVER, *Assistant Professor of Mathematics*

The University of Texas

The Seventh

HERBERT ELLSWORTH SLAUGHT

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INTRODUCTION

In the present paper* we discuss the beginnings of the theories of factorization and congruences, involving rational integers, using some of the simplest concepts in the theories of commutative semigroups (in particular, Abelian groups) and commutative rings. (As an example of the simplicity of the tools used, we nowhere employ the general basis theorem for Abelian groups.) In the main, we develop only those special results from semigroup or group theory that we find necessary in order to obtain our number theoretic results. We thereby obtain much more coherence in our account than has been apparent to us in expositions we have observed, published before this, in which nothing but elementary algebra was employed by the writers.

Up to the point where we introduce the concept of products of semigroups, at least, we hope this account will be understood by readers who have had little more than a first course in number theory or modern algebra. However, the remainder of the article is more sophisticated and may appeal mostly to experienced mathematicians.

As far as previous efforts to apply abstract algebra to number theory are concerned, we note that H. Weber† made some steps in this direction. For example, he treated the set of residue classes modulo m, using the term *Zahlklassen nach einem Modul*, by applying some elementary results in group theory. G. A. Miller‡ also obtained Fermat's and Wilson's theorems as well as several other elementary results by the use of simple properties of groups, and in particular of cyclic groups. Vandiver§ treated the theory of finite rings and semirings from a standpoint of developing results with direct application to number theory. In his courses in algebra and number theory at The University of Texas, he adopted the viewpoint of abstract algebra and group theory in discussing even

* About one-third of Vandiver's work on this paper was done under a Senior Postdoctoral Fellowship awarded to him by the National Science Foundation, and about two-thirds of the work done by Vandiver and about two-thirds of the work done by Weaver on this article was done under Basic Research Grant 3697, which was awarded to them by the National Science Foundation.

The writers wish to express their thanks to R. D. Allentharp who examined the manuscript with great care and made a number of valuable suggestions.

† Lehrbuch der Algebra, Bd. 2, Zweite Auflage, Braunschweig, Friedrich Vieweg und Sohn, vol. 60–68, 1899, pp. 302–314. On page 60 Weber remarks, "Das Wichtigste Beispiel einer endlichen commutativen Gruppe bieten die Reste der natürlichen Zahlen nach einem beliebeigen Modul, wenn sie durch die gewöhnliche Multiplication mit einander verbunden werden." Weber meant to limit himself here to residues prime to the modulus, as otherwise the residues modulo m do not, in general, form a group.

‡ Annals of Math. II, vol. 4, 1903, pp. 188–190; this MONTHLY, vol. 12, 1905, pp. 41–43; this MONTHLY, vol. 18, 1911, pp. 204–209.

§ Trans. Amer. Math Soc., vol. 13, 1912, pp. 293–304; Annals of Math., II, vol. 18, 1917, pp. 105–114; Proc. Nat. Acad. Sci., vol. 20, 1934, pp. 579–584; Bull. Amer. Math. Soc., vol. 40, 1934, pp. 914–920; Proc. Nat. Acad. Sci., vol. 21, 1935, pp. 162–165; Proc. Nat. Acad. Sci., vol. 23, 1937, pp. 552–555; this MONTHLY, vol. 46, 1939, pp. 22–26; Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, vol. 85, 1940, pp. 71–86; Annals of Math., II, vol. 48, 1947, pp. 22–28.

the most elementary parts of number theory. This point of view gradually developed in a period of some 25 years. In addition to this, Weaver|| further developed these ideas in his classes at The University of Texas during the last five years and published some of the results obtained therefrom.

R. Fueter¶ applied some of the ideas of abstract algebra, including modules and ideals, as well as groups, in connection with a development of elementary number theory. He also employed such tools in a treatment of the theory of cyclotomic fields.

E. Hecke* gave an exposition of a considerable part of the theory of Abelian groups and applied the results to the elementary theory of numbers as well as to parts of the theory of algebraic numbers.

H. Hasse† considered the ring of residue classes modulo m and applied elementary group theory and ring theory to developing its properties. He assumed the properties of Abelian groups without proofs for his applications to the theory of the ring mentioned. Later he‡ again considered the ring of residue classes modulo m and used less of group theory than formerly, but he explained results he obtained by arithmetical methods, in terms of groups.

Now, under multiplication, the nonzero elements of the ring of residue classes form a semigroup which is not always a group. Aside from the authors of the present paper, however, none of the investigators above mentioned used semigroups in their accounts of elementary number theory. Also none of them considered the non-unit elements in the ring of residue classes modulo m where matters appear quite complicated unless some of the theory of semigroups is employed. The use of that system, and the introduction of possibly new concepts concerning it, with applications to factorization problems, in our opinion, constitutes the most novel part of our treatment. However, in parts of number theory where addition and multiplication are both involved in certain ways, the theory of semigroups is of little or no value, as in the theory of continued fractions, which we do not discuss here.

It may be that a reader of this paper is mainly interested in the number theoretical phase of it and may be of the opinion that it would have been much simpler to develop what we did here in the theory of congruences, directly, without the use of abstract algebra. However, most of the tools we have set up from the latter theory may be applied extensively, with little or no change, in the theories of finite fields, finite rings, and algebraic numbers. In fact, in writing this paper we have had, among other ideas, this end in view. Obviously, a purely arithmetic approach could not achieve this.§

|| Math. Mag., vol. 25, 1952, pp. 125–136; this MONTHLY, vol. 63, 1956, pp. 387–391.

¶ Synthetische Zahlentheorie, Dritte Auflage, Walter de Gruyter and Co., Berlin, 1950.

* Vorlesungen über die Theorie der Algebraischen Zahlen, Akademische Verlagsgesellschaft, Leipzig, 1923.

† Zahlentheorie, Akademie-Verlag Berlin, 1949.

‡ Vorlesungen über Zahlentheorie, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1950.

§ We note also that the article contains a considerable number of definitions, some of which are used very little. However, in further developments of the present ideas it would be very convenient to use them often.

The literature on elementary number theory is, of course, immense, and the literature on abstract algebra, particularly groups and their generalizations, likewise. Consequently, when we make references in this paper to the work of some other author on some particular idea, *this does not mean that we necessarily regard him as the first mathematician who published such an idea. The reference is given so that the reader might be able, because of it, to augment his knowledge of the topic being discussed.* If some reader thinks it would have been particularly illuminating to have referred to the work of some author we did not mention, we might be able to list such references in a supplement to the present article, if we are advised of these important omissions.

Of course, as time goes on, and particularly recently, more and more applications of modern algebra are being made to various parts of mathematics, pure and applied. The present contribution, of course, comes under the heading of an application to number theory.

Starting with a set of axioms for ordinary algebra and some consequences from them in the results given in section 1.1 below, the present article, we think, is self-contained in reference both to abstract algebra and number theory, aside from solution of the problems. The results we needed from abstract algebra in order to develop theorems in number theory are introduced in the text as we needed them, including the necessary definitions. Even aside from the algebraic approach, the proofs of the theorems we give often seem to contain elements of novelty.

The proofs of Theorems 6.9.12 and 6.12.14 are not simple. This has been also true of all other treatments of these topics we have noted elsewhere, and the difficulties seem to be inherent in the nature of the subjects.

Chapter I

ASSOCIATIVE ALGEBRAIC SYSTEMS

1.1. Some references to associative algebra. We have already treated the foundations of associative algebra, which include the foundations of the theory of integers, in four papers published in *Mathematics Magazine*, each under the title *A development of associative algebra and an algebraic theory of numbers*, appearing as follows:

(I)—Vandiver, vol. 25, 1952, pp. 233–250.
(II)—Vandiver, vol. 27, 1953, pp. 1–18.
(III)—Vandiver and Weaver, vol. 29, 1956, pp. 135–149.
(IV)—Vandiver and Weaver, vol. 30, 1956, pp. 1–8; Errata, vol. 30, 1957, p. 219.

In view of this, our present account will be naïve, but we shall give definitions and proofs in a form we think is usually acceptable in present-day number theory and algebra.

1.2. Some definitions. A *semigroup* $\mathfrak{S}$ is an algebraic system with *operation* $(\cdot)$ and *equivalence* $(=)$ such that for each A, B, $C\in\mathfrak{S}$, $A\cdot B=X$ has a unique solution $X\in\mathfrak{S}$, and $(A\cdot B)\cdot C=A\cdot(B\cdot C)$. The operation symbol is usually omitted between elements of $\mathfrak{S}$. If $\mathfrak{S}$ has an element E such that for each $A\in\mathfrak{S}$, $AE=EA=A$, then E is called the *identity* of $\mathfrak{S}$. It is unique. If for each A_1, $A_2\in\mathfrak{S}$, $A_1A_2=A_2A_1$, $\mathfrak{S}$ is called *commutative* or *Abelian*.

If a semigroup $\mathfrak{G}$ has an identity E, and for each $B\in\mathfrak{G}$, the equations $BX=E$ and $YB=E$ have solutions, X, $Y\in\mathfrak{G}$, then it is easy to show that $X=Y$, and $\mathfrak{G}$ is called a *group*. It follows easily that X is unique, and it is called the *inverse* of B and is denoted by B^{-1}. If an element U in a semigroup $\mathfrak{S}$ has an inverse, U is called a *unit* in $\mathfrak{S}$. If $A_1A_2\cdots A_n=C$, then the A's are *divisors* or *factors* of C, where C and the A's are in $\mathfrak{S}$. If $n\geqq 1$, then C is called a product of the A's.

The *order* of a semigroup $\mathfrak{S}$ is the number of distinct elements of $\mathfrak{S}$. If $A\in\mathfrak{S}$, the set of nonequivalent powers of A forms a *cyclic* semigroup $\mathfrak{C}$, and A is said to be a *generator* of $\mathfrak{C}$. If $\mathfrak{C}$ is finite, it contains a maximal subgroup $\mathfrak{G}$ which is also cyclic.* The orders of $\mathfrak{C}$ and $\mathfrak{G}$ are called, respectively, the *order* and *period* of A.

Let $\mathfrak{S}$ be a set with equivalence $(=)$ and operations addition $(+)$ and multiplication $(\times)$ (the latter sign is usually omitted) such that

* *Cf.* relation (8.1.2) that follows.

(i) $\mathfrak{S}$ is a semigroup relative to $+$ and $=$.
(ii) $\mathfrak{S}$ is a semigroup relative to $\times$ and $=$.
(iii) Whenever $S_1, S_2, S_3 \in \mathfrak{S}$, then $S_1(S_2+S_3)=S_1S_2+S_1S_3$, and $(S_2+S_3)S_1 = S_2S_1+S_3S_1$.
(iv) Substitution† as in ordinary arithmetic holds in $\mathfrak{S}$.

Under these conditions $\mathfrak{S}$ is called a *semiring*. The set of units, if there are any, of the multiplicative semigroup is called the set of *units* of the semiring. A semiring, whose additive semigroup (ASG) is a commutative group, is called a *ring*. The additive identity of a ring is denoted by 0. Suppose there is a ring $\mathfrak{R}$ whose multiplicative semigroup (MSG), the additive identity excluded, forms a commutative group. Such a ring is called a *field*.

The additive inverse of A, if it exists, in a semiring is denoted by $-A$; $A-B$ is used interchangeably with $A+(-B)$.

Let $\mathfrak{J}$ be a non-empty subset of a ring $\mathfrak{R}$ such that for any $I_1, I_2 \in \mathfrak{J}$ and $R \in \mathfrak{R}$, $I_1-I_2 \in \mathfrak{J}$, $RI_1 \in \mathfrak{J}$, and $I_1R \in \mathfrak{J}$; then $\mathfrak{J}$ is called an *ideal* of $\mathfrak{R}$. It is clear that $\mathfrak{J}$ is a subring of $\mathfrak{R}$, since $I_1-I_1=0$, $-I_1=0-I_1$, and $I_1-(-I_2)$ are also in $\mathfrak{J}$. Ideals have been extensively studied and have proved valuable in obtaining the structure of algebras. Hereafter in this paper we shall use small Greek letters to denote elements of a semiring except for semirings of rational integers.

We shall now give some examples of the concepts defined above. The sets will be taken from the set of rational numbers. As for semigroups, such examples are common. Let m be a positive integer; then the set of all positive multiples of m forms a semigroup under addition and also under multiplication. Under addition, the set mk with k ranging over all integers forms a group. However, the set of rational numbers, not including zero, forms a group under multiplication.

Perhaps the simplest example of a semiring is the set of natural numbers. The set of all positive multiples of a positive integer m also forms a semiring.

If a natural number m is multiplied by each integer we obtain a ring. The set of rational numbers forms a field.

If a and b are given integers, then the set $ax+by$ with both x and y ranging independently over the set of integers satisfies the conditions in our definition of an ideal in the ring of integers.

1.3. Three theorems on semigroups. We shall prove some theorems from group theory which we shall need several times.

If $\mathfrak{S}$ is a semigroup, $\mathfrak{R}$: $R_1, R_2, \cdots$ is a subset, finite or infinite, of $\mathfrak{S}$, and $S \in \mathfrak{S}$, we denote by $S\mathfrak{R}$, the set SR_i, $i=1, 2, \cdots$.

THEOREM 1.3.1. *If $\mathfrak{G}$ is a group of order n and $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$ of order d, then d divides n. If A generates a cyclic group $\mathfrak{G}$ of order n and $A^r=E$, where E is the identity of $\mathfrak{G}$, then n divides r.*

† As there exist semirings in which addition is not commutative, the order of the terms should be preserved here.

The first part of Theorem 1.3.1 is due to Lagrange. To prove it we shall show that the distinct elements of $\mathfrak{G}$ are the elements of disjoint sets, each containing d elements: $\mathfrak{H}$, $G_2\mathfrak{H}$, $G_3\mathfrak{H}$, $\cdots$, $G_{n/d}\mathfrak{H}$. $\mathfrak{H}$ contains d elements by hypothesis. We select $G_2 \notin H$, if such exists. Then $G_2\mathfrak{H}$ contains d elements, since from $G_2H = G_2H'$, we have $G_2^{-1}G_2H = G_2^{-1}G_2H'$ and $EH = EH'$ and therefore $H = H'$. $G_2\mathfrak{H}$ and $\mathfrak{H}$ are disjoint; otherwise from $G_2H = H'$ we would have $G_2 \in \mathfrak{H}$. We select $G_3 \notin \mathfrak{H}$, $G_3 \notin G_2\mathfrak{H}$ if such exists and find, similarly to the above discussion, that $G_3\mathfrak{H}$ contains d elements, no one of which is in $\mathfrak{H}$. And no element of $G_3\mathfrak{H}$ is in $G_2\mathfrak{H}$; otherwise, we would have $G_3 \in G_2\mathfrak{H}$. Since $\mathfrak{G}$ is finite, $\mathfrak{G}$ is exhausted by a number n/d of such sets, and we have the first part of our theorem.

To prove the second statement of the theorem, we use Theorem 2.2.8 below, the proof of which is independent of Theorem 1.3.1, and write $r = bn + c$, $0 \leq c < n$ and find, if $c \neq 0$, $A^r = A^{bn}A^c = EA^c = A^c = E$, contrary to hypothesis that A has order n. Hence n divides r.

THEOREM 1.3.2. *A semigroup $\mathfrak{G}$ is a group if and only if the equations $AX = B$ and $YA = B$ have solutions X and Y in $\mathfrak{G}$, for each $A, B \in \mathfrak{G}$.*

To prove this, suppose $\mathfrak{G}$ is a group. Then $A^{-1}B$ and BA^{-1} are solutions, respectively, of $AX = B$ and $YA = B$. To prove the converse, we let $A, B \in \mathfrak{G}$ and E and X be solutions, respectively, of the equations $YA = A$ and $AX = B$. Then $EB = E(AX) = (EA)X = AX = B$. Similarly if E' and Y' are solutions, respectively, of $AX = A$ and $YA = B$, we have $BE' = (Y'A)E' = Y'(AE') = Y'A = B$. Hence, for B arbitrary we have $EB = BE' = B$. Whence we have the identity $EE' = E' = E$. And from the solvability of $AX = YA = E$, we have the existence of an inverse for each $A \in \mathfrak{G}$. Our Theorem 1.3.2 follows.

An element C of a semigroup $\mathfrak{S}$ is said to be *left* (*right*) *cancellable* provided that if $A, B \in \mathfrak{S}$ and $CA = CB$ ($AC = BC$), then $A = B$. A *cancellable element* is one which is both right and left cancellable. If all elements of $\mathfrak{S}$ are cancellable then $\mathfrak{S}$ is called a *cancellative semigroup.*

Now let $\mathfrak{S}_1$ be a finite semigroup and consist of the distinct elements

$$A_1, A_2, \cdots, A_k. \tag{1.3.3}$$

We have

THEOREM 1.3.4. *If C is left cancellable in $\mathfrak{S}_1$, the set*

$$CA_1, CA_2, \cdots, CA_k \tag{1.3.5}$$

is a permutation of (1.3.3). *Any finite cancellative semigroup is a group.*

The proof follows immediately from the fact that if $CA_i = CA_j$ and $i \neq j$, then $A_i = A_j$ since C is left cancellable, which is a contradiction. Hence the set (1.3.5) gives k distinct elements in $\mathfrak{S}_1$, which gives a permutation of the elements in (1.3.3). The second part of the theorem follows from our definition of cancellative semigroup and Theorem 1.3.2.

Chapter II

SEMIGROUPS WITH A UNIQUE BASIS, AND PRIMES

2.1. General uniquely factorable semigroups. If a semigroup $\mathfrak{S}$ contains an element P, not the identity of $\mathfrak{S}$, which has no divisor other than possibly itself and the identity, if it exists in $\mathfrak{S}$, then P is called a *prime in* $\mathfrak{S}$. Nonunits which are not primes are called *composites.* If a nonunit D of $\mathfrak{S}$ is expressed as a product of primes, this expression is called a *decomposition of D into primes.*

Now suppose that $\mathfrak{S}$ is commutative and contains an identity E and that $\mathfrak{S}$ further satisfies:*

(i) $\mathfrak{S}$ contains a subset $\mathfrak{P}$ of primes.
(ii) If $S \in \mathfrak{S}$, and $S \neq E$, then S has exactly one decomposition into primes, aside from the order of the primes.

Then $\mathfrak{S}$ is said to be a *uniquely factorable semigroup*, and the prime elements of $\mathfrak{S}$ are said to be *independent* and to form together with E a *basis* for $\mathfrak{S}$. We notice that $\mathfrak{S}$ is a cancellative semigroup; also $\mathfrak{S}$ is infinite, for it follows from (i) and (ii) that each non-E element of $\mathfrak{S}$ generates an infinite cyclic semigroup.

2.2. Unique factorization in the multiplicative semigroup of natural numbers. Let $\mathfrak{N}$ denote the set of natural numbers with multiplication as an operation and ordinary equality as the equivalence. It follows from (I) of the series of papers in the *Mathematics Magazine* mentioned earlier, that $\mathfrak{N}$ is a multiplicative cancellative semigroup. The ordinary primes of $\mathfrak{N}$ satisfy (i) above, by definition. We shall use the word *prime* henceforth in this section only to denote natural number primes.

We shall now prove that the *number of distinct primes in the multiplicative semigroup of natural numbers is infinite.* To establish this we consider a prime p, and we shall determine a prime greater than p. Let n be the product of the distinct primes less than or equal to p. If $n+1$ is prime, the result is proved. If it is composite, it may be expressed as a product of primes, *viz.*, $n+1=p_1p_2 \cdots p_s$. It is impossible for one of these p's to coincide with one of the primes less than or equal to p. For if so, then there is a p_i such that $1=p_im$, with m an integer, but $|p_im|>1$. Hence each $p_i>p$. We now consider

THEOREM 2.2.1. *The natural numbers form a uniquely factorable cancellative semigroup under multiplication which has a unique infinite basis consisting of the set of prime numbers, together with unity.*

Proof. If we have an $m>1$, we may show that it can be expressed as the product of a finite number of prime factors; that is, the primes are the generators of the semigroup of natural numbers >1. If m is not prime, let $p_1>1$ be

* M. Ward, Annals of Math., II, vol. 36, 1935, pp. 36–39. A. H. Clifford, Annals of Math., II, vol. 39, 1938, pp. 594–610. The (ii) part of our definition prevents $\mathfrak{S}$ from having more than one unit. We shall later generalize the idea of unique factorization to apply to semigroups with more than one unit and also the idea of prime such that a prime times any unit is a prime.

the smallest divisor, not 1, of m. It is evidently prime. Write $m=p_1m_1$. If m_1 is prime, we have our expression; if not, let $m_1=p_2m_2$ with p_2 prime. This process cannot continue indefinitely since $p_i \geqq 2$ and $\prod_{i=1}^{k} p_i \geqq 2^k$, which is greater than m for k sufficiently large; *i.e.*, by the binomial theorem $2^m=(1+1)^m=1+m+t$, $t>0$; so $2^m>m$. Hence

$$m = p_1p_2 \cdots p_s,$$

where each p is prime, and s is some integer. Two decompositions $m=p_1p_2 \cdots p_s =p_1'p_2' \cdots p_t'$ are said to be the same if and only if $s=t$ and the p's are a permutation of the (p')'s.

We shall now show that no element of $\mathfrak{N}$ has two factor decompositions in $\mathfrak{N}$ into primes. Assume the contrary and let n be the smallest positive integer which has two different decompositions into prime factors. We set

$$n = p_1p_2 \cdots p_s = q_1q_2 \cdots q_t,$$

where

$$p_i \neq q_j; \qquad i = 1, 2, \cdots, s; \qquad j = 1, 2, \cdots, t,$$

since otherwise we obtain by cancellation an integer

$$n' < n$$

which has two distinct decompositions into primes. Let $p_1>q_1$; the argument is similar for $p_1<q_1$. Let

$$k = (p_1 - q_1)p_2p_3 \cdots p_s = p_1p_2 \cdots p_s - q_1p_2p_3 \cdots p_s.$$

Obviously $0<k<n$. Hence by the definition of n we have

$$\begin{aligned} k &= q_1q_2 \cdots q_t - q_1p_2p_3 \cdots p_s \\ &= q_1(q_2 \cdots q_t - p_1p_3 \cdots p_s), \end{aligned}$$

but q_1 is distinct from any prime appearing in the decomposition obtained from the other form of k. This gives $k<n$, and having two different expressions as the product of primes, contradicts the assumption. Since we have proved previously that the number of primes is infinite, and obviously unity (which was not considered in the above argument) forms the only other basis element, this gives Theorem 2.2.1.

If $a>0$ and $b>0$ have no factor greater than 1 in common, then they are said to be *relatively prime* or *prime to each other*. The *greatest common divisor* of two positive integers a and m will be designated henceforth as (a, m). In the statements 2.2.1–2.2.5 inclusive, the small letters employed will represent positive integers.

Problem 2.2.2. If n is arbitrary, show how to determine n consecutive composites.

The following corollaries follow immediately from Theorem 2.2.1.

COROLLARY 2.2.3. *If a and b are relatively prime and if ak is divisible by b, then k is divisible by b.*

COROLLARY 2.2.4. *If several positive integers are each prime to a, their product is prime to a.*

Problem 2.2.5. Show that if $m>0$ is not the kth power of a positive integer, then m is not the kth power of a rational fraction.

Problem 2.2.6. In a certain text it is argued that a special case $m=2$, $k=2$ of the last result may be proved without use of Theorem 2.2.1 because we cannot have two irreducible fractions equal with different numerators. Is this valid?

THEOREM 2.2.7 *If p is prime, the binomial coefficient*

$$\binom{p}{n}$$

is divisible by p, for $0<n<p$.

Proof. Since

$$\binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!} = d,$$

an integer, then

$$p(p-1)\cdots(p-n+1) = n!d.$$

The decomposition of the left-hand member into prime factors contains p. On the right, the decomposition of $n!$ cannot contain p since each factor of it is less than p. Hence d is divisible by p.

We shall often need Theorem 2.2.8, below.

THEOREM 2.2.8. *If $a\neq 0$ and m are integers, unique integers r and q exist such that*

$$m = aq + r; \qquad 0 \leqq r < |a|. \tag{2.2.9}$$

We first use induction to prove this theorem for a and m both positive. If $m=1$, $a=1$, we have (2.2.9) with $q=1$, $r=0$. If $m=1$, $a>1$, we have (2.2.9) with $q=0$, $r=1$. Assume that (2.2.9) holds for $m=k>0$ and all $a>0$. Then $k+1=aq+r+1$. Since $0\leqq r<a$, either $0<r+1<a$ or $0<r+1=a$. If $r+1<a$, (2.2.9) is satisfied for $m=k+1$. If $r+1=a$, we have $k+1=a(q+1)+0$, and (2.2.9) holds with $r=0$; so our theorem holds for m, a both positive. If m is a multiple of a, solutions of (2.2.9) are obvious for all such m. Let $\bar{q}$, $\bar{r}$ be a solution of (2.2.9) for m', a', positive values of m, a. Then clearly if m is not a multiple

of a, we have, for the cases $m=-m'$, $a=-a'$; $m=-m'$, $a=a'$; $m=m'$, $a=-a'$; the solutions, respectively, of (2.2.9): $\bar{q}+1$, $a'-\bar{r}$; $-\bar{q}-1$, $a'-\bar{r}$; $-\bar{q}$, $\bar{r}$. Suppose that $m=aq_1+r_1=aq_2+r_2$ with $0\leqq r_1<|a|$, $0\leqq r_2<|a|$. Then $aq_1+r_1=aq_2+r_2$ and $r_1-r_2=a(q_2-q_1)$. Whence $r_1=r_2$, and r in (2.2.9) is unique. If $m=aq_1+r$ $=aq_2+r$, we have $aq_1=aq_2$ and $q_1=q_2$.

Problem 2.2.10. Using the result of Theorem 2.2.8, we obtain for positive a_1 and a_2, $a_1>a_2$: $a_1=a_2q_1+a_3$, for $a_2>a_3\geqq 0$; $a_i=a_{i+1}q_i+a_{i+2}$, $a_i>a_{i+1}>a_{i+2}\geqq 0$ whence ultimately we have an n such that $a_n>0$, and $a_{n-1}=a_nq_{n-1}$. Show that a_n is (a_1, a_2). This is known as the *Euclidean Algorithm* for finding (a_1, a_2).

Chapter III

IDEALS AND CONGRUENCES IN THE RING OF RATIONAL INTEGERS

3.1. Principal ideals involving the rational integers. Now consider the subrings of the ring of rational integers $\mathfrak{R}$ and in particular the subrings which are ideals. Let $\mathfrak{J}$ be such an ideal and suppose that d is the smallest positive integer in it. Then by definition the ideal necessarily contains all integers in the form md where m ranges over all rational integers. Suppose the ideal contains an integer q different from any of these. By Theorem 2.2.8 we have $q=ad+r$, $0<r<d$. Now q belongs to our ideal and so does ad; consequently $q-ad=r$ belongs to the ideal, but $r<d$. This is a contradiction, however; consequently, *all the integers in the ideal have the form md, and the ideal is called principal.* Clearly if a and b are integers and h and k vary independently over the set of integers, the set $ah+bk$ is an ideal. Since this ideal is principal, if s is the smallest positive integer in it, it follows that there is a particular h_1 and k_1 such that

$$ah_1 + bk_1 = s. \tag{3.1.1}$$

3.2. Elements of arithmetic congruence theory. Due to the fact that in each of the ideals just considered each element may be put in the form ad, it is very convenient to use the concept of *congruence.* Thus we write

$$a \equiv b \pmod{m},$$

which reads "a is congruent to b modulo m," or $a=b$ plus some multiple of m. This is equivalent to the statement "a equals b plus an element in the ideal consisting of multiples of m." The sign $\not\equiv$ is read "is not congruent to," or "is incongruent to." All moduli will be assumed >1 unless we state otherwise.

Examples: $7\equiv 2 \pmod 5$; $-3\equiv 4 \pmod 7$; $15\equiv 0 \pmod 5$; $14\not\equiv 1 \pmod{11}$.

Unless it is otherwise stated, from now on all small Latin letters employed in connection with formulas will denote integers.

Using the definition of congruence we easily obtain a proof of

THEOREM 3.2.1. *If* $a\equiv b$ *and* $c\equiv d \pmod m$ *then* $a+c\equiv b+d \pmod m$ *and* $ac\equiv bd \pmod m$.

This theorem includes the result that if $a\equiv b \pmod m$ then $na\equiv nb \pmod m$.

But from the latter we cannot always conclude the former; thus

$$2 \times 4 \equiv 2 \times 1 \pmod{6}, \text{ but } 4 \not\equiv 1 \pmod{6}.$$

Here we extend the notion of "prime to" by defining n as prime to an integer k if and only if $|n|$ is prime to $|k|$, and similarly for the notion of "divisible by." We shall have occasion to employ frequently the result of

THEOREM 3.2.2. *If* $na \equiv nb \pmod{m}$ *and* n *is prime to* m, *then* $a \equiv b \pmod{m}$.

Proof. From the hypothesis, $n(a-b) = mr$, and since n is prime to m, we have $a-b$ divisible by m by using Corollary 2.2.3. Hence $a-b \equiv 0 \pmod{m}$ and $a \equiv b \pmod{m}$.

If $a \equiv r \pmod{m}$, then r is said to be a *residue* of a, modulo m. If $0 \leqq r < m$, then r is said to be the *least residue* of a modulo m.

Problem 3.2.3. If m is odd, show that any integer a is congruent to one of the integers $0, \pm 1, \pm 2, \cdots, \pm(m-1)/2$. This one is called the *least absolute residue* of a, modulo m.

Problem 3.2.4. Extend Theorem 3.2.2 to the case where n and m have the greatest common divisor d by restricting the modulus in the result.

We shall use Theorem 2.2.1 to obtain the following useful theorem.

THEOREM 3.2.5. *If* $a \equiv k \pmod{m_1}$ *and* $a \equiv k \pmod{m_2}$ *with* $(m_1, m_2) = 1$, *then* $a \equiv k \pmod{m_1 m_2}$.

Keeping Theorem 2.2.1 in mind, throughout the proof, we use the definition of congruence to obtain

$$a - k = m_1 h = m_2 j = p_1^{i_1} p_2^{i_2} \cdots p_c^{i_c}$$

with the p's relatively prime in pairs. Hence since $(m_1, m_2) = 1$, if we separate the prime divisors of m_1 from those of m_2 in the right of the above equation, we see that m_2 divides h by Corollary 2.2.3; so we may write $h = h_1 m_2$ and obtain $a - k = m_1 h_1 m_2$; whence

$$a \equiv k \pmod{m_1 m_2}.$$

3.3. An algorithm for a solution of a linear congruence. We shall now call attention to an algorithm* for the solution of the linear congruence 3.3.1 below. We shall be particularly interested in showing how to find, as briefly as possible, an x with $0<x<m$ such that with $b\not\equiv 0 \pmod{m}$,

$$ax \equiv b \pmod{m}, \tag{3.3.1}$$

a and m being positive integers such that $(a, m)=1$, with $0<a<m$.

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* Said algorithm differs considerably from the well-known method depending on the expansion of m/a as a continued fraction, since contrary to the other method, it definitely belongs to the theory of congruences, at least in the case of a prime power modulus, and the theory behind the method which follows is far simpler. *Cf.* Vandiver, this MONTHLY, vol. 31, 1924, pp. 137–140. In setting up the algorithm in a particular case we note that we may select the k's so as to make the absolute values of the remainders as small as possible. For example, for the case $81x\equiv 1 \pmod{257}$, from

$$257 = 81 \times 3 + 14$$

we see that 81 is close to 6×14; so we use

$$\begin{aligned} 6 \times 257 &= 81 \times 19 + 3,\\ 257 &= 3 \times 86 - 1.\end{aligned}$$

Whence

$$\begin{aligned} 81(19 \times 86) &\equiv -1 \pmod{257},\\ 81 \times 165 &\equiv 1 \pmod{257}.\end{aligned}$$

In our general argument we employed more than one modulus when the given modulus was not a power of a prime. However, by trial we can sometimes carry the work to a finish by using one modulus only. For example, consider

$$31x \equiv 1 \pmod{120}.$$

We have

$$\begin{aligned} 2\cdot 120 &= 7\cdot 31 + 23,\\ 2\cdot 120 &= 23\cdot 11 - 13,\\ 2\cdot 120 &= 13\cdot 19 - 7,\\ 120 &= 7\cdot 17 + 1.\end{aligned}$$

Hence

$$31(7\cdot 11\cdot 19\cdot 17) \equiv 1 \pmod{120}.$$

This example indicates that our main difficulties in connection with such a procedure are likely to appear when the modulus contains several small prime factors. As another problem of this type we investigate

$$1723x \equiv 1 \pmod{4028}.$$

Without setting down the actual figures, we see immediately that the remainder when 4028 is divided by 1723 is close to $\frac{1}{3}$ of 1723. Hence we set

$$\begin{aligned} 3\cdot 4028 &= 1723\cdot 7 + 23,\\ 4028 &= 23\cdot 175 + 3,\\ 4028 &= 3\cdot 1343 - 1;\end{aligned}$$

and we have

$$1723(7\cdot 175\cdot 1343) \equiv 1 \pmod{4028}.$$

Assume first that $m=p^n$, where p is prime and k_1 is an integer. We use Theorem 2.2.8 to obtain

$$k_1m = aq_1' + r_1',$$

where k_1 is selected* so as not to be a multiple of a; and r_1' is selected so that it is the least absolute residue of k_1m, modulo a. If, however, r_1' is divisible by p, put

$$k_1m = a(q_1' \pm 1) + r_1' \mp a$$

according as r_1' is negative or positive; hence $|r_1' \mp a| < a$. Now $r_1' \mp a$ is prime to p, for if we assume $r_1' \mp a \equiv 0 \pmod{p}$, then $a \equiv 0 \pmod{p}$, contrary to hypothesis. Hence we may write, in any case,

$$k_1m = aq_1 + r_1,$$

where $0 < |r_1| < a$, and r_1 is prime to p.

Similarly, after selecting an r_2 as a certain residue of k_2m modulo r_1, prime to p in the same manner that r_1 was selected as a certain residue of k_1m, modulo a, and prime to p, we have

$$k_2m = r_1q_2 + r_2,$$

where $0 < |r_2| < |r_1|$ and r_2 is prime to m, k_2 being an integer $\not\equiv 0 \pmod{r_1}$. Proceeding in this way we obtain ultimately

$$k_sm = r_{s-1}q_s + r_s,$$

where $|r_s| = 1$. Hence in general we have

$$k_im = r_{i-1}q_i + r_i,$$

$i=1, 2, \cdots, s$, and $r_0=a$, $|r_s|=1$. Also k_i is an integer, $\not\equiv 0 \pmod{r_{i-1}}$, and r_i is prime to m.

We may then write

$$r_{i-1}q_i \equiv - r_i \pmod{m}.$$

Setting $i=1, 2, \cdots, s$, and multiplying all the congruences together, we have by Theorem 3.2.1

$$a\prod_{i=1}^{s} q_i \prod_{c=1}^{s-1} r_c \equiv (-1)^s r_s \prod_{c=1}^{s-1} r_c \pmod{m}.$$

Using Theorem 3.2.2, since $\prod_{c=1}^{s-1} r_c$ is prime to m, we have

$$a\prod_{i=1}^{s} q_i \equiv (-1)^s r_s \pmod{m}.$$

* Vandiver, Proc. Nat. Acad. Sci., vol. 21, 1935, p. 163.

So, noting that $|r_s| = 1$, we have as a solution of (3.3.1)

$$x = (-1)^s b r_s \prod_{i=1}^{s} q_i. \tag{3.3.2}$$

For the case $m = p$, a prime, the algorithm takes a simple form since for $n = 1$ each r_i above is necessarily prime to p.

Consider (3.3.1) again with $m = m_1 m_2 \cdots m_t$; $m_i = p_i^{n_i}$; $i = 1, 2, \cdots, t$; $(m_i, m_j) = 1$, with i and j any pair of distinct integers in the set $1, 2, \cdots, t$. Further, write $h_i = m/p_i^{n_i}$, $i = 1, 2, \cdots, t$. Then by the above-described method we obtain $y_1, y_2, \cdots, y_t$, such that

$$(ah_i)y_i \equiv b \pmod{m_i},$$

$i = 1, 2, \cdots, t$. The quantity

$$x = h_1 y_1 + \cdots + h_t y_t \tag{3.3.3}$$

satisfies (3.3.1), as we see on using the result of Theorem 3.2.5. When we wish a solution from (3.3.2) or (3.3.4) with $0 < x < m$ in (3.3.1), we may reduce (3.3.2) or (3.3.3) modulo m.

Chapter IV

THE RING OF RESIDUE CLASSES MODULO m

4.1. Residue classes. We shall now define the notion of residue classes, modulo m. For a fixed integer $m>0$ we notice that the m sets defined by $a+km$ give the set of all integers where k ranges over all integers and a ranges independently over $0, 1, \cdots, m-1$. Corresponding, then, to each value of a, there is an infinite set of integers each congruent to a, modulo m, which set we shall denote by C_a, and we consider

$$C_0, C_1, \cdots, C_{m-1}, \tag{4.1.1}$$

which are mutually exclusive by Theorem 2.2.8. *With these definitions in mind, in the rest of this paper we shall pass freely from any equation involving the C_a's to a congruence, modulo m, in which C_a is replaced by a, and vice versa.* We postulate $C_a=C_b$ if C_a and C_b represent the same set. We also postulate $C_a+C_b=C_k$ where $0\leqq k<m$ whenever $a+b\equiv k \pmod m$. Similarly $C_aC_b=C_h$ where $0\leqq h<m$ whenever $ab\equiv h \pmod m$. Then these C's obviously form an Abelian group under addition, and an Abelian semigroup under multiplication. Also the distributive law and substitution hold. Consequently the C's form a ring, which is called *the ring of residue classes modulo m.*

4.2. An application of semigroups to a linear congruence. In applying the C's to obtaining arithmetical results we first consider the multiplicative semigroup formed by the elements of (4.1.1). Referring to the proof of Theorem 1.3.4, take the set (1.3.3) mentioned there and suppose it to be the set (4.1.1) above, and let the C of (1.3.5) be an element C_a of (4.1.1) with $(a, m)=1$. Then C_a is a cancellable element of (4.1.1), so that Theorem 1.3.4 gives the result

THEOREM 4.2.1. *The conditional congruence in x,*

$$ax \equiv b \pmod m; \qquad (a, m) = 1, \tag{4.2.2}$$

has a unique solution x, $0\leqq x<m$.

4.3. Absolutely distinct solutions of polynomial equations in commutative rings. We shall now apply a result in ring theory to congruence theory. Let $\mathfrak{R}$ be a commutative ring with a multiplicative identity, called a *unity* element. Two elements α and β in $\mathfrak{R}$ are said to be *absolutely distinct* if $\alpha-\beta$ is not zero and not a zero divisor. They are said to be *semi-equal* if $\alpha-\beta$ is a zero divisor, a *zero divisor* in $\mathfrak{R}$ being an element $\gamma\neq 0$, such that $\gamma\sigma=0$ with $\sigma\neq 0$. Also, if $h(x)$ is a polynomial with coefficients in $\mathfrak{R}$, a solution of $h(x)=0$ in $\mathfrak{R}$ is called a *zero* of $h(x)$ in $\mathfrak{R}$. We then have

THEOREM* 4.3.1. *The equation with* $\alpha_n \neq 0$ (*said to be of degree* n)

$$\alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0 = 0 \tag{4.3.2}$$

has no more than n absolutely distinct roots in $\mathfrak{R}$, *where* $\mathfrak{R}$ *is a commutative ring with a unity element, and the* α'*s are in* $\mathfrak{R}$ *with* $n > 0$.

For a proof, suppose the statement not true and let k be the degree of a polynomial $f(x)$ of least degree with coefficients in $\mathfrak{R}$ for which the statement is not true. Obviously $k>1$. Let $\theta_1, \theta_2, \cdots, \theta_{k+1}$ be absolutely distinct zeros of $f(x)$ in $\mathfrak{R}$. By an easy extension of ordinary algebraic division we have the identity

$$f(x) = (x - \theta_1)g(x) + \gamma,$$

where γ is in $\mathfrak{R}$, and when we set $x=\theta_1$, this gives $\gamma=0$. The above relation shows that the leading coefficient of $g(x)$ is α_n, and $g(x)=0$ for $x=\theta_i$, $i=2, 3, \cdots, k+1$, since the θ's are absolutely distinct. But the degree of $g(x)$ is less than that of $f(x)$.

Two integers a and b are said to be *absolutely incongruent* modulo m if $a-b$ is prime to m. They are said to be *semi-congruent* modulo m if $a-b$ has a factor f in common with m, such that $m>f>1$.

If we now take $\mathfrak{R}$ to be the ring of residue classes modulo m in Theorem 4.3.1, we immediately derive the

COROLLARY 4.3.3. *The congruence with* $a_n \not\equiv 0 \pmod{m}$ (*said to be of degree* n)

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \equiv 0 \pmod{m} \tag{4.3.4}$$

has no more than n absolutely incongruent solutions in x modulo m; $n>0$.

4.4. A theorem on the product of the distinct elements of a finite Abelian group. We shall now obtain a result (Theorem 4.4.9) concerning commutative groups which has considerable application to congruences.

Let $\mathfrak{G}$ be a commutative group of order $2n$. Let $A_1 \in \mathfrak{G}$. Then there exists an $A_1' \in \mathfrak{G}$ such that, if $M \in \mathfrak{G}$,

$$A_1 A_1' = M. \tag{4.4.1}$$

We will now consider an $A_2 \in \mathfrak{G}$ distinct from A_1 and A_1', if such an A_2 exists. As before, there exists an A_2', with $A_2' \in \mathfrak{G}$ such that

$$A_2 A_2' = M. \tag{4.4.2}$$

Clearly $A_1' \neq A_2'$, for assuming them equal yields $A_1 = A_2$. Similarly $A_2' \neq A_1$. In this way we continue until the elements of $\mathfrak{G}$ are exhausted and obtain a set of r equations of the type (4.4.2). We shall now separate these into two classes, one class if not null containing only the relations of the type (4.4.2) in which A_i is distinct from A_i' so that we may write, after changing subscripts, if necessary,

* Vandiver, Proc. Nat. Acad. Sci., vol. 21, 1935, p. 163.

(4.4.3) $$A_i A_i' = M,$$

$i=1, 2, \cdots, s$, and the other class, if not null, containing those of the A's such that

(4.4.4) $$A_j^2 = M,$$

There are $2n-2s$ of these elements. Let $t=2n-2s$.

Let E be the identity of $\mathfrak{G}$. Then if we set $M=E$ in (4.4.3) and (4.4.4), and notice that elements of order >2 satisfy (4.4.3), and elements of order 1 or 2 satisfy (4.4.4), we see that $\mathfrak{G}$ has an odd number of elements of order 2, since E is the only element of order 1. Let H be one of them. Then we obtain from (4.4.4) an element A_{2s+1} such that

(4.4.5) $$A_{2s+1}(A_{2s+1}H) = HM.$$

Now $A_{2s+1}H$ by definition of H is an element distinct from A_{2s+1}. Select an element A_{2s+2}, defined in (4.4.4), which is distinct from A_{2s+1} and $A_{2s+1}H$ if any such element exists. Then

(4.4.6) $$A_{2s+2}(A_{2s+2}H) = HM.$$

Using the reasoning used above to establish (4.4.2), we see that $A_{2s+2}H \neq A_{2s+1}$, $A_{2s+2}H \neq A_{2s+1}H$, $A_{2s+2}H \neq A_{2s+2}$. In this way we continue until the elements thus obtained from (4.4.4) are exhausted, and we may then write

(4.4.7) $$A_{2s+c}(A_{2s+c}H) = HM,$$

$c=1, 2, \cdots, k$, with each A_{2s+c} one of the A_j's in (4.4.4). In view of (4.4.3) and the way the A_j's were defined in (4.4.4), the two factors in the left-hand members of the relations (4.4.3) together with those in the left of (4.4.7) give each of the distinct elements of $\mathfrak{G}$ just once; that is, since $2k+2s=2n$, then $k=t/2$. Since $\mathfrak{G}$ is commutative we can then multiply together the s relations (4.4.3) and the relations (4.4.7), obtaining, if $B_1, B_2, \cdots, B_{2n}$ are the distinct elements of $\mathfrak{G}$,

(4.4.8) $$\prod_{h=1}^{2n} B_h = M^n H^{t/2}.$$

In the above argument we excepted the cases where either of the classes was null. If the class (4.4.3) is null, then (4.4.8) holds also, since here $s=0$ and $t=2n$. If the class (4.4.4) is null, then (4.4.8) holds for $t=0$, with H^0 defined as E. We then have

THEOREM 4.4.9. *Let $\mathfrak{G}$ be a finite Abelian group of order $2n$; then the relation* (4.4.8) *holds with $t \geqq 0$. Here $B_1, B_2, \cdots, B_{2n}$ are the distinct elements of $\mathfrak{G}$; H is some element of order 2 in $\mathfrak{G}$, $M \in \mathfrak{G}$, and t is the number of elements A_j which satisfy* (4.4.4).

Chapter V

THE ADDITIVE GROUP OF THE RING OF RESIDUE CLASSES MODULO m

5.1. On the generators of cyclic groups with application to the additive group modulo m. As we noted in remarks made below our relation (4.1.1), the ASG of the ring of residue classes modulo m is an Abelian group. Also, this group $\mathfrak{F}(m)$ is obviously cyclic with one of its generators being C_1.

We shall now consider the subgroups of the general cyclic group and shall prove the useful

THEOREM 5.1.1. *A subgroup of a cyclic group $\mathfrak{C}$ of order n and generator A is cyclic of order d where d divides n, and corresponding to any such d there is a unique subgroup of $\mathfrak{C}$ of order d. If $A^{n/d}$ generates a cyclic subgroup of order d, then $A^{kn/d}$ is a generator if and only if k is prime to d.*

Let $\mathfrak{G}$ be a cyclic group of order n and generator A with E its identity element. We shall show first that any subgroup $\mathfrak{G}_1 \subset \mathfrak{G}$ of order d has the form

$$A^c, A^{2c}, \cdots, A^n = E \tag{5.1.2}$$

where $n/d = c$ and where c is the smallest integer such that $A^c \in \mathfrak{G}_1$. If $A^k \in \mathfrak{G}_1$ then $k = qc + r$ with $0 \leqq r < c$. Hence by Theorem 1.3.2, $A^r \in \mathfrak{G}_1$. If $r > 0$ this contradicts the definition of c. Hence $r = 0$; so all the elements of $\mathfrak{G}_1$ are of the form A^{sc}. Also $A^n = E$ so $A^n \in \mathfrak{G}_1$ which gives $n \equiv 0 \pmod{c}$. Hence if d divides n there is a cyclic subgroup of $\mathfrak{G}$ of order d. There cannot be another subgroup $\mathfrak{G}_2$ of order d since A^c is unique in $\mathfrak{G}_1$.

We now seek all the generators of (5.1.2). Let $(h, d) = t$, $h = h_1 t$, and $d = d_1 t$. Then, since $(A^c)^d = E$, we have $(A^{hc})^{d_1} = (A^c)^{h_1 t d_1} = (A^c)^{d h_1} = E^{h_1} = E$. Clearly if A^{hc} is a generator of $\mathfrak{G}_1$, $d_1 = d$ and $(h, d) = 1$. On the other hand, if $(h, d) = 1$ and A^{hc} has order r, $A^{chr} = E$, and, by Theorem 1.3.1, d divides hr and therefore r, by Corollary 2.2.3. By definition of *generator*, $d = r$ and A^{hc} is a generator of $\mathfrak{G}_1$ if $(h, d) = 1$. This gives the second statement in the Theorem.

If $rs = m$; $r > 1$, $s > 1$; with $(r, s) = 1$, then $\mathfrak{F}(m)$ can be represented as the *sum* of the two additive cyclic groups defined by C_{rk}; $k = 0, 1, \cdots, s-1$, and C_{sl}; $l = 0, 1, \cdots, r-1$; *that is, $C_{rk} + C_{sl}$ with k and l ranging as above give distinct elements which coincide with the elements of* $\mathfrak{G}$. (This additive notation for groups is described in more detail in section 6.6.) To prove this it is sufficient to show that the integers $rk + sl$, with k and l ranging as above, are congruent in some order to the integers $0, 1, \cdots, m-1$. In the first place, any integer a may be represented in this form, for by Theorem 4.2.1 it follows that there is an x and y such that

$$a = rx + sy, \tag{5.1.3}$$

and by Theorem 2.2.8 we may write $x=x_1+sq_1$, $0\leqq x_1<s$; $y=y_1+rq_2$, $0\leqq y_1<r$; which gives by substitution in (5.1.3),

$$a \equiv rx_1 + sy_1 \pmod{m}, \tag{5.1.4}$$

with x_1 in the range $0, 1, \cdots, s-1$, and y_1 in the range $0, 1, \cdots, r-1$. No two such representations modulo m coincide. For if we substitute x_2 in place of x_1, and y_2 in place of y_1, in (5.1.4), with the same restrictions on x_2 and y_2 as we placed on x_1 and y_1, respectively, then upon subtracting the resulting relation from (5.1.4), we have

$$r(x_1 - x_2) + s(y_1 - y_2) \equiv 0 \pmod{m};$$

whence

$$r(x_1 - x_2) \equiv 0 \pmod{s};$$

or since $(r, s)=1$, $x_1=x_2$. Similarly $y_1=y_2$, which proves the last result stated in italics above.

We shall now define a very important arithmetical function, corresponding to each positive integer m, known usually as *Euler's ϕ function* or the *indicator* or *totient* of m. It is defined as $\phi(m)=1$ for $m=1$ and as the number of integers less than m and prime to it for $m>1$.

Consider again the relation (5.1.4) with the conditions mentioned below it. Assume that for $0\leqq a<m$ there exist $u=x_1$ and $v=y_1$ such that $(u, s)=1$, $(v, r)=1$, and

$$ru + sv \equiv a \pmod{m}. \tag{5.1.5}$$

Then $(a, s)=1$, since from (5.1.5) $ru\equiv a \pmod{s}$, with $(r, s)=1$, $(u, s)=1$. Similarly $(a, r)=1$. Hence $(a, m)=1$. Conversely (5.1.5) for $(a, m)=1$ gives $(u, s)=1$, and $(v, r)=1$. For it gives $ru\equiv a \pmod{s}$, and, since $(r, s)=1$, $(u, s)=1$; similarly for r in place of s we obtain $(v, r)=1$. Also, there cannot be another such representation of a, modulo m in (5.1.5) since x_1 and y_1 are unique in (5.1.4).

If we consider the additive group $\mathfrak{F}(m)$ of residue classes modulo m and apply the result in (5.1.5) to the cyclic subgroups $\mathfrak{F}(r)$ and $\mathfrak{F}(s)$ of orders r and s respectively, then (5.1.5) gives the result that any generator of $\mathfrak{F}(m)$ can be expressed uniquely as the sum of a generator of $\mathfrak{F}(r)$ and a generator of $\mathfrak{F}(s)$. However, it is clear that any cyclic group whatever can be represented by means of an additive cyclic group of residue classes modulo m. In (5.1.5) we assumed that $r>1$ and $s>1$, but we now state a result which is also obviously true for $r=1$ or $s=1$, namely:

THEOREM 5.1.6. *Any generator of an additive cyclic group $\mathfrak{F}(m)$ of order m may be expressed uniquely as the sum of some generator of $\mathfrak{F}(r)$ and some generator of $\mathfrak{F}(s)$ where $m=rs$, $(r, s)=1$, and conversely the sum of a generator of $\mathfrak{F}(r)$ and one of $\mathfrak{F}(s)$ gives one of $\mathfrak{F}(m)$, no two distinct sums giving the same generator of $\mathfrak{F}(m)$.*

(Generalizations of Theorem 5.1.6 have led to a number of results in the theory of finite rings.)

5.2. Simple properties of the totient. From the above result we obtain, by counting the number of distinct generators in each of the groups mentioned, the

COROLLARY 5.2.1. *If $m=rs$, $(r, s)=1$, then*

$$\phi(m) = \phi(r)\phi(s). \tag{5.2.2}$$

If $m=p_1^{d_1}p_2^{d_2}\cdots p_t^{d_t}$ with the p's distinct primes, then by using (5.2.2) successively,

$$\phi(m) = \prod_{i=1}^{t} \phi(p_i^{d_i}). \tag{5.2.3}$$

Also, as is easily seen,

$$\phi(p_i^{d_i}) = p_i^{d_i-1}(p_i - 1), \tag{5.2.4}$$

and (5.2.3) and (5.2.4) give

THEOREM* 5.2.5. *We have, if $m>1$,*

$$\phi(m) = m\prod_{i=1}^{t} (1 - (1/p_i)), \tag{5.2.6}$$

where $p_1, p_2, \cdots, p_t$ are the distinct prime divisors of m.

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* Euler gave a proof of the existence of an infinity of primes using Theorem 5.2.5: assume the number of primes is finite with p the greatest and set as the product of all the primes,

$$n = 2\cdot 3\cdot 5\cdots p.$$

Then we have

$$\phi(n) = (2-1)(3-1)\cdots(p-1),$$

which is a contradiction, since by definition of n, $\phi(n)=1$. So a prime p_1 must exist with $p<p_1<n$. This is slightly more general than the result obtained, from the usual proof, that $p<p_1<n+2$.

Let $\mathfrak{G}$ be a cyclic group of order n with a cyclic subgroup $\mathfrak{G}_1$ of order d where d divides n. $\mathfrak{G}_1$ exists and is unique by Theorem 5.1.1. Then by Theorem 5.1.2, $\mathfrak{G}_1$ has just $\phi(d)$ distinct generators. Also, any element of $\mathfrak{G}$ generates some cyclic group. Whence

THEOREM† 5.2.7. *We have*

$$\sum_d \phi(d) = n, \tag{5.2.8}$$

where the summation extends over all the distinct divisors d of n.

We may now note another property of the additive cyclic group modulo m. Let p be prime; then any integer m can be expressed uniquely in the form

$$m \equiv a_0 + a_1 p + \cdots + a_{n-1} p^{n-1} \pmod{p^n}, \tag{5.2.9}$$

where each a belongs in the set $0, 1, \cdots, p-1$. To prove this we may, by Theorem 2.2.8, express m uniquely in the form $m = a_0 + k_1 p$ where a_0 has the required property. We then express k_1 in a similar form, giving $m = a_0 + a_1 p + k_2 p^2$. Proceeding by induction we obtain (5.2.9).

† G. A. Miller, this MONTHLY, vol. 12, 1905, pp. 41–43.

Chapter VI

THE MULTIPLICATIVE GROUP $\mathfrak{G}(m)$ OF RESIDUE CLASSES C_a MODULO m; $(a, m)=1$

6.1. The theorems of Euler and Wilson. We shall now discuss some applications of the results in semigroups we obtained previously. Consider the multiplicative semigroup of residue classes modulo $m>1$. We first note that the set, denoted by $\mathfrak{G}(m)$,

$$(6.1.1)\qquad C_{r_1}, C_{r_2}, \cdots, C_{r_e}; \qquad (r_i, m) = 1$$

with $e=\phi(m)$, forms a group, under multiplication. This follows immediately from Theorem 1.3.4 and Theorem 3.2.2.

We shall now employ the group (6.1.1) and apply Theorem 4.4.9 to it. We find here that for $m>2$, $\phi(m)$ is of the type $2n$, by (5.2.4), so that for a given a with $(a, m)=1$, (4.4.8) becomes for our special group

$$(6.1.2)\qquad \prod_{i=1}^{e} C_{r_i} = C_{m-1}^{t/2} C_a^{d},$$

where $d=\phi(m)/2$ and where t is the number of solutions in C_x of the relation

$$(6.1.3)\qquad C_x^2 = C_a,$$

since there is an element of order two in $\mathfrak{G}(m)$, and it may be taken as C_{m-1}, which is not C_1 for $m>2$.

Let $a=1$; we obtain from (6.1.2)

$$(6.1.4)\qquad \prod_{i=1}^{e} C_{r_i} = C_{m-1}^{t/2}.$$

Whence for $m\neq 2$,

$$(6.1.5)\qquad \prod_{j=1}^{e} r_j \equiv (-1)^{t/2} \pmod{m},$$

where t is now the number of incongruent solutions of $x^2\equiv 1 \pmod{m}$.

Using (6.1.4), (6.1.5) in (6.1.2) gives $C_a^d=C_1$ or C_{m-1} and by squaring, $C_a^{\phi(m)} = C_1$. Whence

$$(6.1.6)\qquad a^{\phi(m)} \equiv 1 \pmod{m},$$

for $m>2$, and it is obvious for $m=2$. This is Euler's generalization of Fermat's theorem:

$$(6.1.7)\qquad a^{p-1} \equiv 1 \pmod{p},$$

which follows from (6.1.6) when $m=p$, a prime, since it is obvious that $\phi(p) =p-1$. Also, if $m=p$ in (6.1.5), we find Wilson's theorem:

$$(p-1)! \equiv -1 \pmod{p} \tag{6.1.8}$$

since there are not more than two solutions of $x^2 \equiv 1 \pmod{p}$ by the Corollary 4.3.3. Using the known number of solutions in x in $x^2 \equiv 1 \pmod{m}$ in (6.1.5), then we have a generalization of (6.1.8) due to Gauss.

6.2. Criterion for the solution of a quadratic congruence modulo p. Again from (6.1.2) and (6.1.8), with $m=p$, we have $C_a^{(p-1)/2} = C_1$ or C_{p-1} according as $C_x^2 = C_a$ has solutions or has no solutions in C_x; whence

THEOREM 6.2.1. *The congruence*

$$x^2 \equiv a \pmod{p},$$

with p an odd prime and $(a, p)=1$, has solutions or has no solutions in x according as

$$a^{(p-1)/2} \equiv \pm 1 \pmod{p}. \tag{6.2.2}$$

Putting $a=-1$ in (6.2.2) gives, since p is odd,

$$-1 \equiv (-1)^{(p-1)/2} \pmod{p}$$

when $p \equiv 3 \pmod{4}$, and

$$1 \equiv (-1)^{(p-1)/2} \pmod{p}$$

when $p \equiv 1 \pmod{4}$, so that

$$x^2 \equiv -1 \pmod{p} \tag{6.2.3}$$

has solutions if and only if $p \equiv 1 \pmod{4}$.

6.3. The Minkowski-Thue theorem on linear congruences. Suppose $m>1$, and $(a, m)=1$, and let k denote the least integer $>\sqrt{m}$. Consider the numbers of the form $(ay+x)$ where x and y each range independently over the set $0, 1, \cdots, k-1$. As $k^2>m$, then it follows that at least two numbers of this form are congruent modulo m, and we may set

$$ay_1 + x_1 \equiv ay_2 + x_2 \pmod{m},$$

with either $y_1 \neq y_2$ or $x_1 \neq x_2$. Whence

$$a(y_1 - y_2) \equiv x_2 - x_1 \pmod{m}, \tag{6.3.1}$$

with both $x_1 \neq x_2$, $y_1 \neq y_2$, since if either $y_1 = y_2$ or $x_1 = x_2$, with $(a, m)=1$, our assumption, "either $y_1 \neq y_2$ or $x_1 \neq x_2$," is violated. Hence, setting $u = |y_1 - y_2|$, $v = |x_1 - x_2|$, we have

$$au \equiv \pm v \pmod{m}, \tag{6.3.2}$$

with $0 < u \leqq k-1$, $0 < v \leqq k-1$. And we have

THEOREM 6.3.3. (*Minkowski-Thue*).* *Let $m>1$, and let k denote the least integer $>\sqrt{m}$. Then for any given integer a with $(a, m)=1$ we have nonzero integers u and v such that each does not exceed $k-1$, and* (6.3.2) *holds.*

6.4. The expression of $p=4n+1$ as the sum of two squares. Take a as satisfying (6.2.3) in (6.3.2) with $m=p=4n+1$, and square both members of (6.3.2); this gives

$$v^2 + u^2 \equiv 0 \pmod{p} \tag{6.4.1}$$

with $|u|$ and $|v|$ each $<\sqrt{p}$, whence $0<u^2+v^2<2p$, and we have from (6.4.1)

THEOREM 6.4.2 (*Fermat*). *If p is any prime of the form $4n+1$ then*

$$p = a^2 + b^2. \tag{6.4.3}$$

Problem 6.4.4. Prove that the representation (6.4.3) is unique aside from order and sign.

6.5. Repetitive sets with application to Euler's theorem. Consider the distinct elements

$$A_1, A_2, \cdots, A_t \tag{6.5.1}$$

of a finite subset of a commutative semigroup $\mathfrak{S}$ and suppose there exists an element C in $\mathfrak{S}$ such that

$$CA_1, CA_2, \cdots, CA_t \tag{6.5.2}$$

are equivalent in some order to the elements of (6.5.1); then (6.5.1) is called a *repetitive set in $\mathfrak{S}$ with multiplier C*. It is obvious that the set of multipliers of a repetitive set forms a semigroup. It is also clear, by Theorem 1.3.4, that if $\mathfrak{S}$ is finite then the set of all its elements forms a repetitive set with respect to any cancellable element of $\mathfrak{S}$ as a multiplier.

If $\mathfrak{R}$ is a repetitive set and C a multiplier, then C need not be in $\mathfrak{R}$; neither is $\mathfrak{R}$ necessarily a semigroup, as is seen from the example of the set of classes C_2 and C_3 modulo 5, with multiplier C_4. Now C_4 is not in the set; the product C_2C_3 is likewise not in the set, so that the repetitive set in this case does not form a semigroup.

It is possible to generalize this notion. Suppose that the elements of the set (6.5.2) are equal in some order to the distinct elements

$$N_1A_1, N_2A_2, \cdots, N_tA_t \tag{6.5.3}$$

* This theorem has been attributed to Thue, alone, by many writers, but it is a special case of a classical theorem of Minkowski's, which appeared in the year 1896, and this seems to have been some years before Thue published anything on it. *Cf.* Vandiver, Bull. Amer. Math. Soc., vol. 22, 1915, pp. 61–68, where some references are given and where more general congruences of this type were also obtained. *Cf.* also A. Brauer and R. L. Reynolds, Can. Jour. of Math., vol. 3, 1951, pp. 367–374.

where the N's belong to a subsemigroup $\mathfrak{S}'$ of $\mathfrak{S}$. We say then that (6.5.1) is a *repetitive set in* $\mathfrak{S}$ *with respect to* $\mathfrak{S}'$ *and with multiplier* C.

Repetitive sets of the type first mentioned are called *ordinary*. If we multiply together the quantities (6.5.3), then we obtain, using (6.5.2)

$$C^t \prod_{i=1}^{t} A_i = \prod_{i=1}^{t} N_i A_i.$$

Now further assume $\mathfrak{S}$ is a cancellative semigroup; then we obtain

$$C^t = \prod_{i=1}^{t} N_i. \tag{6.5.4}$$

Let $\mathfrak{S}$ be a group; then the elements of $\mathfrak{S}$ form an ordinary repetitive set with any element as multiplier. If $A \in \mathfrak{S}$ and $\mathfrak{S}'$ above is taken to be the identity E, (6.5.4) becomes

$$A^n = E \tag{6.5.5}$$

with n the order of the group.

We note that (6.1.1) forms a repetitive set of order $\phi(m)$ with any element of (6.1.1) as multiplier. Consequently by (6.5.5) if $(a, m) = 1$,

$$C_a^{\phi(m)} = C_1; \tag{6.5.6}$$

whence we obtain another proof of Euler's theorem, (6.1.6).

For our present uses we shall in the next few sections develop a few theorems concerning commutative rings and finite groups, particularly cyclic groups.

6.6. A basis theorem for finite Abelian groups.

THEOREM 6.6.1. *Let* $a_1a_2 \cdots a_r = m$ *be the order of an Abelian group* $\mathfrak{G}$ *with the* a*'s prime each to each. Then any element* C *of* $\mathfrak{G}$ *may be expressed uniquely as*

$$C = A_1 A_2 \cdots A_r \tag{6.6.2}$$

with

$$A_i^{a_i} = E; \qquad i = 1, 2, \cdots, r, \tag{6.6.3}$$

and with E *the identity of* $\mathfrak{G}$.

We shall first illustrate the method of proof by assuming $r=2$ in (6.6.2). We use addition as the group operation as this is convenient here. This means, for example, that if we have, in the multiplicative notation, an expression in $\mathfrak{G}$ like S^aT^b, we write it in the additive form as $aS+bT$. Also, in particular, if we have, in the multiplicative form $R^k=E$, then this assumes the additive form $kR=0$, and if S has order r and $c \equiv d \pmod{r}$, then $cS=dS$. We define A_1 as

$$A_1 = m_1^{k_1} C \tag{6.6.4}$$

and A_2 as

$$A_2 = m_2^{k_2} C \tag{6.6.5}$$

with

$$(m/a_i) = m_i$$

and

$$k_i = \phi(a_i),$$

for $i=1, 2$. If we add (6.6.4) and (6.6.5) we obtain

$$A_1 + A_2 = C(m_1^{k_1} + m_2^{k_2}). \tag{6.6.6}$$

The coefficient of C in the above is congruent to 1, modulo a_1, and is congruent to 1, modulo a_2, by (6.5.6), and by Theorem 3.2.5 it is congruent to 1 modulo a_1a_2, since $(a_1, a_2)=1$. Hence we have (6.6.2) for $r=2$,

$$C = A_1 + A_2.$$

For r general in (6.6.2) we write

$$A_i = Cm_i^{k_i},$$

and proceeding as we did for the case $r=2$, we find

$$C = C\sum_{i=1}^{r} m_i^{k_i} = \sum_{i=1}^{r} A_i. \tag{6.6.7}$$

Now the coefficient of C in the left-hand member is $\equiv 1 \pmod{m}$ since $m_i^{k_i} \equiv 1 \pmod{a_i}$ for $i=1, 2, \cdots, r$, and we may use again the result in Theorem 3.2.5 and the relation (6.5.6). Hence we have (6.6.2). To prove the uniqueness of this representation let us assume another representation of this kind exists so that

$$B_1 + B_2 + \cdots + B_r = A_1 + A_2 + \cdots + A_r;$$

then for each i in the set $1, 2, \cdots, r$, we have

$$m_i^{k_i} \sum_{j=1}^{r} B_j = m_i^{k_i} \sum_{j=1}^{r} A_j,$$

which gives by (6.6.3), written in additive notation,

$$B_i = A_i$$

for any i; whence follows Theorem 6.6.1.

We use the notation employed in the proof of Theorem 6.6.1 relative to a modulus $m=\prod_{i=1}^{r} a_i$ and wish to find an integer s such that for given integers s_i, $i=1, 2, \cdots, r$, $s\equiv s_i \pmod{a_i}$. We write

$$\sum_{i=1}^{r} m_i^{k_i} s_i, \tag{6.6.8}$$

and we see that this is an s desired. This explicit solution* of the so-called *Chinese problem of reminders* we shall find a powerful tool in much of our remaining work in this paper.

6.7. A criterion for cyclic groups. We shall now obtain further properties of groups. Here we consider any finite group $\mathfrak{F}$ of order n having the property that for every d dividing n then $X^d=E$ has at most d solutions X in $\mathfrak{F}$. Using the same scheme employed in the proof of Theorem 5.2.7, we shall show that $\mathfrak{F}$ is cyclic. If there is a cyclic subgroup $\mathfrak{C}\subset\mathfrak{F}$ of order d, then d divides n, and it has exactly $\phi(d)$ generators by Theorem 1.3.1 and Theorem 5.1.1. Also, there cannot be another cyclic group of order d in $\mathfrak{F}$, otherwise $X^d=E$ would have more than d solutions in $\mathfrak{F}$; *i.e.*, if A is a generator, then $A, A^2, \cdots, A^d$ give distinct solutions. But every element of $\mathfrak{F}$ generates some cyclic subgroup of $\mathfrak{F}$, and there must be a cyclic subgroup of $\mathfrak{F}$ corresponding to each divisor d of n, since if none existed for some d_1 with $n\equiv 0 \pmod{d_1}$, then the order of $\mathfrak{F}$ would be less than n, as a consequence of the relation $\sum_d \phi(d)=n$. Therefore for $d=n$, some element generates $\mathfrak{F}$ itself, which makes $\mathfrak{F}$ cyclic. Conversely, by Theorem 5.1.1, if $\mathfrak{F}$ is cyclic, then $X^d=E$ has no more than the d solutions:

$$B^{n/d},\ B^{2n/d},\ \cdots,\ B^{dn/d},$$

where B generates $\mathfrak{F}$. Consequently we may state the

THEOREM† 6.7.1. *A finite group $\mathfrak{F}$ of order n and identity E is cyclic if and only if for every d dividing n the relation*

$$X^d = E$$

has at most d solutions X in $\mathfrak{F}$.

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* Vandiver, Annals of Math., II, vol. 18, 1917. pp. 115–119, Theorem I. Also, in other publications he used this form in connection with a composite ideal modulus in an algebraic field of which the m related to (6.6.8) is a special case.

† H. Zassenhaus, The Theory of Groups, Chelsea Publishing Co., New York, New York, 1949, p. 74.

6.8. The number of solutions of $ax \equiv b \pmod{m}$, $(a, m) = d$, with application to cyclic groups. We shall now prove

THEOREM 6.8.1. *The congruence*

$$ax \equiv d \pmod{n} \tag{6.8.2}$$

has $(a, n) = r$ incongruent solutions or no solutions according as r does or does not divide d. If it has solutions, then for $a = a_1 r$, $n = kr$, $d = d_1 r$ and t a solution of $a_1 t \equiv d_1 \pmod{k}$, the r solutions are given by

$$t,\ t + k,\ \cdots,\ t + (r - 1)k.$$

Proof. If (6.8.2) is written in the form $ra_1x - d \equiv 0 \pmod{kr}$, we see that it has a solution if r divides d, and not otherwise. Now assume $d = d_1 r$ and write (6.8.2) as $r(a_1x - d_1) \equiv 0 \pmod{rk}$. This has a solution since $a_1 x \equiv d_1 \pmod{k}$ has a solution by Theorem 4.2.1 and Corollary 2.2.3. Write this unique solution, t modulo k in the r forms, $x = t + sk$, $s = 0, 1, \cdots, r-1$, each less than n. Clearly these give exactly the incongruent solutions of (6.8.2), since each x which satisfies (6.8.2) also satisfies $a_1x - d_1 \equiv 0 \pmod{k}$, and each $t + sk$ satisfies $r(a_1x - d_1) \equiv 0 \pmod{rk}$ and therefore (6.8.2).

We shall now apply Theorem 6.8.1 to the theory of cyclic groups. Let us consider the solution of $X^a = B$ in a cyclic group $\mathfrak{G}$. Then we may put, if R generates $\mathfrak{G}$, $B = R^d$. We may set also $X = R^y$ with y to be determined. Hence our equation becomes

$$R^{ay} = R^d, \qquad R^{ay-d} = E, \tag{6.8.3}$$

and from Theorem 1.3.1, $ay \equiv d \pmod{n}$ where n is the order of $\mathfrak{G}$. Our problem, then, is to find the values of y which are less than or equal to n, and these are given by Theorem 6.8.1. Hence we have

THEOREM 6.8.4. *In a cyclic group $\mathfrak{G}$ of order n the equation $X^a = R^d$ has r solutions or no solutions X according as r divides d or does not divide d; $r = (a, n)$; R is a generator of $\mathfrak{G}$.*

Note that the group of Theorem 6.8.4 is *abstractly identical* with that of the additive group of residue classes defined by the residues modulo n in Theorem 6.8.1, in the sense that if we identify multiplication in $\mathfrak{G}$ with addition modulo n and the element R^a with C_a, where $0 \leqq a < n$, then these groups have identical operations tables, with respect to multiplication and addition, respectively. Such a relation between two semigroups, and hence groups, is called an *isomorphism.* If each element of the semiring $\mathfrak{S}$ is related to exactly one element of the semiring $\mathfrak{S}'$, and conversely, in such a way that whenever $\alpha, \beta \in \mathfrak{S}$ are related, respectively, to $\alpha', \beta' \in \mathfrak{S}'$, it follows that $\alpha\beta$ and $\alpha + \beta$ are related, respectively, to $\alpha'\beta'$ and $\alpha' + \beta'$; then this relationship is also called an *isomorphism*, and the semirings are said to be *isomorphic.*

6.9. Primitive roots modulo m. Consider the residue classes, modulo p, a prime, excluding the zero class, represented by

$$C_1, C_2, \cdots, C_{p-1}. \tag{6.9.1}$$

These form a group under multiplication of order $p-1$, as we remarked after (6.1.1). For n any divisor of $(p-1)$ the equation $C_x^n = C_1$ has no more than n distinct solutions C_x since the congruence $x^n \equiv 1 \pmod p$ has no more than n incongruent solutions, by the Corollary 4.3.3. If we use Theorem 6.7.1 it follows that (6.9.1) forms a cyclic group.

The integer a is said to *belong to the exponent d*, modulo m, if

$$a^d \equiv 1 \pmod m;$$

but

$$a^k \not\equiv 1 \pmod m$$

for $0<k<d$; in particular if $d=\phi(m)$, then a is said to be a *primitive root* modulo m. We have just proved that at least one primitive root modulo m exists when m is prime.

We note the following result which follows immediately from Theorem 6.8.4.

Let g be a primitive root modulo the prime p; then

$$x^a \equiv g^d \pmod p$$

has $r=(a, p-1)$ incongruent solutions or no solution according as r divides d or does not divide d.

This is sometimes called the fundamental theorem of binomial congruences, modulo p.

The question as to primitive roots modulo m, when m is composite, is more complicated. We have by (6.1.6) for $m=p^n$, p prime, a prime to p, for p odd,

$$a^{p^{n-1}(p-1)} \equiv 1 \pmod{p^n}; \tag{6.9.2}$$

so we consider the residue classes, with $e=\phi(p^n)$,

$$C_{r_1}, C_{r_2}, \cdots, C_{r_e}, \tag{6.9.3}$$

where the r's are the integers less than p^n and prime to it. They form a group as we have noted earlier, which we shall now show is cyclic for p odd. It is easy to show that the solutions in a commutative group $\mathfrak{K}$ of order n of the equation $X^r=E$, E the identity of $\mathfrak{K}$, form a group if r divides n. Hence if s divides $p-1$ and $0\leqq k<n$, each of the sets of solutions of

$$C_x^{sp^k} = C_1, \tag{6.9.4}$$

$$C_y^s = C_1, \tag{6.9.5}$$

$$C_z^{p^k} = C_1, \tag{6.9.6}$$

forms a group. Now C_y is a solution of (6.9.5) if and only if

$$y^s \equiv 1 \pmod{p^n}. \tag{6.9.7}$$

By the Corollary 4.3.3 this has no more than s absolutely incongruent solutions in y. Let $a_1, a_2, \cdots, a_h$ be incongruent solutions with $h>s$. Then for some i and j,

$$a_i = a_j + p^t r, \qquad (r, p) = 1 \tag{6.9.8}$$

with $n>t>0$, and by expansion

$$a_i^s \equiv a_j^s + s a_j^{s-1} p^t r \pmod{p^{t+1}}, \tag{6.9.9}$$

and since $a_i^s \equiv a_j^s \equiv 1 \pmod{p^n}$, we find $s a_j^{s-1} p^t r \equiv 0 \pmod{p^{t+1}}$, or $r \equiv 0 \pmod{p}$, a contradiction. Hence (6.9.5) has no more than s solutions in C_y.

Now consider the solutions in C_z of (6.9.6). C_z is a solution of (6.9.6) if and only if, with $0 \leqq k < n$,

$$z^{p^k} \equiv 1 \pmod{p^n}. \tag{6.9.10}$$

Each solution z of (6.9.10) has the form $z=1+tp^v$, $v \geqq 1$, $(t, p)=1$, or $t=0$, since by Fermat's theorem, $z^{p^k} \equiv z \pmod{p}$. The relation, since p is odd,

$$(1 + tp^v)^{p^r} \equiv 1 + tp^{r+v} \pmod{p^{r+v+1}} \tag{6.9.11}$$

holds for $r=1$, as we see when we expand and use Theorem 2.2.7. If (6.9.11) holds for $r \geqq 0$, we see that it will hold for $(r+1)$ in lieu of r by raising each member to the power p. Hence it holds for any $r \geqq 0$. If we set $r=k$ and $v=n-k+c$ in (6.9.11), with c to be determined, we obtain $(1+tp^{n-k+c})^{p^k} \equiv 1+tp^{n+c} \pmod{p^{n+c+1}}$.

Hence for $(t, p)=1$, z satisfies (6.9.10) if and only if $c \geqq 0$. By Theorem 2.2.8, $t=t_1+p^k g$ with $p^k>t_1 \geqq 0$. We note that $z=1+tp^{n-k+c} \equiv 1+(t_1+p^k g)p^{n-k+c} \equiv 1+t_1 p^{n-k+c} \pmod{p^n}$. Hence for all $c \geqq 0$ this cannot have more than the p^k solutions $0, 1, \cdots, p^k-1$, in t_1, and therefore (6.9.10) has no more than p^k solutions in z.

Now the solutions of (6.9.4) form a group whose order divides sp^k and by Theorem 6.6.1 each such solution is the product of an element which satisfies (6.9.5) by one which satisfies (6.9.6). We have just proved that (6.9.5) has no more than s solutions and (6.9.6) no more than p^k solutions; hence (6.9.4) has no more than sp^k solutions, and by Theorem 6.7.1, the group (6.9.3) is cyclic. Otherwise expressed, for p an odd prime, a primitive root modulo p^n exists.

Obviously primitive roots exist modulo 2 and 4, respectively, but not modulo 2^k with $k>2$, since $a^2 \equiv 1 \pmod{8}$ for any odd a.

Finally, let $m=2p^n$, where p is an odd prime. Employ a primitive root r modulo p^n. Let g be that one of the numbers r and $r+p^n$ which is odd. If g belongs to the exponent e modulo m, then

$$r^e \equiv g^e \equiv 1 \pmod{p^n},$$

whence e is divisible by $\phi(p^n)=\phi(m)$. But $e \leqq \phi(m)$. Hence $e=\phi(m)$ and g is a primitive root modulo m.

We have already noted that no primitive roots exist for 2^t, $t>2$; however, as before, we designate the group (6.9.3) by $\mathfrak{G}(m)$, and we may complete the proof* of

THEOREM 6.9.12. *The groups* $\mathfrak{G}(2)$, $\mathfrak{G}(4)$, $\mathfrak{G}(p^n)$, $\mathfrak{G}(2p^n)$, p *odd, are cyclic. If* $t \geqq 3$, *then the group* $\mathfrak{G}(2^t)$ *has the property that any given element in it may be expressed in the form* AB *with* A *a unique element in a cyclic group of order* 2, *and* B *a unique element in a cyclic group of order* 2^{t-2}, *where the generators of these cyclic groups are, respectively, the residue classes* C_a, *with* $a \equiv -1 \pmod{2^t}$, *and* C_b, *with* $b \equiv 5 \pmod{2^t}$.

Proof. $\mathfrak{G}(2)$ and $\mathfrak{G}(4)$ are obviously cyclic. $\mathfrak{G}(p^n)$ and $\mathfrak{G}(2p^n)$ were proved cyclic above.

For the proof of the theorem concerning $\mathfrak{G}(2^t)$, $t>2$, we shall first show that

$$5^{2^k} \not\equiv 1 \pmod{2^t} \text{ for } t \geqq 3 \text{ and } k < t-2, \tag{6.9.13}$$

but

$$5^{2^{t-2}} = 1 + 2^t u, \text{ where } u \text{ is odd and } t \geqq 3. \tag{6.9.14}$$

Since $25=1+8\cdot 3$, then the last equation is satisfied by $t=3$. By induction it is true in general, as we observe that

$$5^{2^{t-1}} = (1+2^t u)^2 = 1 + 2^{t+1}u + 2^{2t}u^2 = 1 + 2^{t+1}u(1+2^{t-1}u).$$

Hence we have (6.9.14). Now if 5 belongs to the exponent n modulo 2^t, then n divides 2^{t-1} by Theorem 1.3.1 and (6.5.6). It then follows from (6.9.14) that 5 belongs to the exponent 2^{t-2} modulo 2^t, and we have (6.9.13). We return to the first equation above, and we shall show that if $2^{t-2}=c$, and $t \geqq 3$,

---

* A simpler proof, using group theory, of the fact that primitive roots modulo p^n and $2p^n$ exist when p is odd, may be given by the use of the criterion that an Abelian group of order h is cyclic if and only if for every prime q dividing h, the relation

$$x^q = E,$$

where E is the identity of the group, has exactly q solutions. However, as of now, we know of no proof of this which does not involve the use of the basis theorem for Abelian groups, which we are avoiding in the present work.

$$
\begin{array}{c}
5,\ 5^2,\ \cdots,\ 5^c, \\
-5,\ -5^2,\ \cdots,\ -5^c
\end{array}
\tag{6.9.15}
$$

give all the 2^{t-1} odd incongruent residues modulo 2^t. Since 5 belongs to 2^{t-2} modulo 2^t, the elements of the first line alone are incongruent modulo 2^t and similarly for the second line. If we have $5^a \equiv -5^b \pmod{2^t}$, a and b each in the range $0, 1, \cdots, c$, then if $a > b$,

$$5^{a-b} \equiv -1 \pmod{2^t}, \qquad 5^{2(a-b)} \equiv 1 \pmod{2^t}.$$

Again, since 5 belongs to the exponent 2^{t-2}, modulo 2^t, then $2(a-b) \equiv 0 \pmod{2^{t-2}}$, $a-b \equiv 0 \pmod{2^{t-3}}$; whence $a-b = 2^{t-3}$, and $5^{2^{t-3}}+1 \equiv 0 \pmod{2^t}$. However, this is impossible since $5^{2^{t-3}}-1 \equiv 0 \pmod 4$; hence $5^{2^{t-3}}+1 \not\equiv 0$ (modulo 2^t). This proves Theorem 6.9.12, as the generators of the cyclic groups of orders 2 and 2^{t-2} are, respectively, the residue classes determined by (-1) and 5, modulo 2^t.

6.10. A theorem on the cyclic subgroup of $\mathfrak{G}(m)$ of maximal order, with application to Carmichael numbers. Since the above discussion shows that we do not have primitive roots corresponding to all moduli, we may consider the problem of determining the order of a *maximal cyclic group* (a cyclic group of maximum order) contained in $\mathfrak{G}(m)$. This is an extension of the problem of finding primitive roots, as when a primitive root g, $0 \leqq g < m$ exists for a modulus m, then C_g generates the cyclic group of residue classes $\mathfrak{G}(m)$ corresponding to integers prime to m, and this group is maximal. R. D. Carmichael* defined a numerical function $\lambda(m)$ having the following properties:

$$
\begin{aligned}
\lambda(2^t) &= \phi(2^t) \text{ if } t = 0, 1, 2; \\
\lambda(2^t) &= \tfrac{1}{2}\phi(2^t) \text{ if } t > 2; \\
\lambda(p^t) &= \phi(p^t) \text{ if } p \text{ is an odd prime}; \\
\lambda(2^t p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n}) &= h, \text{ with each } p \text{ an odd prime},
\end{aligned}
$$

where h is the least common multiple of $\lambda(2^t), \lambda(p_1^{t_1}), \cdots, \lambda(p_1^{t_n})$.

We shall now apply some relations used in the proof of Theorem 6.6.1 to the obtaining of results, in effect, in the theory of the additive cyclic group of the residue classes modulo m, as well as in the theory of the multiplicative group $\mathfrak{G}(m)$, where, if the p's are distinct primes,

$$m = p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n}.$$

If we set $a_i = p_i^{t_i}$ in the statement of Theorem 6.6.1 with $r = n$ and let $\mathfrak{G}$ be the additive group of the residue classes C_a, $a = 0, 1, \cdots, m-1$, then (6.6.7), with the remark following it, gives, using congruences,

* Bull. Amer. Math. Soc., vol. 15, 1908–9, pp. 221–222.

$$a \equiv a \sum_{i=1}^{n} (m/p_i^{t_i})^{\phi(p_i^{t_i})} \pmod{m}, \tag{6.10.1}$$

for a given a. Setting

$$a \equiv r_i \pmod{p_i^{t_i}}, \qquad i = 1, 2, \cdots, n, \tag{6.10.2}$$

we obtain

$$a \equiv \sum_{i=1}^{n} r_i(m/p_i^{t_i})^{\phi(p_i^{t_i})} \pmod{m}. \tag{6.10.3}$$

The converse of this result was given in (6.6.8).

Let a belong to the exponent k modulo m, and take (6.10.2) to the kth power and assume that r_i belongs to the exponent e_i modulo $p_i^{t_i}$, $i=1, 2, \cdots, n$. Then we have

$$r_i^k \equiv 1 \pmod{p_i^{t_i}}; \qquad i = 1, 2, \cdots, n. \tag{6.10.4}$$

Hence k is a multiple of e_i, $i=1, 2, \cdots, n$. Suppose in particular that we select an a in (6.10.1) such that r_i belongs to $\lambda(p_i^{t_i})$ for $i=1, 2, \cdots, n$, which is always possible by Theorem 6.9.12. Then k is the L. C. M. of these λ's, which is $\lambda(m)$. Then a belongs to $\lambda(m)$ modulo m, *so that C_a generates a cyclic subgroup of order $\lambda(m)$ in the group of residue classes $\mathfrak{G}(m)$ of* (6.1.1). Further, in the above, the e_i, $i=1, 2, \cdots, n$ are divisors of $\lambda(p_i^{t_i})$, respectively, by Theorem 1.3.1. Consequently we have

$$b^{\lambda(m)} \equiv 1 \pmod{m}, \tag{6.10.5}$$

for any b with $(b, m)=1$. We may then state

THEOREM 6.10.6. *If $\mathfrak{G}(m)$ is the group of residue classes given in* (6.1.1), *then a cyclic subgroup of maximal order has $\lambda(m)$ as its order.*

We shall next apply the above result to the following well-known problem. For what integers n is $a^{n-1}\equiv 1 \pmod{n}$ for all a's such that $(a, n)=1$? Suppose n is such an integer, and let some a, say a_1, belong to the exponent $\lambda(n)$, which is always possible, as we have seen. Then it follows that $n\equiv 1 \pmod{\lambda(n)}$ by Theorem 1.3.1. On the other hand, if $n\equiv 1 \pmod{\lambda(n)}$, then by (6.10.5), if $(a, n)=1$, it follows that $a^{\lambda(n)}\equiv 1 \equiv a^{n-1} \pmod{n}$. So we have the criterion: $a^{n-1}\equiv 1 \pmod{n}$ *for all a's such that* $(a, n)=1$, *if and only if* $n\equiv 1 \pmod{\lambda(n)}$. Below the number 2,000 there are only three values for composite n, namely, $561=3\cdot 11\cdot 17$; $1105=5\cdot 13\cdot 17$; $1729=7\cdot 13\cdot 19$. Such numbers we shall call *Carmichael numbers*, following the example of several writers.*

* R. D. Carmichael, Bull. Amer. Math. Soc., vol. 16, 1909–10, pp. 237–238; this MONTHLY, vol. 19, 1912, pp. 22–27.

6.11. Integral homogeneous symmetric functions defined over repetitive sets in commutative rings with units. Here we shall discuss a type of function in a commutative finite ring which has a multiplicative identity denoted by 1. We define the *indicator* of $\mathfrak{R}$ as the number of distinct units in its multiplicative semigroup. If $F(\gamma_0, \gamma_1, \cdots, \gamma_j)$ is a function with the γ's in a ring, then F is called an *integral homogeneous symmetric function* of degree d, written IHSF provided it satisfies:

(i) F is a polynomial in the γ's, which is unchanged aside from the order of the terms by any permutation of the γ's.

(ii) $F(\sigma\gamma_0, \sigma\gamma_1, \cdots, \sigma\gamma_j) = \sigma^d F(\gamma_0, \gamma_1, \cdots, \gamma_j)$, $\sigma \in R$.

Consider an IHSF of degree d formed by an ordinary repetitive set,† say

$$F(\beta_1, \beta_2, \cdots, \beta_k).$$

If β is a multiplier of the repetitive set, then

$$F(\beta\beta_1, \beta\beta_2, \cdots, \beta\beta_k) = F = \beta^d F,$$

and

$$(\beta^d - 1)F = 0. \tag{6.11.1}$$

It is easy to show that no unit μ satisfies $\mu F = 0$ for $F \in \mathfrak{R}$, $F \neq 0$. Now if it is possible to select β and d so that $\beta^d - 1$ is a unit in $\mathfrak{R}$, then $F = 0$. This can be done in a number of special cases.

For example, we may consider the application of the above ideas to the residue classes modulo m. We can select many subsets of this system which form repetitive sets and apply the relation (6.11.1) to obtain various results. However, we may obtain theorems giving us more information concerning exceptional cases when we consider the whole ring modulo m.

Under multiplication the set of distinct residue classes modulo m forms a repetitive set with any C_a, $(a, m) = 1$, as multiplier.

Let

$$m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where the p's are distinct primes, and let S be an IHSF of the distinct residue classes modulo m. Then by (ii) above, we have if d is the degree of S and $(b, m) = 1$, $SC_b^d = S$. Hence

$$S(C_b^d - C_1) = 0. \tag{6.11.2}$$

Now assume that d is not a multiple of $\phi(p_i)$, where $\phi(p_i)$ represents the indicator of p_i, $i = 1, 2, \cdots, k$, and let C_{b_i} be such that b_i is a primitive root modulo p_i. Then

† The idea of an ordinary repetitive set in any finite ring was defined in Vandiver, Annals of Math., II, vol. 18, 1916–17, pp. 105–114, and the definition may obviously be extended to finite sets in any ring.

$$d = r\phi(p_i) + d_i,$$

where $0 < d_i < \phi(p_i)$, and we have

$$b_i^{d_i} \not\equiv 1 \pmod{p_i},$$

for $i = 1, 2, \cdots, k$. Then by the Chinese problem of remainders select a c so that

$$c \equiv b_i \pmod{p_i^{a_i}}. \tag{6.11.3}$$

Hence $c^d - 1$ is prime to m if d is not a multiple of $(p_i - 1)$, $i = 1, 2, \cdots, k$, and by (6.11.2), we have the

THEOREM* 6.11.4. *An IHSF of all the distinct residue classes modulo m is equal to the zero class, except possibly when the degree of the function is divisible by one of the integers* $(p_i - 1)$, *where* $i = 1, 2, \cdots, k$, *and* $p_1, p_2, \cdots, p_k$ *are the distinct prime factors of m.*

The reader may easily verify that the above theorem also holds if we restrict the residue classes in the IHSF mentioned to be the set $\mathfrak{G}(m)$.

We also obtain from (6.11.1) the following theorem, which we give without proof:

THEOREM 6.11.5. *Any IHSF of the elements of an ordinary repetitive set in a finite commutative ring* $\mathfrak{R}$ *with unity element, whose degree is prime to the indicator of* $\mathfrak{R}$, *is zero in* $\mathfrak{R}$, *provided there exists a multiplier* η *of the set such that* $\eta(\eta - 1)$ *is a unit in* $\mathfrak{R}$.

6.12. Direct product of semigroups with application to $\mathfrak{G}(m)$. We shall now describe the concept of direct products of semigroups. Suppose that a *countable* (that is, such that the non-equivalent elements can be put into one-to-one correspondence with the positive integers) semigroup $\mathfrak{S}$ contains a set of subsemigroups

$$\mathfrak{S}_1, \mathfrak{S}_2, \cdots \tag{6.12.1}$$

such that

(i) $\mathfrak{S}_i$ and $\mathfrak{S}_j$ have at most the identity E, if such exists in $\mathfrak{S}$, in common for $i \neq j$.
(ii) If $S_i \in \mathfrak{S}_i$, $S_j \in \mathfrak{S}_j$, and $i \neq j$, then $S_iS_j = S_jS_i$.
(iii) If $S \in \mathfrak{S}$ then S is expressible in the form, for some m,

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* *Loc. cit.*, p. 36.

$$S = \prod_{i=1}^{m} S_{k_i}, \quad S_{k_i} \in \mathfrak{S}_{k_i}, \tag{6.12.2}$$

with each $\mathfrak{S}_{k_i}$ some element in the set (6.12.1) and where such pair of different S_{k_i}'s are contained in different $\mathfrak{S}_{k_i}$'s.

(iv) Whenever an element of $\mathfrak{S}$ has factorizations as in (iii),

$$\prod_{i=1}^{a} S_{c_i} = \prod_{i=1}^{b} S_{d_i}, \tag{6.12.3}$$

where no element in either factorization is the identity for $\mathfrak{S}$, if such exists, then $a=b$, and the S_{c_i}'s are the S_{d_i}'s in some order.

Then $\mathfrak{S}$ is said to be a *direct product** of the semigroups (6.12.1). If the semigroups in (6.12.1) are all cyclic, then a set consisting of a generator for each of the semigroups in (6.12.1) is called a basis† for $\mathfrak{S}$.

Many types of patterns in number theory and algebra are related to these concepts. For example, let us consider again Theorem 2.2.1 and its proof, and we shall show in detail how the decomposition into primes fits into our present scheme and the MSG of positive integers is, in effect, an example of a representation of a semigroup which is a direct product of other semigroups. Our subsets (6.12.1) become in this case the infinite set consisting of the semigroup formed by unity and the cyclic semigroups formed by powers of p where p varies over the set of primes. As each natural number >1 can be expressed as a product of primes, then the remarks concerning (6.12.2) hold. The conditions, in connection with our relation (6.12.3), also hold since factorization is unique; hence in the language we are using at present, Theorem 2.2.1 may be stated in the form, *The semigroup consisting of the natural numbers under multiplication may be expressed as the direct product of subsemigroups consisting of unity, and the cyclic semigroups generated by the primes p where p varies over the set of primes.*

As another example, let us consider $\mathfrak{G}$ of Theorem 6.6.1. This result is equivalent to the theorem that, using the notation in the statement of the theorem, $\mathfrak{G}$ is the direct product of semigroups, or, in this case, of groups of orders $a_1, a_2, \cdots, a_r$ having the property stated in (6.6.3). Here, contrary to what we observed in connection with Theorem 2.2.1, the identity element is an element belonging to each one of the subgroups mentioned. Here, too, the number of subgroups referred to is finite whereas we employed an infinity of subsemigroups in Theorem 2.2.1. Again in Theorem 6.6.1, as is known, the subgroups are not necessarily cyclic, as in Theorem 2.2.1.

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* The usual definition of *direct product* of groups is a special case of our definition. We also notice that if an identity is adjoined to each of the $\mathfrak{S}_i$'s in our definition which fails to have one, the order of $\mathfrak{S}$ for the finite case is the product of the orders of the $\mathfrak{S}_i$'s.

† Vandiver, Proc. Nat. Acad. Sci., vol. 20, 1934, pp. 583–584, used in effect, a similar definition of basis.

Even in the case where we consider the semigroup formed by the natural numbers under multiplication we can express it as a direct product of subsemigroups in different ways. Instead of taking the identity group as one of the factors in the product, we can consider, say, the whole semigroup as the product of the cyclic subsemigroups defined by each prime except that in the case of the prime 2, we consider the semigroup consisting of all positive powers of 2 together with unity.

As another application of these concepts, we may consider the semigroup formed by the non-zero rational integers, under multiplication. It is not difficult to see that this system forms a cancellative semigroup, which may be expressed as the direct product of the group generated by (-1) and the cyclic semigroups formed by the positive powers of the primes. Here the basis consists of (-1) and the primes.

If we have given a semigroup with operation addition and with a basis, we have a generalization of the idea of linear independence when we consider the coefficients in the forms as belonging to the semiring of natural numbers. By extending the ideas of group operators to semigroups we may extend the domain of the coefficients to other semirings.

We shall now show that $\mathfrak{G}(m)$, the group of residue classes C_a with $(a, m)=1$, can be expressed as the direct product of certain cyclic groups, a special representation which is important in the theory of group characters. In textbooks on number theory and modern algebra this theorem is not usually given except in connection with a discussion of characters. It is given here in direct and complete form without any reference to that theory. To establish the result we shall use congruences instead of the residue classes directly and consider the following number as a modulus:

$$m = 2^{a_0} p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}, \tag{6.12.4}$$

where, in the above, each p is an odd prime. Suppose first that $a_0>2$. If $i=1, 2, \cdots, k$, with g_i a primitive root, modulo $p_i^{a_i}$, then, by (6.6.8) there exist integers r_i and s_i such that

$$r_i = g_i + s_i p_i^{a_i}, \qquad r_i = 1 + t_i(m/p_i^{a_i}). \tag{6.12.5}$$

Similarly we have by the proof of Theorem 6.9.12,

$$r_0 = 5 + u\cdot 4\cdot 2^{a_0-2}, \qquad r_0 = 1 + t_0(m/2^{a_0}), \tag{6.12.6}$$

$$r = -1 + v\cdot 4\cdot 2^{a_0-2}, \qquad r = 1 + t(m/2^{a_0}). \tag{6.12.7}$$

We shall now show that if we take

$$r^c r_0^{c_0} \cdots r_k^{c_k} \tag{6.12.8}$$

and let c, c_0, and c_i, $i=1, 2, \cdots, k$, range independently, respectively, over 0 and 1; 0, 1, $\cdots$, $2^{a_0-2}-1$; and 0, 1, $\cdots$, $\phi(p_i^{a_i})-1$, then the $\phi(m)$ elements (6.12.8) are congruent modulo m to the $\phi(m)$ integers less than and prime to m. To prove this, suppose that for some d and some e within the ranges for the c's

$$r^d r_0^{d_0} \cdots r_k^{d_k} \equiv r^e r_0^{e_0} \cdots r_k^{e_k} \pmod{m}. \tag{6.12.9}$$

If we reduce (6.12.9) modulo 2^{a_0}, we obtain by (6.12.6) and (6.12.7),

$$r^d r_0^{d_0} \equiv r^e r_0^{e_0} \pmod{2^{a_0}}. \tag{6.12.10}$$

Now we know by Theorem 6.9.12 that the residue classes C_h modulo 2^{a_0}, h odd, $a_0>2$ form under multiplication a group which is the direct product of two cyclic groups, one of order 2, generated by C_j, $j=2^{a_0}-1$, and the other of order 2^{a_0-2}, generated by C_5. Using (6.12.6) and (6.12.7) we have, from (6.12.10), because of the ranges of the d's and e's,

$$d = e, \qquad d_0 = e_0. \tag{6.12.11}$$

Now reduce (6.12.9) modulo $p_n^{a_n}$ where n is some integer in the set 1, 2, $\cdots$, k. We find from (6.12.5)

$$r^{d_n} \equiv r^{e_n} \pmod{p_n^{a_n}}. \tag{6.12.12}$$

Because of the fact that g_n is a primitive root modulo $p_n^{a_n}$ in (6.12.5) and because of the ranges for the d's and e's, we find $d_n=e_n$, $n=1, 2, \cdots, k$. Hence the two expressions in (6.12.9) are identical, and by having the d's range as stated we obtain the result that the group of residue classes $\mathfrak{G}(m)$ is the direct product of $k+2$ cyclic groups of residue classes, the C_r's with the r's given in (6.12.5), (6.12.6), and (6.12.7) being the generators, since also

$$\phi(m) = \phi(2^{a_0}) \prod_{i=1}^{k} \phi(p_i^{a_i}).$$

Let now $a_0=0$. We find by the use of (6.12.5) and the method used above that our $\mathfrak{G}(m)$ is the direct product of just k cyclic groups. If $a_0=1$, then we note that $\phi(m)=\phi(m/2)$ and that we obtain the generators of k cyclic groups by taking r_i odd in each case in (6.12.5) as is always possible.

Now suppose that $a_0=2$ in (6.12.4). We now select an s, $0 \leqq s < m-1$,

$$s \equiv -1 \pmod{4}, \qquad s \equiv +1 \pmod{m/4}. \tag{6.12.13}$$

Then we see, using our previous methods, that $\mathfrak{G}(m)$ is in the present case the direct product of $k+1$ cyclic groups having the k generators C_{r_i} obtained from (6.12.5) and the single generator C_s obtained from (6.12.13). We may then state the

THEOREM* 6.12.14. *Let m be defined as in* (6.12.4). *The group of residue classes $\mathfrak{G}(m)$ may be expressed as the direct product of cyclic groups as follows: If $a_0=0$, it is the direct product of k cyclic groups, with generators C_{r_i}, and r_i is defined in* (6.12.5). *If $a_0=1$, $\mathfrak{G}(m)$ is again the direct product of k cyclic groups with generators obtained from* (6.12.5), *provided the r_i's are selected as odd. If $a_0=2$, then $\mathfrak{G}(m)$ is the direct product of $k+1$ cyclic groups with k generators obtained from* (6.12.5) *and a single generator obtained from* (6.12.13). *If $a_0>2$ in* (6.12.4), *then our group can be expressed as the direct product of $k+2$ cyclic groups, and the generators may be obtained from* (6.12.5), (6.12.6), *and* (6.12.7).

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* Theorem 6.12.14 does not in general give the direct product of groups whose orders are all powers of primes, as may be obtained by using the fundamental basis theorem for Abelian groups (not employed in this paper). However, Theorem 6.12.14 may be used as the first step in determining these latter types of subgroups if such determination is desired. *Cf.* Hasse (*loc. cit.*, p. 2), pp. 40–43.

Chapter VII

QUADRATIC RECIPROCITY

7.1. Simple properties of the Legendre symbol. Consider (6.2.2). In view of this criterion we shall find it convenient to introduce the notation which immediately follows.

If p is an odd prime and $(a, p)=1$, then, in view of Theorem 6.2.1 we define the symbol

$$\left(\frac{a}{p}\right)$$

to be ± 1, and also such that

$$a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) (\text{mod } p).$$

The symbol $(\frac{a}{p})$ just defined, which we shall also write as (a/p), is called Legendre's symbol, or the quadratic character of a, modulo p. (Some writers use $(a|p)$ for this symbol, rather than $(\frac{a}{p})$, but we prefer the latter, since its properties given in Problem 7.1.2 (below) resemble to some extent properties of ordinary fractions.) We have easily, from this definition, and (6.2.2),

$$\left(\frac{1}{p}\right) = 1, \qquad \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}. \tag{7.1.1}$$

Problem 7.1.2. Prove that, if $(a_1a_2, p)=1$, p an odd prime, then

$$\left(\frac{a_1a_2}{p}\right) = \left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right), \tag{7.1.3}$$

$$\left(\frac{a_1}{p}\right) = \left(\frac{a_1 + kp}{p}\right). \tag{7.1.4}$$

To investigate $(\frac{2}{p})$ we consider*

$$2 \times 1,\ 2 \times 2,\ 2 \times 3,\ \cdots,\ 2 \times p' \ (p' = (p-1)/2),$$

modulo p. Hence if $a=1, 2, \cdots, p'$, the least absolute residue r_a of $2a$ is even if $2a<p/2$ and odd if $2a>p/2$, so that

$$2a \equiv (-1)^{|r_a|} \, |r_a| \pmod p \tag{7.1.5}$$

where $|r_a| < p/2$. Also, the resulting r's are distinct since the even ones are incongruent and also the odd ones, and $2p'<p$. Multiplying together the p' congruences (7.1.5), we have

$$2^{p'}(p'!) \equiv (-1)^R(p'!) \pmod p$$

* This simple proof of (7.1.6) was communicated to Vandiver by Dr. W. L. G. Williams.

where $R = \sum_{r=1}^{p'} |r_a|$, and since $(p'!)$ is prime to p we have

$$2^{p'} \equiv (-1)^{(p^2-1)/8} \pmod{p},$$

since $R = 1+2+ \cdots +p' = (p^2-1)/8$; and

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}. \tag{7.1.6}$$

As is easily shown, this may be otherwise expressed by saying that the quadratic character of 2 is $(+1)$ if p is of the form $8n+1$ or $8n+7$, and is (-1) if p is of the form $8n+3$ or $8n+5$.

7.2. Gauss' lemma. We shall now use (6.5.4) to prove the following theorem:

THEOREM 7.2.1 (*Gauss' lemma*). *If the negative residues among the least absolute residues of*

$$a,\ 2a,\ \cdots,\ p'a,$$

modulo p, are μ in number then

$$\left(\frac{a}{p}\right) = (-1)^{\mu}. \tag{7.2.2}$$

Let the semigroup $\mathfrak{S}'$ mentioned relative to (6.5.4) consist of the residue classes C_1 and C_{p-1} and

$$C_1,\ C_2,\ \cdots,\ C_{p-1}$$

be the semigroup $\mathfrak{S}$ mentioned there. Then we shall show that

$$C_1,\ C_2,\ \cdots,\ C_{p'},$$

is a repetitive set with respect to the subgroup consisting of C_1 and C_{p-1}, with any C_a, $(a, p)=1$, as multiplier. If we consider

$$C_aC_1,\ C_aC_2,\ \cdots,\ C_aC_{p'},$$

then we write

$$C_aC_s = C_{t_1}$$

with $t_1 > p'$. We note that $C_{t_1} = C_{p-1}C_{p-t_1}$.

On the other hand, consider the cases when

$$C_aC_s = C_{t_2}$$

with $t_2 < p'$. Then it follows that the $(p-t_1)$'s and the t_2's constitute the integers $1, 2, \cdots, p'$ in some order. For if $p-t_1' \equiv t_2' \pmod{p}$, then $t_1' + t_2' \equiv 0 \pmod{p}$, which is impossible, since each of the t's is $<p/2$. Then Theorem 7.2.1 immediately follows from (6.5.4).

7.3. The law of quadratic reciprocity. We shall now prove (proof due to Zeller-Frobenius*)

THEOREM 7.3.1. (*The Law of Quadratic Reciprocity.*) *If p and q are distinct odd primes, then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)/2\cdot(q-1)/2}. \tag{7.3.2}$$

By Theorem 7.2.1 we have

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\mu+\nu}, \tag{7.3.3}$$

where

$$\left(\frac{q}{p}\right) = (-1)^{\nu},$$

with ν corresponding to μ in Theorem 7.2.1 when p and q are interchanged.

Consider the number of solutions (x, y) of

$$py - qx = s, \tag{7.3.4}$$

where $-q/2 < s < p/2$ and $1 \leqq x \leqq p'$, $1 \leqq y \leqq q'$, and $p' = (p-1)/2$ and $q' = (q-1)/2$. Then we shall show that the number of solutions (x, y) satisfying (7.3.4) is equal to $\mu+\nu$. In (7.3.4)

let λ denote the number of solutions (x, y) such that $s > p/2$;

let μ_1 denote the number of solutions (x, y) such that

$$0 < s < p/2; \tag{7.3.5}$$

let ν_1 denote the number of solutions (x, y) such that

$$0 > s > -q/2; \tag{7.3.6}$$

let σ denote the number of solutions (x, y) such that $s < -q/2$.

Clearly, the totality of solutions (x, y) is $(p-1)/2 \cdot (q-1)/2$ so we have

$$\frac{p-1}{2}\cdot\frac{q-1}{2} = \lambda + \mu_1 + \nu_1 + \sigma.$$

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* This proof, in several ways the simplest known of the theorem, and depending only on Gauss' Lemma, has appeared in a number of texts. However, we have observed, so far, just one text in English that gives it: W. J. LeVeque, Topics in Number Theory, Addison-Wesley Publishing Company, Inc., Reading, Mass., vol. I, 1956, pp. 69–71. Geometric forms of this proof have been given by Hasse (*loc. cit.*, p. 2), pp. 56–57 and pp. 94–95, D. H. Lehmer, this MONTHLY, vol. 64, 1957, pp. 103–106.

But if (x, y) is a solution of (7.3.4) for $-q/2 < s_1 < p/2$, and $x' = [(p+1)/2] - x$ and $y' = [(q+1)/2] - y$, then we note that (x', y') is also a solution of (7.3.4) for $s = s_2 = [(p-q)/2] - s_1$, and s_2 is in the prescribed range for s_1. Also, we note that if (x, y) is a solution such that $s_1 > p/2$; then (x', y') is a solution such that $s_2 < -q/2$. Hence $\lambda = \sigma$, and

$$\frac{p-1}{2} \cdot \frac{q-1}{2} = \mu_1 + \nu_1 + 2\lambda;$$

so that

$$\frac{p-1}{2} \cdot \frac{q-1}{2} \equiv \mu_1 + \nu_1 \pmod{2}. \tag{7.3.7}$$

Consider μ_1; then since $0 < s < p/2$, if we take x in the range $1, 2, \cdots, p'$, then (7.3.4) gives $py < (q \cdot p/2) + p/2$ or $y < q' + 1$. Hence we have $qx \equiv -s \pmod{p}$ with $s > 0$ so that by the definition of μ we have $\mu_1 = \mu$. Similarly if we consider ν_1 and take y in the range $1, 2, \cdots, q'$, the same method shows that $x < p' + 1$, and we have $py \equiv s \pmod{q}$ with s negative and $(-s)$ in the range $1, 2, \cdots, q'$. This gives $\nu_1 = \nu$. Hence, by (7.3.3) and (7.3.7), we have a proof of Theorem 7.3.1.

Problems

Problem 7.3.8. Prove that if p has the form $1+8n$, or $3+8n$, then there exist integers x and y such that

$$p = x^2 + 2y^2.$$

Also, if p has any other form, this equation has no solutions.

Problem 7.3.9. Prove the following generalization of Theorem 6.3.3. Let $m > 1$ and

$$ef > m, \qquad e > 1, \qquad m \geqq f;$$

then for an a with $(a, m) = 1$ there exist an x and y such that

$$ay \equiv \pm x \pmod{m}$$

with

$$0 < x < e, \qquad 0 < y < f.$$

Problem 7.3.10. Determine the value of $(81/257)$; $(37/103)$; $(965/1093)$. Decide if the following congruences have solutions: $x^2 \equiv 15 \pmod{101}$; $7x^2 \equiv 3 \pmod{41}$, $x^2 \equiv 673 \pmod{1093}$.

Problem 7.3.11. Give another proof* of Theorem 5.2.7 by considering the n expressions

$$\frac{n}{a}, \tag{7.3.12}$$

$a=1, 2, \cdots, n$, and reducing each to its lowest terms. Show that corresponding to each distinct divisor d of n there are $\phi(d)$ such fractions with numerator d.

If p is prime and a solution of $x^2 \equiv a \pmod{p}$ exists, then a is called a quadratic residue of p, otherwise a is called a quadratic nonresidue of p.

Problem 7.3.13. If p is an odd prime and r is a primitive root modulo p, then show that the least nonnegative residues mod p of $r^2, r^4, \cdots, r^{p-1}$ are the quadratic residues of p.

Problem 7.3.14. Prove that if m^2+1 is divisible by an odd prime p, then p is of the form $4n+1$.

Problem 7.3.15. Prove that -3 is a quadratic residue of $p=3n+1$ and a quadratic non-residue of $p=3n-1$.

Problem 7.3.16. If $p=10n+b$ is a prime, for what positive integers b is it true that $(5/p)=1$?

Problem 7.3.17. If $p=44n+b$ is a prime, for what positive integers b is it true that $(11/p)=1$?

Problem 7.3.18. Suppose m is a positive integer greater than 1, a belongs to s modulo m, b belongs to t modulo m, and $(s, t)=1$. Show that ab belongs to st modulo m.

Problem 7.3.19. Suppose m is a positive integer greater than 1 and a belongs to e modulo m. Show that a^s belongs to $e/(e, s)$ modulo m. Use this result to find the primitive roots modulo 19, given that 3 is a primitive root modulo 19.

Problem 7.3.20. Suppose that x is a primitive root modulo the odd prime p such that $(x, p-1)=1$. Does there exist a positive integer y less than p such that

$$y^x \equiv x \pmod{p}? \qquad \text{(R. P. Kelisky)}$$

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* Note by Vandiver: One day in the year 1928 when I visited Prof. R. Fueter in Zürich he showed me some notes made by a former student of Kummer (whose name I have forgotten) of the latter's lectures on elementary number theory. The proof of our Theorem 5.2.7 indicated in Problem 7.3.11 was included in the notes.

Problem 7.3.21. Suppose that p is an odd prime and $r=(p-1)/2$ is a primitive root modulo p. Show that there exists a positive integer x less than p such that

$$r^x \equiv x \pmod{p}. \quad \text{(R. P. Kelisky—R. C. Osborne)}$$

Problem 7.3.22. The congruence

$$x^n \equiv a \pmod{p}$$

has g solutions or no solutions according as $a^{(p-1)/g} \equiv 1 \pmod{p}$ holds or not; $g=(n, p-1)$, $(a, p)=1$. (Hint: use Theorem 6.8.4.)

Problem 7.3.23. Show that a primitive root r modulo p^n, where p is an odd prime, is also a primitive root modulo p, and conversely, provided p^2 does not divide $(r^{p-1}-1)$.

Problem 7.3.24. Show that if g is a primitive root modulo the odd prime p and if $g^{p-1}-1$ is divisible by p^2, then g is not a primitive root modulo p^n for $n \geqq 2$.

Problem 7.3.25. Show that if g is a primitive root modulo the prime $p=4n+1$, then $(-g)$ is also a primitive root modulo p.

Problem 7.3.26. Show that if g is a primitive root modulo the prime $p=4n+3$, then $(-g)$ belongs to the exponent $(p-1)/2$ modulo p.

Problem 7.3.27. Show that if a belongs to the exponent $(p-1)/2$ for the prime modulus $p=4n+3$, then a is a primitive root modulo p.

Problem 7.3.28. Show that if $p=8n+3$ and $q=(p-1)/2=4n+1$ are both primes, then 2 is a primitive root modulo p.

Chapter VIII

THE SEMIGROUP OF THE NONUNITS OF THE MULTIPLICATIVE SEMIGROUP OF RESIDUE CLASSES MODULO m

8.1. On the groups contained in finite cyclic semigroups. Let A be an element of a semigroup, and suppose that the semigroup of distinct powers of A is finite:

$$A,\ A^2,\ \cdots,\ A^k,\ \cdots,\ A^{s-1}, \qquad A^s = A^k, \qquad s > k > 0. \tag{8.1.1}$$

Now it is possible to show that the elements

$$A^k,\ A^{k+1},\ \cdots,\ A^{s-1} \tag{8.1.2}$$

form a cyclic group* $\mathfrak{G}$. For a proof we note first that if $s-k$, the order of (8.1.2), is 1, $s-k$ has no prime divisors, and $\mathfrak{G}$ is obviously cyclic. Suppose $s-k>1$. We have for $t \geqq 0$

$$A^{s-k}A^{k+t} = A^{s+t} = A^{k+t}.$$

Assume that

$$A^{r(s-k)}A^{k+t} = A^{k+t}. \tag{8.1.3}$$

It then follows that

$$\begin{aligned} A^{(r+1)(s-k)}A^{k+t} &= A^{s-k}A^{r(s-k)}A^{k+t} \\ &= A^{s-k}A^{k+t} \\ &= A^{k+t}. \end{aligned} \tag{8.1.4}$$

Then by induction (8.1.4) holds for all r's. Clearly there exists an n such that $A^{n(s-k)} \in \mathfrak{G}$; so from (8.1.4) $A^{n(s-k)}$ is an identity for $\mathfrak{G}$. We also have

$$\begin{aligned} (A^{n(s-k)+1})^j &= A^{nj(s-k)}A^j = EA^j \\ &= A^{n(s-k)+j}. \end{aligned}$$

And the distinct elements of $\mathfrak{G}$ are $A^{n(s-k)+i}$, $i=1, 2, \cdots, (s-k)$. And for t in the above range, the inverse of $A^{n(s-k)+t}$ is $A^{n(s-k)+s-k-t}$. Hence $\mathfrak{G}$ is a cyclic group.

8.2. The cyclic semigroups generated by nonunit residue classes. The nonunit residue classes modulo m were investigated by Weaver and E. T. Parker.† Some of their properties will be given below.

We first notice that the congruence $ax \equiv 1 \pmod{m}$ has no solutions if and only if a is divisible by a factor of m; hence the nonunits modulo m are the

* F. C. Biesele, An Introduction to the Theory of Semi-Groups, M. A. thesis, The University of Texas, June 1933, p. 9.

† Weaver, Math. Mag., vol. 25, 1952, pp. 125–136; E. T. Parker, Proc. Amer. Math. Soc., vol. 5, 1954, pp. 612–616.

elements whose least positive residues have one or more prime divisors in common with m. Further, $(r_1r_2, m) = 1$ if and only if $(r_1, m) = 1$ and $(r_2, m) = 1$. Hence the nonunits modulo m form a semigroup under multiplication.

We wish to discuss the order and period of a nonunit class C_t modulo m. We first consider an example. Let $m = 24$ and $C_t = C_{10}$. Then (8.1.1) becomes C_{10}, $(C_{10})^2 = C_4$, $(C_{10})^3 = C_{16}$, with $(C_{10})^4 = (C_{10})^3$. C_{10} has period 1 and order 3. We observe that if we write the decomposition of 24 into primes: $24 = 2^3 \cdot 3$, the period n of C_{10} is the exponent to which 10 belongs modulo $(24/2^3) = 3$, and if j is the least power of 10 which 2^3 divides, then the order of C_{10} is $j+n-1$. These properties of C_{10} modulo m are general and are stated below as Theorem 8.2.3.

If l is an integer and b is the least positive integer such that $l^b \equiv 0 \pmod{d}$, then b is called the nullifying exponent of l modulo d. In the above, $b = j$.

We return to the question of the order and period of the nonunit C_t. Let $m = m_1k$, with $(m_1, k) = 1$ where each prime divisor of m which divides one of the pair t, k divides the other also. Let b be the nullifying exponent of t modulo k and t belong to n modulo m_1. Then allowing 1 as a possible modulus in the congruences below, we have

$$(8.2.1) \qquad t^b \equiv 0 \pmod{k}, \qquad t^n \equiv 1 \pmod{m_1};$$

whence

$$(8.2.2) \qquad t^b(t^n - 1) \equiv 0 \pmod{m_1k}, \qquad t^{b+n} \equiv t^b \pmod{m}.$$

On the other hand if t has order b_1+n_1-1 and period n_1, we have $t^{b_1}(t^{n_1}-1)$ divisible by km_1 and by Corollary 2.2.3,

$$t^{b_1} \equiv 0 \pmod{k}, \qquad t^{n_1} \equiv 1 \pmod{m_1}.$$

Hence, by definition of b and n, $b_1 \geqq b$, $n_1 \geqq n$, but by definition of order and period, $b_1+n_1 \leqq b+n$. Therefore $b_1 = b$ and $n_1 = n$. We have proved

THEOREM* 8.2.3. *Let C_t be a nonunit modulo m, with $m = m_1k$ and $(m_1, k) = 1$ where each prime divisor of m which divides one of the pair t, k divides the other also; then if b is the nullifying exponent of t modulo k and t belongs to n modulo m_1, C_t has period n and order $b+n-1$.*

The reader will find it interesting to illustrate the theorem with $m = 36$ and $t = 2, 10, 9, 18$.

8.3. Unique factorization modulo m. Earlier, we defined direct product in such a way that we had a unique representation (6.12.2) of each element of $\mathfrak{S}$. We wish to extend this notion of unique representation. To do this, we first define the idea of associates, and extend the definition of prime.

---

* This proof is a simplification of the proofs of Theorems 3 and 4 in Weaver (*loc. cit.*, p. 48), pp. 130–131, both of which results are included in our present Theorem 8.2.3.

Let $\mathfrak{S}$ be a commutative semigroup with $\mathfrak{U}$ as its non-null subgroup of units. If $S \in \mathfrak{S}$, the elements $S \cdot U$, where U varies over $\mathfrak{U}$ are called *associates* of one another. S is obviously an associate of itself. If P is a nonunit of $\mathfrak{S}$ such that P has only its own associates for nonunit divisors, then P is called a *prime* of $\mathfrak{S}$.

$\mathfrak{S}$ is called *uniquely factorable, aside from unit elements*, if

(i) $\mathfrak{S}$ contains a subset $\mathfrak{P}$ of primes.
(ii) Each nonunit of $\mathfrak{S}$ is expressible as a product of a unit and powers of distinct primes.
(iii) If N is a nonunit of $\mathfrak{S}$ and $N = U\prod_{i=1}^{r} P_i^{c_i} = V\prod_{j=1}^{s} Q_j^{d_j}$ are factorizations of N with U and V units and the P_i's and Q_j's, distinct primes, respectively, then $r = s$ and each $P_i^{c_i}$ is an associate of some $Q_j^{d_j}$.

We emphasize from our definition that this factorization is unique only up to associates of the powers of the primes.

If p is a natural prime divisor of the modulus m, and $pu \equiv s_1 \pmod{m}$ for C_u a unit, then $(s_1, m) = p$; hence the residue class containing s_1 is divisible only by associates of that containing p, and by unit classes, and the class containing s_1 is a prime as defined above. However, if C_q is a prime, then since C_q has only its own associates for nonunit divisors, it follows that $(q, m) = p'$ where p' is a natural prime divisor of m, and C_q is an associate of a class containing a natural prime divisor of m. Hence the set $\mathfrak{P}$ of primes P modulo m is the set consisting of the associates of each class containing a natural prime divisor of m.

Since the least positive residues modulo m represent a complete set of residue classes, it follows from arithmetic unique factorization that each nonunit element of the ring of residue classes modulo m can be factored into a unit multiplied by powers of distinct primes.

Next we prove the

LEMMA 8.3.1. *If p^b, for $b > 0$, is the largest power of the natural prime p which divides m, then no two distinct elements of the set $C_p, (C_p)^2, \cdots, (C_p)^b$ are associates. If there are other elements in the semigroup generated by C_p, they are associates of $(C_p)^b$.*

If p^j and p^i are contained in associated classes, where $1 \leqq j \leqq i \leqq b$, then for some $C_g \in \mathfrak{U}$,

$$p^i \equiv gp^j \pmod{m}, \qquad p^{i-j}g^{-1} \equiv 1 \pmod{m/(p^j, m)},$$

and p^{i-j} is contained in a unit class modulo $m/(p^j, m)$. But since $j \leqq i \leqq b$ it follows that $i = j$, and no two distinct elements of the set $C_p, (C_p)^2, \cdots, (C_p)^b$ are associates. To prove the second part of the lemma, we note first that if $m = p^b$, the set just mentioned exhausts the semigroup generated by C_p. If $m = p^b m_1$, $(p, m_1) = 1$ and $m_1 \neq 1$, let $u_1 = m_1 + p$. Then $(u_1, m_1) = (u_1, p) = 1$, and

$(u_1, m) = 1$; consequently u_1 is in a unit class. Therefore by multiplying together $p^b = p^b$ and $u_1^i = (m_1 + p)^i$ we obtain

$$p^{b+i} \equiv (u_1)^i p^b \pmod{m}$$

for $i \geqq 1$.

To prove that each nonunit C_n of the multiplicative semigroup modulo m satisfies (iii), let C_n have the factorizations as in (ii):

$$V \prod_{i=1}^{x} P_i^{a_i} = W \prod_{j=1}^{y} Q_j^{b_j}. \tag{8.3.2}$$

Since for each P_i and Q_j, natural prime divisors p_i and q_j of m and integers contained in unit classes v_i and w_j exist such that $p_i^{a_i} v_i$ and $q_j^{b_j} w_j$ are contained in $P_i^{a_i}$ and $Q_j^{b_j}$, respectively, (8.3.2) gives

$$v \prod_{i=1}^{x} p_i^{a_i} v_i \equiv w \prod_{j=1}^{y} q_j^{b_j} w_j \pmod{m}, \tag{8.3.3}$$

with v, w contained in units. We see that each natural prime that divides one member of (8.3.3) divides the other also, and we may assume that $x = y$ and that (8.3.3) is written in such an order that $p_i = q_j$ when subscripts are $=$, and it follows from the lemma that if $a_i \geqq b_i \geqq b$ where p_i^b is the largest power of p_i dividing m, then $p_i^{a_i} v_i$ and $p_i^{b_i} w_i$ are contained in associated classes. On the other hand, if $a_i \leqq b_i \leqq b$, we can write a congruence from the two members of (8.3.3):

$$p_i^{a_i} e \equiv p_i^{b_i} f \pmod{m},$$

where $(p_i, e) = (p_i, f) = 1$. Hence

$$e \equiv p_i^{b_i - a_i} f \pmod{m/(p_i^{a_i}, m)},$$

and $b_i = a_i$ since p_i divides $m/(p_i^{a_i}, m)$ for $a_i < b$. And C_n has a unique factorization into primes and we have

THEOREM* 8.3.4. *The residue classes modulo m, $m > 1$, form a uniquely factorable multiplicative semigroup, aside from unit divisors.*

* Weaver, Math. Mag., vol. 25, 1952, pp. 132–134.

Chapter IX

THE SEMIRING FORMED BY CERTAIN GENERALIZED RESIDUE CLASSES

9.1. Some finite commutative semirings. In another paper† the foundations of a theory of certain semirings were developed, and a set of postulates was given which covered not only these finite semirings but also the elementary arithmetic of the natural numbers itself. First the set of natural numbers was defined in the usual way. We then considered the infinite set of symbols:

$$A_1, A_2, A_3, \cdots. \tag{9.1.1}$$

We next introduced six postulates to cover the ideas of identity, equality, parentheses, substitution, induction, addition, and multiplication. Concerning the set (9.1.1), we initially made no statements whatever concerning possible equality between members of it. Then, using our postulates, we proved that this system formed a semiring. At no time in the proofs of these theorems did we assume that elements of the set (9.1.1) were distinct; we were free to make any assumption we pleased concerning equality or inequality among the set of elements. We then introduced inequality for the first time and used the symbol $\neq$.

We next considered the set (9.1.1) together with the six postulates, and assumed that for some natural number, not 1, and denoted by m, we had, if m' is the immediate successor of m in the set of natural numbers,

$$A_{m'} = A_1, \tag{9.1.2}$$

under the assumption that

$$A_1 \neq A_k, \tag{9.1.3}$$

if k denotes a natural number in the set

$$2, 3, \cdots, m.$$

We then proved

$$A_k + A_m = A_k; \tag{9.1.4}$$

that is, A_m acted as an additive identity in our set (9.1.1) under addition. We also obtained

$$A_kA_m = A_mA_k = A_m. \tag{9.1.5}$$

It is easy to show that the set (9.1.1), with the conditions (9.1.2), (9.1.3), and (9.1.4) holding, *is isomorphic with the set of residue classes modulo m as defined in* (4.1.1). As we have already noted, this set forms a ring.

† Vandiver, Math. Mag., vol. 1952, pp. 233–250. In the present paper we do not give proofs of the results stated from here on.

9.2. The generalized residue classes. We shall now discuss a semiring which, from our present viewpoint, is an extension of this ring of residue classes. Refer to the set (9.1.1) and assume that

$$A_1, A_2, A_3, A_4, A_5, A_6$$

are unequal, but that $A_7 = A_3$, and that the six postulates mentioned above still hold; then it follows that

$$A_6 + A_1 = A_2 + A_1.$$

Yet we cannot cancel the A_1's; that is, the law of cancellation under addition does not hold in this algebra.

In (9.1.1) the elements repeat in cycles, relative to addition, the elements in each cycle equaling A_3, A_4, A_5, and A_6 in order. There is no element having the property of the zero element. Also, the cancellation law of multiplication does not hold in general. Further, what corresponds to division is not always possible. This semiring cannot be *imbedded* in a ring (a semiring is said to be imbedded in another if it is isomorphic to a subsemiring of the latter), and hence the ring is not the fundamental system for associative distributive algebra. *But from the point of view we have employed here, the semiring just defined is a natural extension of the ring of residue classes*; in fact if we take the elements of the cycle, A_3, A_4, A_5, A_6, and combine them under addition and multiplication subject to the condition $A_7 = A_3$, we have a ring isomorphic to the ring of residue classes modulo 4.

Suppose we consider the set (9.1.1) again and assume $A_1, A_2, \cdots, A_{j-1}$ are distinct but that $A_j = A_i$ for some $i < j$. A semiring of the type just discussed exists for any such i and j. Such semirings reduce to rings only when $i = 1$. These systems have so far been studied very little, although the theory of semirings in general is now receiving considerable attention from investigators.

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FACTORIAL $\frac{1}{2}$: A SIMPLE GRAPHICAL TREATMENT*

FREDERICK MOSTELLER AND D. E. RICHMOND,
Harvard University and Williams College

The great secret of successful interpolation or extrapolation is the choice of a good scale for plotting the function or the independent variable. Since the straight line is the only curve that provides easy and reliable interpolation, a good scale is one that tends to produce a straight line. We use this principle twice to obtain an approximate numerical definition for $\frac{1}{2}!$.

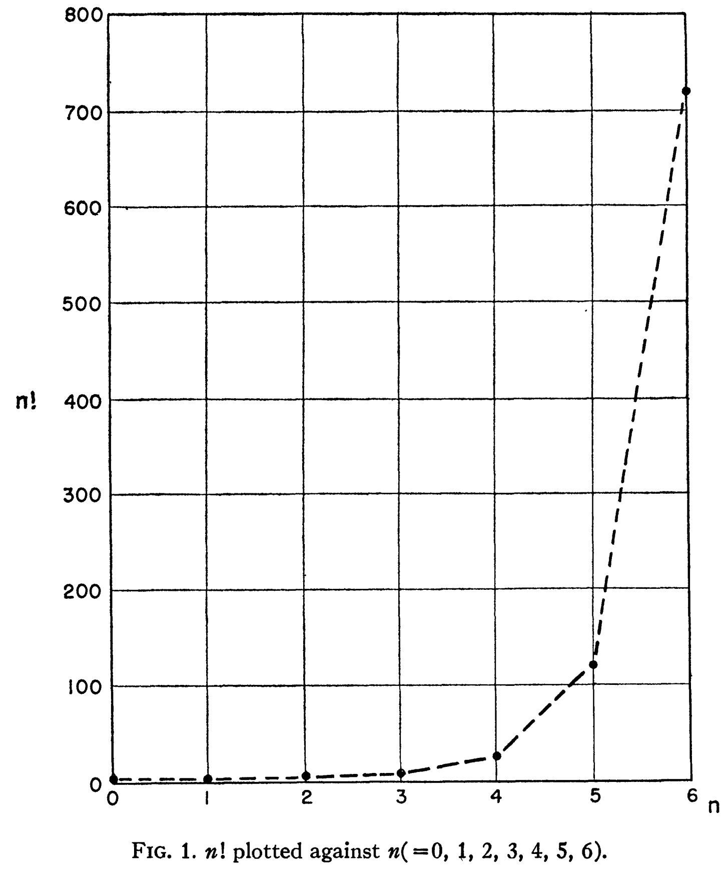

FIG. 1. $n!$ plotted against $n(=0, 1, 2, 3, 4, 5, 6)$.

Elementary students, after having heard about $n!$ for integers, often wish to know its value when n is not an integer. Our discussion is keyed to the definition and evaluation of $\frac{1}{2}!$, and more generally of $(n+\frac{1}{2})!$, where n is an integer. While the discussion could be extended beyond the half-integers, its merit lies in its

* This research was facilitated by a grant from the Ford Foundation and by the Laboratory of Social Relations of Harvard University.

intuitive appeal—to pursue this extension would be excessive.

We begin by plotting $f(n)=n!$ against n for a few values of n (see Fig. 1).

The request to define $(n+\frac{1}{2})!$ is equivalent to asking how to interpolate certain values on a curve connecting the points plotted in Figure 1 for integral values of n. One feels that a complete set of interpolated points should create a smooth curve, but words like "smooth" are vague, except when points are collinear. Then the natural step is to put in the extra points by linear interpolation.

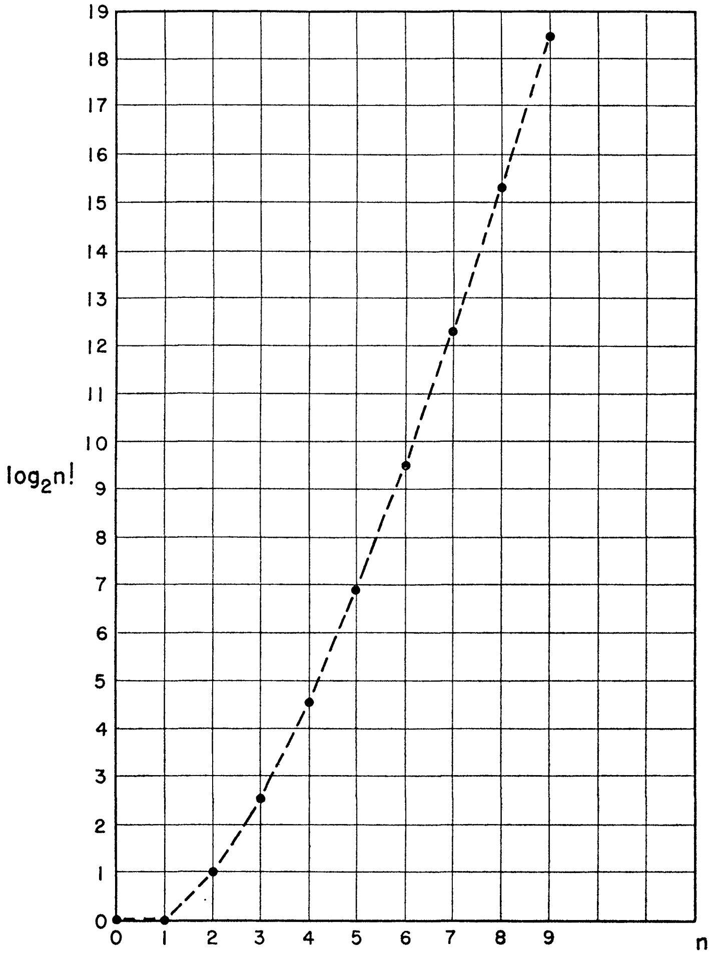

FIG. 2. Plot of $\log_2 n!$ against $n(=0, 1, \cdots, 9)$.

Our trouble is that the function $f(n)$ rises so fast that we cannot get a good "purchase" on it. If a function increases too fast for comfort, it is natural to take its logarithm, so we plot $\log_2 n!$ against n (see Fig. 2).

The curve passed by the eye through these new points seems to be getting straighter as n increases. We check this by looking at the slopes of the chords between n and $n+1$ and between $n+1$ and $n+2$; namely:

$$\log (n+2)! - \log (n+1)! = \log (n+2),$$
$$\log (n+1)! - \log n! = \log (n+1).$$

We observe that as n gets large, the difference in the slopes of adjacent chords

$$\log (n+2) - \log (n+1) = \log \left(1 + \frac{1}{n+1}\right)$$

becomes very small. Therefore, as n gets larger, the polygonal curve formed by the chords is getting straighter and straighter. This straightness means that we can evaluate $\log (n+\frac{1}{2})!$ by going far enough out on the curve, so that linear interpolation will do the trick.

So for large integral n we would have approximately

$$\log \left(n + \frac{1}{2}\right)! = \frac{1}{2} [\log n! + \log (n+1)!]$$

or, the equivalent approximation

$$\left(n + \frac{1}{2}\right)! = \sqrt{n!(n+1)!} = n!\sqrt{n+1}. \tag{1}$$

Since the fundamental relation

$$(n+1)! = (n+1)n! \tag{2}$$

serves to define $n!$ for integral values of $n \geqq 0$ (given $1! = 1$), it is natural to try to retain it for nonintegral values of n. We therefore write

$$\frac{3}{2}! = \frac{3}{2}\cdot\frac{1}{2}!,$$
$$\frac{5}{2}! = \frac{5}{2}\cdot\frac{3}{2}! = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}!,$$
$$\cdots\cdots\cdots\cdots,$$
$$\left(n + \frac{1}{2}\right)! = \frac{(2n+1)(2n-1)\cdots 3}{2^n}\cdot\frac{1}{2}!. \tag{3}$$

By inserting the factors $2n, 2n-2, \cdots, 2$ in the numerator and denominator of the right-hand side of (3) we can rewrite it as

$$\left(n + \frac{1}{2}\right)! = \frac{(2n+1)!}{2^{2n}n!}\cdot\frac{1}{2}!.$$

Then we have

$$\frac{1}{2}! = \frac{2^{2n}n!}{(2n+1)!}\left(n+\frac{1}{2}\right)!. \tag{4}$$

We obtain a sequence of approximations $v_0, v_1, v_2, \cdots$ to the desired $\frac{1}{2}!$ by using (1) as an approximation for $(n+\frac{1}{2})!$ in (4). Then

$$v_n = \frac{2^{2n}(n!)^2\sqrt{n+1}}{(2n+1)!}.$$

It is pleasant to work entirely with integers. Accordingly, we shall compute v_n^2 instead of v_n:

$$v_n^2 = \frac{(n+1)(n!)^4 2^{4n}}{[(2n+1)!]^2}. \tag{5}$$

Each member of this sequence is positive. Since

$$\frac{v_n^2}{v_{n-1}^2} = 1 - \frac{1}{(2n+1)^2}, \qquad n \geqq 1, \tag{6}$$

the sequence decreases monotonically. Hence v_n^2 approaches a limit which will be defined as $(\frac{1}{2}!)^2$.

Equation (6) is equivalent to

$$v_{n-1}^2 = \left(1 + \frac{1}{4n(n+1)}\right) v_n^2, \tag{7}$$

which will be useful later.

We wish to evaluate this limit graphically. We need to adopt a scale on the horizontal axis so that as n goes from 1 to ∞, the graph stays on the paper. The classical device is to plot against $1/n$ or more generally $1/(n+k)$ (k a positive constant), because then as n goes from 1 to ∞, the function is plotted for points on the interval from $1/(1+k)$ to 0. If by good luck we can choose k so that the resulting graph is approximated by a straight line, we can find the required limit of v_n^2 as $n \to \infty$, by finding where this straight line crosses the vertical axis.

The first three points $(1/(1+k), v_1^2)$, $(1/(2+k), v_2^2)$, $(1/(3+k), v_3^2)$ can be lined up exactly by choosing $k = 23/25 = 0.92$. This value of k is near enough to unity to suggest plotting v_n^2 against $1/(n+1)$ (see Fig. 3). (Had $n = 0, 1, 2$ been used for the first three points rather than $n = 1, 2$, and 3, k would have been 16/17 or approximately 0.94, even closer to unity. By not plotting the point (1, 1) corresponding to $n = 0$ in Figure 3, we have been able to enlarge the scale on both axes.)

The result is a satisfactory resemblance to a straight line, especially on the left. Extrapolating to $1/(n+1) = 0$ by eye gives the approximation 0.785 for $[(\frac{1}{2})!]^2$. The straightness is sufficiently encouraging to justify a more careful extrapolation to 0.

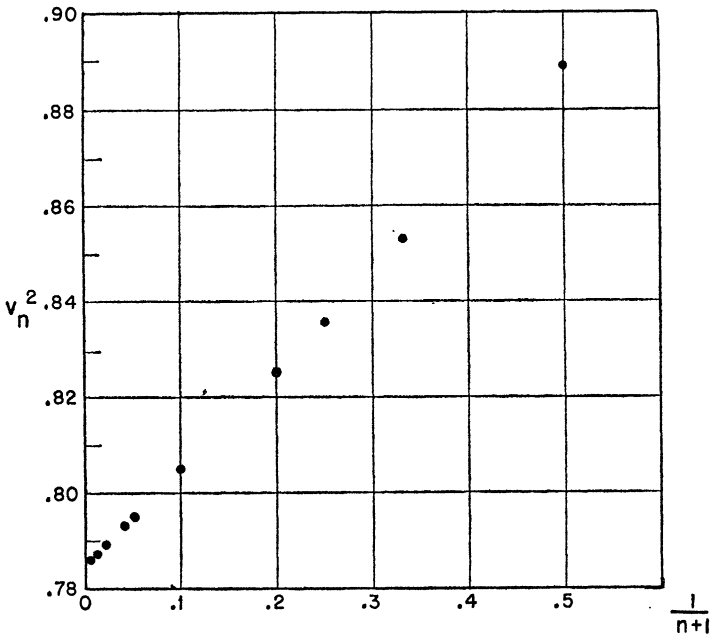

FIG. 3. Plot of v_n^2 against $1/(n+1)$.

TABLE 1

ESTIMATES OF $(\frac{1}{2}!)^2 = v_\infty^2$

n	$\frac{1}{n+1}$	First approximation v_n^2	Second approximation $b_n = v_n^2\left(1 - \frac{1}{4(n+1)}\right)$
1	.5	.88888 88889	.77777 77778
2	.33 · · ·	.85333 33333	.78222 22222
3	.25	.83591 83673	.78367 34694
4	.20	.82559 83875	.78431 84681
9	.10	.80527 22492	.78514 04430
19	.05	.79527 62214	.78533 52686
24	.04	.79329 10176	.78535 81074
49	.02	.78933 49223	.78538 82477
99	.01	.78736 41070	.78539 56968
199	.0050	.78638 05239	.78539 75483
399	.0025	.78588 91906	.78539 80098

$$\frac{\pi}{4} = .78539\ 81634$$

To facilitate the work, we tabulate v_n^2 against $1/(n+1)$ for certain convenient values of n (see Table 1). The last column contains extrapolations to v_∞^2 which will now be explained.

The slope of the line joining the point

$$P_n = \left(\frac{1}{n+1}, v_n^2\right) \quad \text{to} \quad P_{n-1} = \left(\frac{1}{n}, v_{n-1}^2\right)$$

is

$$m_n = \frac{v_{n-1}^2 - v_n^2}{\dfrac{1}{n} - \dfrac{1}{n+1}}.$$

Using equation (7),

$$m_n = n(n+1)\left[v_{n-1}^2 - v_n^2\right] = \frac{v_n^2}{4},$$

a remarkably simple result. The line through P_n with slope m_n intersects the v_n^2 axis at

$$b_n = v_n^2\left[1 - \frac{1}{4(n+1)}\right].$$

The results for b_n have been listed in Table 1 as estimates of $v_\infty^2 = (\frac{1}{2}!)^2$. As we go down the table we note that the value is stabilizing near 0.785398, which is $\pi/4$ to six decimal places. The correct result $(\frac{1}{2})! = \sqrt{\pi}/2$ is therefore suggested. This result is no accident, since there is, in fact, a close connection between our work and Wallis' formula

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$$

which may be written [2]

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{(2n!)^2(2n+1)}$$

$$= \lim_{n\to\infty} \left(\frac{2n+1}{n+1}\right)(v_n^2),$$

so that $\lim_{n\to\infty} v_n^2 = \pi/4$. The interest in this note lies in the elementary methods which have been used to evaluate $(\frac{1}{2})!$, rather than in the result itself.

The remainder of this note is concerned with improvements of the results in Table 1 which may be obtained by plotting against $1/(n+k)$ where $k \neq 1$.

If we parallel the previous discussion, we obtain for the slope of the line from $P_n=(1/(n+k),\ v_n^2)$ to $P_{n-1}=(1/(n+k-1),\ v_{n-1}^2)$, the value

$$m_n = \frac{v_n^2}{4}\,\frac{(n+k-1)(n+k)}{n(n+1)}, \tag{8}$$

and for the intercept on the v_n^2 axis

$$b_n = v_n^2\left[1 - \frac{1}{4(n+1)} + \frac{1-k}{4(n+1)n}\right]. \tag{9}$$

What is the best k to use for a given n? As long as the graph is concave upward to the left of P_n, b_n will be too small. Since b_n increases as k decreases, we shall obtain the best value of b_n among graphs of this type by choosing the smallest k compatible with the requirement of upward concavity.

The graph is concave upward to the immediate left of P_n if the ratio $R = m_n/m_{n+1} \geqq 1$, that is, using (8), if

$$R = \frac{\left(n+\frac{3}{2}\right)^2(n+k-1)}{n(n+1)(n+k+1)} \geqq 1,$$

TABLE 2

BOUNDS FOR $(\frac{1}{2}!)^2 = v_\infty^2$ OBTAINED FROM THE BEST VALUES OF k

$$b_n = v_n^2\left(1 - \frac{1}{4(n+1)} + \frac{1-k}{4(n+1)n}\right)$$

n	Lower bounds $k=\dfrac{7n+9}{8n+9}$	Upper bounds $k=\dfrac{7}{8}$
1	.78431 37255	.79166 66667
2	.78506 66667	.78666 66667
3	.78525 66481	.78585 03401
4	.78532 52954	.78560 84656
9	.78538 89838	.78542 00514
19	.78539 70137	.78540 06696
24	.78539 75746	.78539 94246
49	.78539 80898	.78539 83157
99	.78539 81542	.78539 81821
199	.78539 81622	.78539 81657
399	.78539 816325	.78539 81637

$$\frac{\pi}{4} = .78539\ 816340$$

which implies that $k \geqq (7n+9)/(8n+9)$. Note that if this condition is satisfied for a given n, it is satisfied for all larger values of n and hence for the whole graph to the left of P_n. To obtain the best value of v_∞^2 for a fixed n among graphs with upward concavity, we should therefore use $k_n = (7n+9)/(8n+9)$. Results for some values of n are given in Table 2.

As we know, even this most favorable k gives too small a result for v_∞^2. It is easy to obtain a result which is known to be too *large* by choosing $k \leqq 7/8$, since for all such values of k the graph is concave *downward* to the left of P_n. Among these values, $k = 7/8$ gives the smallest b_n and hence the best upper bound for v_∞^2 among graphs of this type. Values of b_n for $k = 7/8$ have been added to Table 2.

Unfortunately, a choice of k *between* $7/8$ and k_n corresponds to a graph which changes its concavity to the left of P_n so that no further improvement can be made by these elementary methods since we have no way to determine whether the result is too large or too small. Nevertheless, the bounds obtained are rather good.

In both tables the entries are believed to be accurate to the number of places indicated. Tables of logarithms and logarithms of factorials used in these calculations were [**1**, **3**]. The authors wish to express their appreciation to Mrs. Cleo Youtz for carrying out the calculations.

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SINGULAR VALUES OF A MATRIX

ALI R. AMIR-MOÉZ, Queens College, AND ALFRED HORN, University of California, Los Angeles

1. The singular values. There is an analogy between matrices and complex numbers which has been known for a long time. In this analogy, the operation which leads from a matrix A to its adjoint A^* corresponds to passing to the conjugate for complex numbers. Accordingly, Hermitian matrices (those for which $A = A^*$) play the roles of real numbers. Positive semidefinite matrices correspond to nonnegative real numbers, and we shall refer to such matrices as nonnegative matrices. Every matrix A can be written uniquely in the form $H_1 + iH_2$, where H_1 and H_2 are Hermitian. Indeed, $H_1 = (A + A^*)/2$, $H_2 = (A - A^*)/2i$. Unitary matrices correspond to complex numbers of modulus one. Moreover, every unitary matrix can be written in the form e^{iH}, where H is Hermitian and the

exponential is defined by its power series.

There is also an analogue to the polar form of a complex number: every matrix A may be written as $A = UH_1 = H_2U$, where U is unitary, and H_1, H_2 are nonnegative Hermitian. H_1 and H_2 are uniquely determined as the nonnegative Hermitian matrices whose squares are A^*A, AA^* respectively. The unitary part is uniquely defined only when A is nonsingular.

The analogy is deepened when we consider the eigenvalues of the matrices. A Hermitian matrix has only real eigenvalues, and the eigenvalues are all nonnegative if and only if the matrix is nonnegative. The eigenvalues of a unitary matrix have modulus one.

In this paper, we are going to consider the following aspect of the analogy. Since much more is known about the eigenvalues of a Hermitian matrix, it is natural to ask to what extent the eigenvalues of the Hermitian matrices related to A in the above decompositions determine the eigenvalues of A. Let us define the eigenvalues in question.

DEFINITION. *The absolute singular values of A are the eigenvalues of the nonnegative part of the polar decomposition of A. Equivalently they are the nonnegative square roots of the eigenvalues of A^*A.*

The eigenvalues of AA^* are the same as those of A^*A, so that it does not matter which polar decomposition of A we use.

There are two norms for a matrix which are in common use. It is noteworthy that both may be expressed in terms of the absolute singular values ρ_i, where $\rho_1 \geqq \cdots \geqq \rho_n$. The Hilbert norm is defined by

$$\|A\|_H^2 = \sup_{|x|=1} (Ax, Ax),$$

and therefore

$$\|A\|_H^2 = \sup_{|x|=1} (A^*Ax, x) = \rho_1^2.$$

The Frobenius norm is

$$\|A\|_F^2 = \sum_{i,j=1}^{n} |a_{ij}|^2,$$

where a_{ij} is the i, j element of A. Since the ith diagonal element of A^*A is $\sum_{j=1}^{n} |a_{ij}|^2$, we see that

$$\|A\|_F^2 = \text{trace } A^*A = \sum_{i=1}^{n} \rho_i^2.$$

More generally, by the well-known theorem of Schur [7] on the connection between the diagonal elements and the eigenvalues of a Hermitian matrix, we have

$$\rho_n^2 + \cdots + \rho_{n-k+1}^2 \leqq \sum_{p=1}^{k} \sum_{j=1}^{n} |a_{i_p,j}|^2 \leqq \rho_1^2 + \cdots + \rho_k^2, \qquad 1 < k < n,$$

where $i_1, \cdots, i_k$ is any increasing sequence of integers between 1 and n.

DEFINITION. *The real singular values of A are the eigenvalues of $(A+A^*)/2$, and the imaginary singular values of A are the eigenvalues of $(A-A^*)/2i$.*

If A is a normal matrix, then it is unitarily equivalent to a diagonal matrix and consequently it is easy to see that the real, imaginary and absolute singular values of A are merely the real parts, imaginary parts and absolute values of the eigenvalues of A. In the general case the relations between these quantities are given in the following theorems.

THEOREM 1. *Let A be a matrix with eigenvalues λ_i such that $\Re\lambda_1 \geqq \cdots \geqq \Re\lambda_n$ and real singular values $\alpha_1 \geqq \cdots \geqq \alpha_n$. Then*

$$\Re\lambda_1 + \cdots + \Re\lambda_k \leqq \alpha_1 + \cdots + \alpha_k, \qquad 1 \leqq k \leqq n, \tag{1}$$

$$\Re\lambda_1 + \cdots + \Re\lambda_n = \alpha_1 + \cdots + \alpha_n. \tag{2}$$

Conversely, if λ_i are any complex numbers and $\alpha_1 \geqq \cdots \geqq \alpha_n$ are real numbers satisfying (1) *and* (2), *there exists a matrix A with eigenvalues λ_i and real singular values α_i.*

Proof. (The first part of this theorem is due to Ky Fan [2].) If A has eigenvalues λ_i and real singular values α_i, then there exists a triangular matrix

$$T = \begin{pmatrix} \lambda_1 & & & \\ & \cdot & 0 & \\ & * & \cdot & \\ & & & \lambda_n \end{pmatrix}$$

with zeros above the diagonal such that $A = U^*TU$, U unitary. Therefore

$$\frac{A + A^*}{2} = U^*\left(\frac{T + T^*}{2}\right)U.$$

But $(T+T^*)/2$ is Hermitian and has diagonal elements $\Re\lambda_i$ and eigenvalues α_i. Now in [5], Theorem 5, it was proved that (1) and (2) hold if and only if there exists a Hermitian matrix with diagonal elements $\Re\lambda_i$ and eigenvalues α_i. Thus (1) and (2) hold.

Conversely if (1) and (2) hold, let $H=(h_{ij})$ be a Hermitian matrix with eigenvalues α_i and diagonal elements $\Re\lambda_i$. Let

$$A = \begin{pmatrix} \lambda_1 & 0 & & \cdots & 0 \\ 2h_{21} & \lambda_2 & 0 & \cdots & 0 \\ \cdots & & & \cdots & \\ 2h_{n1} & & \cdots & 2h_{n-1,n} & \lambda_n \end{pmatrix}.$$

Then A has eigenvalues λ_i and $(A+A^*)/2=H$. This completes the proof.

THEOREM 2. *A necessary and sufficient condition that there exist a matrix A with imaginary singular values $\beta_1 \geqq \cdots \geqq \beta_n$ and eigenvalues λ_i such that $\mathscr{I}\lambda_1 \geqq \cdots \geqq \mathscr{I}\lambda_n$ is*

$$\mathscr{I}\lambda_1 + \cdots + \mathscr{I}\lambda_k \leqq \beta_1 + \cdots + \beta_k, \qquad 1 \leqq k < n, \tag{3}$$

$$\mathscr{I}\lambda_1 + \cdots + \mathscr{I}\lambda_n = \beta_1 + \cdots + \beta_n. \tag{4}$$

Proof. The proof is similar to that of Theorem 1.

THEOREM 3. *A necessary and sufficient condition that there exist a matrix A with absolute singular values $\rho_1 \geqq \cdots \geqq \rho_n$ and eigenvalues λ_i such that $|\lambda_1| \geqq \cdots \geqq |\lambda_n|$ is*

$$|\lambda_1 \cdot \cdots \cdot \lambda_k| \leqq \rho_1 \cdot \cdots \cdot \rho_k, \qquad 1 \leqq k \leqq n, \tag{5}$$

$$|\lambda_1 \cdot \cdots \cdot \lambda_n| = \rho_1 \cdot \cdots \cdot \rho_n. \tag{6}$$

Proof. The necessity was proved by Weyl [8], and the sufficiency is due to Horn [4].

We turn now to the question of the relationship between the eigenvalues of A and the eigenvalues of the unitary part of the polar decomposition of A. Conditions (5) and (6) are equivalent to the following condition: For each $k \leqq n$, the absolute value of the product of any k eigenvalues of A (with different subscripts) lies in the convex set generated by the products k at a time of the absolute singular values. We have a corresponding result in the next theorem.

THEOREM 4. *Let A be a nonsingular matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$, and let $\gamma_1, \cdots, \gamma_n$ be the eigenvalues of the unitary part U of the polar decomposition of A. Then for each $k \leqq n$, the product of any k of the λ_i lies in the convex cone generated by the products k at a time of the γ_i.*

Proof. A convex cone (with vertex at the origin) in the complex plane is a set of numbers which is closed under linear combinations with nonnegative coefficients. Let $A = UH$ be a polar decomposition of A. Suppose $Ax=\lambda x$, $x \neq 0$. Then $\lambda(x, Hx)=(Ax, Hx)=(UHx, Hx)=\sum_{i=1}^n \gamma_i |(Hx, u_i)|^2$, where $\{u_i\}$ is the orthonormal sequence of eigenvectors of U. Since A is nonsingular, so is H, and therefore $(x, Hx)>0$. Therefore λ lies in the convex cone generated by the γ_i. This completes the proof for $k=1$.

If $k>1$, let $A_{(k)}$ be the kth adjugate (exterior product) [6] of A. Then $A_{(k)}=U_{(k)}H_{(k)}$, and $U_{(k)}$ is unitary and $H_{(k)}$ is nonnegative. Furthermore the eigenvalues of $A_{(k)}$ are the products k at a time of the eigenvalues of A, and similarly for $U_{(k)}$. Therefore the statement for $k=1$ implies the statement for all $k \leqq n$. The proof is now complete.

We now consider the question of the converse of Theorem 4. For $n=2$, it is indeed true that if λ_1, λ_2 are nonzero complex numbers and γ_1, γ_2 are complex

numbers of modulus one such that

$$\frac{\lambda_1\lambda_2}{|\lambda_1|\,|\lambda_2|} = \gamma_1\gamma_2$$

and the λ_i lie in the convex cone generated by γ_1, γ_2, then there exists a matrix A with eigenvalues λ_1, λ_2 whose unitary part has eigenvalues γ_1, γ_2. However for $k>2$, the conditions of Theorem 4 are not sufficient. For example, if the eigenvalues of the unitary part of A are all real, then the number of positive eigenvalues of A must be the same as the number of positive eigenvalues of its unitary part. Even if we add this condition, we still do not have sufficient conditions.

It is to be remarked, though, that a knowledge of the eigenvalues of the unitary part of A imposes conditions only on the arguments of the eigenvalues of A. That is, if there exists a matrix A with eigenvalues λ_i, whose unitary part has eigenvalues γ_i, and if ρ_i are any positive numbers, then there exists a matrix B with eigenvalues $\rho_i\lambda_i$ whose unitary part has eigenvalues γ_i. This remark as well as the above comment on the lack of validity of the converse of Theorem 3 is due to Professor R. Steinberg, University of California, Los Angeles.

If we assume known both the real and imaginary singular values of A, then the eigenvalues of A will be subject to much stronger conditions than those implied by Theorems 1 and 2. Exactly how much can be said is still unknown. Wielandt [9] has determined the exact region in which an eigenvalue of A can lie if we are given the real and imaginary singular values. But the more difficult problem of determining the exact range of variation of the sequence of all the eigenvalues of A is not yet solved.

A theorem relating all three types of singular values is the following, which corresponds to the Pythagorean theorem.

THEOREM 5. *If A has real, imaginary and absolute singular values α_i, β_i and γ_i respectively, then*

$$\sum_{i=1}^{n} \alpha_i^2 + \sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} \gamma_i^2.$$

Proof. We have

$$\left(\frac{A + A^*}{2}\right)^2 + \left(\frac{A - A^*}{2i}\right)^2 = \left(\frac{AA^* + A^*A}{2}\right).$$

The result now follows from the following facts:

a) The eigenvalues of the square of a matrix are the squares of the eigenvalues of the matrix.

b) The trace of a matrix (sum of its diagonal elements) is equal to the sum of its eigenvalues.

c) The trace of a sum of matrices is the sum of their traces.

2. Singular values of sums and products of matrices. In this section we consider the singular values of a sum or product of matrices. If A and B are matrices, then $(A+B)^*=A^*+B^*$. Therefore the question about the relation between the real or imaginary singular values of $A+B$ and those of A and B reduces to the corresponding question about the eigenvalues of a sum of Hermitian matrices. Much has been learned about this latter problem in recent years. The most general result published so far is the following [1].

THEOREM 6. *Let A and B be Hermitian matrices with eigenvalues $\alpha_1 \geqq \cdots \geqq \alpha_n$ and $\beta_1 \geqq \cdots \geqq \beta_n$, respectively. Let the eigenvalues of $A+B$ be $\gamma_1 \geqq \cdots \geqq \gamma_n$. Let $i_1 \leqq \cdots \leqq i_k$ and $j_1 \leqq \cdots \leqq j_k$ be any sequences of integers between 1 and n such that $i_p+j_p \geqq n+p$ for $p=1, \cdots, k$, where $k \leqq n$. Then*

$$(7) \qquad \alpha_{i_1'}+\cdots+\alpha_{i_k'}+\beta_{j_1'}+\cdots+\beta_{j_k'} \leqq \gamma_{(i_1+j_1-n)'}+\cdots+\gamma_{(i_k+j_k-n)'},$$

where $i_k'=i_k$ and $i_p'=\min(i_p, i_{p+1}'-1)$ for $p=k-1, \cdots, 1$, and the sequences $\{j_p'\}$, $\{(i_p+j_p-n)'\}$ are similarly defined.

Unpublished work of A. Horn and A. Hoffman shows that there exist still more inequalities between the eigenvalues of $A+B$ and those of A and B. These inequalities are also linear. However the exact range of variation of the eigenvalues of $A+B$ has not yet been completely determined.

Wielandt has remarked that for any matrix A, the eigenvalues of the $2n$-rowed Hermitian matrix

$$\tilde{A}=\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

are $\pm\rho_1, \cdots, \pm\rho_n$, where the ρ_i are the absolute singular values of A. Since the operation $A\to\tilde{A}$ is linear, we can apply Theorem 6 to obtain inequalities between the absolute singular values of $A+B$ and those of A and B.

For products of matrices, there is a theorem very much like Theorem 6. If in Theorem 6, we let the α_i, β_i, γ_i be the absolute singular values of A, B and AB respectively, and we replace the sums in (7) by products, then we obtain a theorem due to Amir-Moéz [1].

There remains the question of the relationship between the eigenvalues of the unitary part of AB and the eigenvalues of the unitary parts of A and of B. One might expect the former to lie in a convex cone determined by products of the latter. But this conjecture is refuted by the following examples.

(1) Any rotation in the plane is a product of two reflections. Thus we can have two unitary matrices with eigenvalues ± 1 whose product has eigenvalues $e^{\pm i\theta}$, where θ is any real number.

(2) Let H and K be positive definite matrices which do not commute. The unitary parts of H and K are the identity matrix whose eigenvalues are 1. But if the eigenvalues of the unitary part U of HK were all 1, then U would be the identity matrix and HK would then be positive definite. This contradicts the

hypothesis that $HK \neq KH$, that is, that HK is not Hermitian.

In case U and V are unitary matrices and O does not lie in the convex hull of the eigenvalues of U, then one can prove something about the eigenvalues of UV. In fact if $UVx = \lambda x$, and $|x| = 1$, then $\lambda(x, Ux) = (UVx, Ux) = (Vx, x)$ and therefore $\lambda = (Vx, x)/(x, Ux)$. Let the eigenvalues of U and V be denoted by α_i and β_i respectively. Then (Ux, x) lies in the convex hull of the α_i, and is by hypothesis different from O. It follows that $1/(x, Ux)$ lies in the convex cone generated by the α_i. Thus every eigenvalue of UV is a product of a number in the convex cone generated by the α_i and a number in the convex cone generated by the β_i.

A. Horn and R. Steinberg have settled the question raised above concerning the eigenvalues of the unitary part of a matrix:

THEOREM. *If* $(\lambda_1, \cdots, \lambda_n)$ *and* $(\alpha_1, \cdots, \alpha_n)$ *are n-tuples of complex numbers with* $|\lambda_i| \neq 0$, $|\alpha_i| = 1$, *then there exists a matrix A with eigenvalues* λ_i *such that the unitary part of A has eigenvalues* α_i *if and only if*

$$\prod_{i=1}^{n} \frac{\lambda_i}{|\lambda_i|} = \prod_{i=1}^{n} \alpha_i$$

and either

(a) α_i *lie in a closed half plane with* 0 *on its boundary but not on any line through* 0 *and** $(\arg \lambda_1, \cdots, \arg \lambda_n) \prec (\arg \alpha_1, \cdots, \arg \alpha_n)$, *where we use a branch of the argument which is continuous in the closed half plane, or*

(b) α_i *lie on a line through* 0 *and* $\lambda_i/|\lambda_i|$ *are a rearrangement of the* α_i, *or*

(c) α_i *do not satisfy* (a) *or* (b).

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MODELS OF PROJECTIVE AND EUCLIDEAN SPACE

DAVID GANS, New York University

Although the number of possible models of Euclidean 3-space is without end, and likewise for models of projective 3-space, one finds in the literature no single model which puts both spaces completely before our eyes at the same time, exhibiting their topological relations in a simple and easy-to-visualize manner. Apart from its intrinsic mathematical interest and value, such a miniature would go far toward meeting a sorely-felt need in geometrical education. The customary method of trying to get college students to see the relation between the two spaces, for example, is to have them visualize Euclidean space in terms of the unbounded Cartesian model, and then to add ideal points to this model. While there is nothing wrong with this procedure from an abstract, logical viewpoint, it fails utterly to give the student a visual grasp of projective space. For he is expected to believe, among other things, that the ideal point on a straight line is a definite point which is reached by going infinitely far in either direction along the unbounded line! Surely the Cartesian model is worthless as a pictorial basis for beliefs such as this.

The present article shows that the kind of simultaneous model referred to above can be obtained quite simply by the use of a sphere. In the model which is described, the points, planes, and straight lines of Euclidean 3-space are represented, respectively, by the points interior to the sphere, certain half ellipsoids composed of these points, and certain half ellipses on these ellipsoids. On adding to this model the points on the surface of the sphere, diametrically opposite points being identified, we obtain a model of projective 3-space in which the closed half ellipsoids and closed half ellipses resulting from the specified addition of points represent projective planes and lines, respectively. These three-dimensional models are generalizations of two-dimensional ones, possessing similar advantages, which have already been described in this MONTHLY [2].

1. A transformation T of Euclidean 3-space. Consider the transformation T,

$$(1)\qquad x' = \frac{x}{\sqrt{x^2+y^2+z^2+1}}, \quad y' = \frac{y}{\sqrt{x^2+y^2+z^2+1}},$$
$$z' = \frac{z}{\sqrt{x^2+y^2+z^2+1}},$$

for arbitrary real values of x, y, z. Then

$$(2)\qquad x'^2 + y'^2 + z'^2 = \frac{x^2+y^2+z^2}{x^2+y^2+z^2+1} < 1.$$

Hence, T sends each point (x, y, z) into a point (x', y', z') within the unit sphere S with center at the origin O. From (2) we obtain

$$x^2 + y^2 + z^2 + 1 = \frac{1}{1 - x'^2 - y'^2 - z'^2},$$

and substitution of this in (1) gives the inverse of T:

$$x = \frac{x'}{\sqrt{1 - x'^2 - y'^2 - z'^2}}, \quad y = \frac{y'}{\sqrt{1 - x'^2 - y'^2 - z'^2}}, \quad z = \frac{z'}{\sqrt{1 - x'^2 - y'^2 - z'^2}}. \tag{3}$$

We see that to each point (x', y', z') interior to S there corresponds a point (x, y, z). Being continuous, together with its inverse, T is thus a topological transformation of Euclidean 3-space into the interior of S.

If P is any point at distance $r>0$ from O, it follows from (1) and (2) that the point $P' = T(P)$ is between O and P, and at the distance $r/\sqrt{r^2+1}$ from O. Hence, straight lines through O go into the parts of themselves interior to S, and spheres with center O into smaller spheres with center O. In other words, the compression of space into the interior of S effected by T takes place radially toward O, the amount of contraction depending only on r.

It follows from the foregoing properties that the orthogonality of two straight lines, one of which goes through O, is invariant under T. Except for this special situation, however, T generally distorts angles. Nor is the collinearity of points preserved except for points on straight lines through O. These things will be apparent from the discussion to follow, in which the effect of T on planes and the general straight line are considered.

2. The effect of T on planes. A plane through O,

$$ax + by + cz = 0, \tag{4}$$

is transformed by T into that much of the locus

$$(ax' + by' + cz')/\sqrt{1 - x'^2 - y'^2 - z'^2} = 0, \quad \text{or} \quad ax' + by' + cz' = 0,$$

as lies interior to S. Thus, (4) goes into a part of itself, namely, into the interior of a circular disc with diameter 2 and center at O. Conversely, to each such disc there corresponds a plane through O which is transformed into it.

As a first example of a plane not through O let us consider $x = d$, where $d \neq 0$. T sends this into

$$d = x'/\sqrt{1 - x'^2 - y'^2 - z'^2}, \tag{5}$$

which is that part of the ellipsoid

$$(1 + d^2)\frac{x'^2}{d^2} + y'^2 + z'^2 = 1 \tag{6}$$

for which x' has the same sign as d. Being symmetrical to the coordinate planes,

and having semimajor, semiminor, and semimean axes of lengths 1, 1, $d/\sqrt{1+d^2}$, respectively, this ellipsoid is interior to S, except where it meets S along the great circle in the yz-plane, and is divided into symmetrical hemiellipsoids by the circle. The locus of (5) is that one of these hemiellipsoids (excluding the circle) which lies on the same side of the plane of the circle as does the plane $x=d$. (The other hemiellipsoid, also exclusive of the circle, is the transform of the plane $x=-d$ by T.)

We next show that every other plane not through O is likewise transformed into a hemiellipsoid, moreover, one related to it in size, shape, and position exactly as the hemiellipsoid (5) is related to the plane $x=d$. Hence, consider the plane

$$ax + by + cz = d, \tag{7}$$

where $d\neq 0$ and $a^2+b^2+c^2=1$. Substitution in (3) shows that this plane is transformed into

$$\frac{ax' + by' + cz'}{\sqrt{1 - x'^2 - y'^2 - z'^2}} = d, \tag{8}$$

where, of course, $x'^2+y'^2+z'^2<1$. To determine the nature of this locus it is useful to consider the related locus

$$ax' + by' + cz' = d\sqrt{1 - x'^2 - y'^2 - z'^2} \tag{9}$$

for $x'^2+y'^2+z'^2\leqq 1$. The latter goes through the intersection of $ax'+by'+cz'=0$ and $\sqrt{1-x'^2-y'^2-z'^2}=0$, *i.e.*, of a plane through O and the sphere S. Thus (9) meets S in a great circle. The locus of (8) results from excluding this circle from (9).

Now, (9) is part of the locus of

$$(ax' + by' + cz')^2 = d^2(1 - x'^2 - y'^2 - z'^2), \tag{10}$$

or

$$(a^2 + d^2)x'^2 + (b^2 + d^2)y'^2 + (c^2 + d^2)z'^2 + 2abx'y' + 2acx'z' + 2bcy'z' = d^2. \tag{11}$$

To determine the nature of this locus we shall compare (11) with the general equation

$$Ax^2 + By^2 + Cz^2 + 2Hxy + 2Fxz + 2Gyz + 2Lx + 2My + 2Nz + K = 0. \tag{12}$$

This equation represents an ellipsoid if

$$D \equiv \begin{vmatrix} A & H & F \\ H & B & G \\ F & G & C \end{vmatrix} \neq 0, \qquad \Delta \equiv \begin{vmatrix} A & H & F & L \\ H & B & G & M \\ F & G & C & N \\ L & M & N & K \end{vmatrix} \neq 0,$$

and if the roots of the characteristic equation

$$r^3 - Ir^2 + Jr - D = 0 \tag{13}$$

have the same sign, where $I=A+B+C$ and $J=AB+BC+CA-F^2-G^2-H^2$ (see [1], p. 273).

Applying this to equation (11) we see that

$$A = a^2 + d^2, \quad B = b^2 + d^2, \quad C = c^2 + d^2,$$
$$H = ab, \quad F = ac, \quad G = bc, \quad K = -d^2, \quad L = M = N = 0,$$

from which we find, after substitution and simplification, that

$$D = d^2(a^2 + b^2 + c^2 + d^2), \qquad \Delta = -d^2D,$$
$$I = a^2 + b^2 + c^2 + 3d^2, \qquad J = d^2(2a^2 + 2b^2 + 2c^2 + 3d^2).$$

Since, by hypothesis, $d \neq 0$ and $a^2+b^2+c^2=1$, we see that $D \neq 0$, and hence that $\Delta \neq 0$. The coefficients of equation (13) being real, so are the roots (see [1], p. 259). None of these roots is zero since the constant term $-D \neq 0$, and none is negative since I, J, D are all positive. Equation (13) thus has three positive roots. The locus of (11) is therefore an ellipsoid.

We next show that this ellipsoid is symmetrical to the plane (4). Two points (x_1, y_1, z_1), (x_2, y_2, z_2) are symmetrical to this plane if and only if the four equations

$$x_1 - x_2 = ka, \quad y_1 - y_2 = kb, \quad z_1 - z_2 = kc,$$
$$ax_1 + by_1 + cz_1 = -(ax_2 + by_2 + cz_2)$$

are satisfied. If (x_1, y_1, z_1) is given, these equations determine x_2, y_2, z_2, k uniquely, the value of the last being

$$k = -2(ax_2 + by_2 + cz_2). \tag{14}$$

Hence, to prove the symmetry of the ellipsoid we can assume (x_1, y_1, z_1) is a point on (10), replace x_1, y_1, z_1 in (10) by x_2+ka, y_2+kb, z_2+kc, where k has the value in (14), and show that what results is (10) with x_2, y_2, z_2 replacing x', y', z'. The first of these steps gives

$$(ax_1 + by_1 + cz_1)^2 = d^2(1 - x_1^2 - y_1^2 - z_1^2),$$

and the second,

$$[a(x_2 + ka) + b(y_2 + kb) + c(z_2 + kc)]^2$$
$$= d^2[1 - (x_2 + ka)^2 - (y_2 + kb)^2 - (z_2 + kc)^2],$$

which reduces to

$$(ax_2 + by_2 + cz_2 + k)^2 = d^2[(1 - x_2^2 - y_2^2 - z_2^2) - 2k(ax_2 + by_2 + cz_2) - k^2].$$

On substituting from (14) into this we obtain

$$(ax_2 + by_2 + cz_2)^2 = d^2(1 - x_2^2 - y_2^2 - z_2^2),$$

which shows that (x_2, y_2, z_2) is on the ellipsoid (10). Hence the latter is symmetrical to the plane (4).

Returning to (9) we note that those of its points not on this plane lie on one side of it. For, any such point (x', y', z') of (9) gives to $ax'+by'+cz'$, which is the directed distance from the plane to this point, the value $d\sqrt{1-x'^2-y'^2-z'^2}$, which has the sign of d. This side of the plane (4) is also that on which the plane (7) lies, for any point (x', y', z') on the latter gives to $ax'+by'+cz'$ the value d. Now, the locus (9) is part of the ellipsoid (10), the remaining part is $ax'+by'+cz' = -d\sqrt{1-x'^2-y'^2-z'^2}$, and the latter clearly lies on the opposite side of (4). Hence the locus of (9) is that one of the two symmetrical parts of the ellipsoid produced by the plane (4) which lies on the same side of (4) as does the plane (7). On excluding from (9) its boundary circle we obtain the hemiellipsoid (8) into which the plane (7) is transformed.

It is now clear that the ellipsoid (10) has its center at O, has semimajor and semiminor axes each one unit long, and hence is an ellipsoid of revolution, symmetrical to the line m through O perpendicular to the plane (4). Since the hemiellipsoid (8) is also symmetrical to m, the length of the semimean axis is the unsigned distance from O to the point V where m pierces the hemiellipsoid. The equations of m are $x'=at$, $y'=bt$, $z'=ct$, where t denotes directed distance from O. Substitution in (8) gives

$$\frac{(a^2+b^2+c^2)t}{\sqrt{1-(a^2+b^2+c^2)t^2}} = d \quad \text{or} \quad \frac{t}{\sqrt{1-t^2}} = d,$$

so that t has the same sign as d. Hence

$$t = \frac{d}{\sqrt{1+d^2}}$$

represents the directed distance OV, and the absolute value of this is the length of the semimean axis.

It has been shown that if p is any plane $ax+by+cz=d$, where $d\neq 0$ and $a^2+b^2+c^2=1$, it is transformed by T into a hemiellipsoid h whose shape, size, and position can be described as follows: (1) h is a surface of revolution interior to S, (2) the plane through O parallel to p meets S in the boundary of h, (3) h lies between this plane and p, (4) the line through O perpendicular to p is the axis of symmetry of h, (5) the point lying on this line, between O and p, and distant from O by $|d|/\sqrt{1+d^2}$, is the vertex V of h.

This description provides steps whereby h can be determined when p is given. Conversely, if a hemiellipsoid h interior to S is given whose boundary is a great circle of S and whose semimean axis OV has length $k<1$, there is a plane p which transforms into h. To determine it we extend OV beyond V to a point A such that $OA=k/\sqrt{1-k^2}$. The plane perpendicular to OA at A is p. In other

words, T establishes a 1-1 correspondence between all such hemiellipsoids and all planes not through O.

It is to be noted that a family of parallel planes is transformed into a single circular disc and all the hemiellipsoids "based" on it, that is, having a common boundary with it.

3. The effect of T on straight lines. Some agreements on terms will prove useful. We shall call the segment OV of the hemiellipsoid h its semimean axis, and by its semimajor and semiminor axes we shall mean the semimajor and semininor axes of the ellipsoid of which it is part. Each meridian curve on a hemiellipsoid is half of an ellipse whose major axis is a diameter of S and whose semiminor axis is the segment OV. We shall call the meridian curve a *semiellipse*, and by its semimajor and semiminor axes we shall mean the semimajor and semiminor axes of the ellipse.

Let g be any straight line in space. We first suppose it goes through O. Consider two planes containing g. These planes go through O, and hence each is transformed by T into that part of itself which is interior to S. Thus g is transformed into that part of itself which is interior to S, *i.e.*, a diameter of S exclusive of its endpoints. Conversely, each diameter of S, exclusive of its endpoints, is the transform of some straight line, *i.e.*, the line containing it.

Now suppose that g does not go through O. Let A be the foot of the perpendicular from O to g, π the plane through A perpendicular to OA, and π' the plane through g and OA. Then π will transform into a semiellipsoid h whose vertex V is on line OA, and π' into a circular disc D containing segment OV. Now g, being the intersection of π and π', transforms into the intersection of h and D. Since D goes through OP, the axis of symmetry of h, it must meet h in a meridian curve. Hence g transforms into a semiellipse. Conversely, let s be any semiellipse (exclusive of its endpoints) whose major axis is a diameter of S and whose semiminor axis OV is therefore of length $k<1$. Extend OV beyond V to a point A such that $OA=k/\sqrt{1-k^2}$, and let π be the plane through A perpendicular to OA. If π' is the plane of s, it follows that the line of intersection of π and π' is transformed into s.

Thus, all the straight lines in space transform into the diameters of S (excluding their endpoints) and the semiellipses (excluding their endpoints) having these diameters as major axes, and every such diameter or semiellipse is the transform of some straight line. A family of parallel lines goes into a single diameter and all semiellipses "based" on it, *i.e.*, having it as major axis. When a plane π through O is transformed into the corresponding circular disc, the straight lines of π go into the diameters of the disc and the semiellipses based on them. A family of parallel lines in π goes into a single diameter of the disc and the semiellipses based on it.

4. A topological model of Euclidean space. It may be helpful to summarize our main results. The interior of unit sphere S with center at O is a topological model M of Euclidean 3-space in which

(a) the points interior to S correspond to the points of space;

(b) the discs of unit radius with center at O and the hemiellipsoids based on them correspond to the planes of space (a single disc and the hemiellipsoids based on it corresponding to the family of planes parallel to the disc);

(c) the diameters of S and the semiellipses based on them correspond to the straight lines of space (a single diameter and the semiellipses based on it corresponding to the family of lines parallel to the diameter).

In particular, each disc is a topological model of a Euclidean plane in which a diameter of the disc and the semiellipses based on it correspond to the family of straight lines in the plane parallel to the diameter.

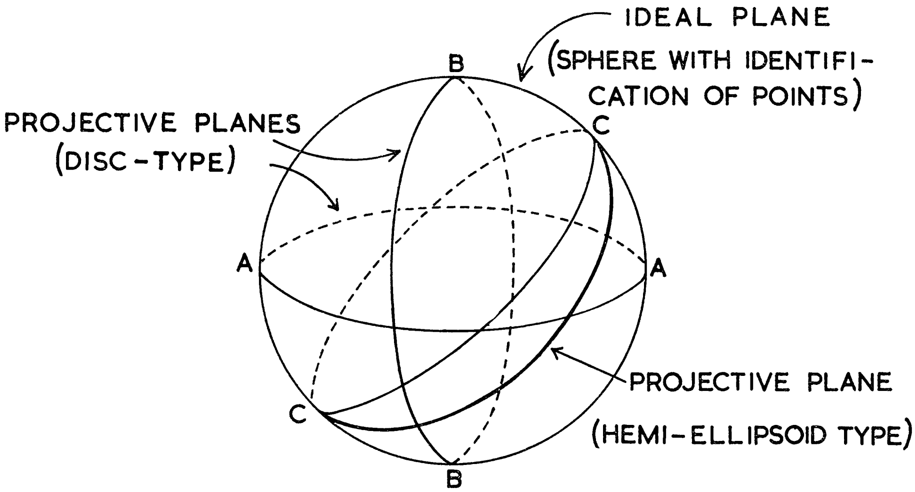

FIG. 1. Spherical-type model of projective 3-space.

5. A topological model of projective space. To the above model M of Euclidean space let us now add the points of S, that is, the points on the surface of S, and regard each two diametrically opposite points as identical. The resulting aggregate is a topological model M' of projective 3-space, for this addition of points is equivalent to the usual extension of Euclidean space by means of ideal points. Thus, with the specified identification of points, the surface of S and its great circles become a projective plane and its projective lines, corresponding to the ideal plane and its ideal lines. By the addition of the points of S the diameters and semiellipses of M are converted into projective lines, the discs and hemiellipsoids into projective planes (Fig. 1). A family in M consisting of a diameter and the semiellipses based on it (corresponding to a family of parallel straight lines in space) is converted into a family of projective lines on one point of S. A family in M consisting of a disc and the hemiellipsoids based on it (corresponding to a family of parallel planes in space) is converted into a family of

projective planes on one projective line of S. It is clear that the projective lines and planes of which we have been talking enjoy not only the foregoing properties but all of the other incidence relations possessed by the projective lines and planes of projective 3-space.

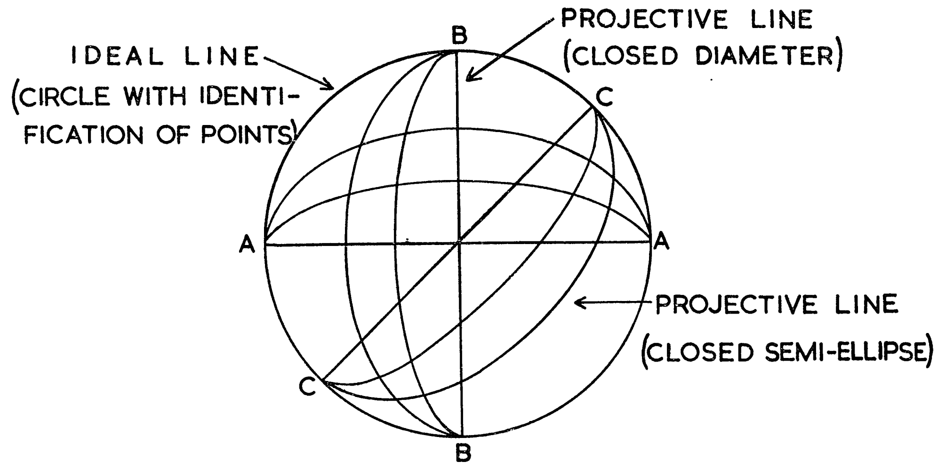

FIG. 2. Disc-type model of projective plane.

Incidentally, in adding to each disc of M its circular boundary we have obtained the two-dimensional projective model mentioned at the end of the introduction of this article. This is shown in Figure 2, where A, B, C are points of M.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

A METRIC FOR THE SPACE OF FUNCTION ELEMENTS

ANDREW M. GLEASON, Harvard University

When considering the problem of analytic continuation for functions of a complex variable, it is convenient to introduce the set $\mathcal{E}$ of all function elements, where a function element is a power series $\sum_{n=0}^{\infty} a_n(\zeta - z)^n$ with positive or

infinite radius of convergence R. $\mathcal{E}$ becomes a topological space when neighborhoods are defined in terms of analytic continuation and the connected components of $\mathcal{E}$ are the Riemann surfaces associated with complete analytic functions. From this fact it is clear that $\mathcal{E}$ is metrizable, but it is not clear that a metric can be easily defined. We shall exhibit a metric for $\mathcal{E}$ which has the very desirable property that the natural projection of $\mathcal{E}$ into the complex plane is a local isometry. Our metric will have the slight disadvantage that two points may be at distance ∞, but this does not vitiate any of the usual theorems about metric spaces; indeed, we can find a finite metric by setting $\rho'(a, b) = \min(1, \rho(a, b))$, but this new metric will not be quite as desirable as the old one.

By a function element we mean a sequence $e = \{z; a_0, a_1, \cdots\}$ of complex numbers where $\lim\sup |a_n|^{1/n} < \infty$. We identify e with the power series $\sum a_n(\zeta - z)^n$ which defines an analytic function (of ζ) in the open disk about z of radius $R(e) = \lim\inf |a_n|^{-1/n}$($= \infty$ possibly); we shall refer to this disk as the e-disk. Let $\mathcal{E}$ be the set of all function elements.

Define complex valued functions on $\mathcal{E}$ by

$$p(e) = z,$$
$$F(e) = a_0, \qquad F'(e) = a_1, \cdots, F^{(n)}(e) = n!a_n, \cdots.$$

Here p is the natural projection of $\mathcal{E}$ onto the complex plane which takes a power series into its center and the functions $F, F', \cdots$ evaluate the function defined by a power series and its derivatives.

We shall say two function elements e_1 and e_2 are *adjacent* if their disks overlap, and in the overlapping region they define the same function. Since the intersection of two disks is always connected, in order that e_1 and e_2 be adjacent it is sufficient that they define functions which agree on some open set.

Define a function on $\mathcal{E} \times \mathcal{E}$ by

$$\begin{aligned} \rho(e_1, e_2) &= R(e_1) + R(e_2) && \text{if } e_1 \text{ and } e_2 \text{ are not adjacent,} \\ &= |p(e_1) - p(e_2)| && \text{if } e_1 \text{ and } e_2 \text{ are adjacent.} \end{aligned}$$

This function takes extended nonnegative real values. We shall prove shortly that it is a metric for the set $\mathcal{E}$. It is entirely clear that ρ is symmetric and $\rho(e_1, e_2) = 0$ if and only if $e_1 = e_2$; hence we need only prove the triangle inequality. We note in passing that, in any event $\rho(e_1, e_2) \leqq R(e_1) + R(e_2)$, since when disks overlap their centers are closer than the sum of the radii.

LEMMA 1. *For all $e_1, e_2 \in \mathcal{E}$, $R(e_1) + \rho(e_1, e_2) \geqq R(e_2)$.*

Proof. There is nothing to prove unless $\rho(e_1, e_2) < R(e_2)$. In this case $p(e_1)$ falls in the e_2-disk because $\rho(e_1, e_2) = |p(e_1) - p(e_2)|$ and e_1 is obtained by developing the e_2-function about the point $p(e_1)$ since e_1 and e_2 are adjacent. The e_1-disk certainly contains the largest disk D with center $p(e_1)$ and contained in the e_2-disk. Since D has radius $R(e_2) - |p(e_1) - p(e_2)| = R(e_2) - \rho(e_1, e_2)$ we have $R(e_1) \geqq R(e_2) - \rho(e_1, e_2)$.

LEMMA 2. *Let D_1, D_2, D_3 be disks in the complex plane with centers z_1, z_2, z_3 and radii R_1, R_2, R_3, respectively. If $|z_1-z_2|<R_1+R_2$, $|z_2-z_3|<R_2+R_3$, and $|z_1-z_2|+|z_2-z_3|<R_1+R_3$, then $D_1\cap D_2\cap D_3$ is not void.*

Proof. This is trivial if $z_2\in D_1$ and $z_2\in D_3$. On the other hand, we cannot have both $z_2\notin D_1$ and $z_2\notin D_3$ since then $|z_1-z_2|\geqq R_1$ and $|z_2-z_3|\geqq R_3$ contradicting the third hypothesis. By symmetry, we may assume then, $z_2\in D_1$ but $z_2\notin D_3$. Consider the point x on the segment z_2z_3 such that $|x-z_3|=R_3$. We have $R_1+R_3>|z_1-z_2|+|z_2-z_3|=|z_1-z_2|+|z_2-x|+|x-z_3|$ so $R_1>|z_1-z_2|+|z_2-x|\geqq|z_1-x|$ and therefore $x\in D_1$. Moreover $|x-z_2|=|z_2-z_3|-R_3<R_2$ so $x\in D_2$. Since $x\in\overline{D}_3$ by construction and $D_1\cap D_2$ is a neighborhood of x, $D_1\cap D_2\cap D_3$ is not void.

We are now in a position to prove the triangle inequality for ρ; that is, $\rho(e_1, e_2)+\rho(e_2, e_3)\geqq\rho(e_1, e_3)$. There are three cases. If e_2 is not adjacent to e_1, then $\rho(e_1, e_2)+\rho(e_2, e_3)=R(e_1)+R(e_2)+\rho(e_2, e_3)\geqq R(e_1)+R(e_3)\geqq\rho(e_1, e_3)$ using Lemma 1, and similarly if e_2 is not adjacent to e_3. Finally we consider the possibility that e_2 is adjacent to both e_1 and e_3. In that case $\rho(e_1, e_2)=|p(e_1)-p(e_2)|$ and $\rho(e_2, e_3)=|p(e_2)-p(e_3)|$, so if $|p(e_1)-p(e_2)|+|p(e_2)-p(e_3)|\geqq R(e_1)+R(e_3)$ the required result follows immediately. On the other hand, if $|p(e_1)-p(e_2)|+|p(e_2)-p(e_3)|<R(e_1)+R(e_3)$, then Lemma 2 applies to the e_1-, e_2-, and e_3-disks and we conclude that these disks mutually overlap. Since both e_1 and e_3 agree with e_2 on the common part, e_1 and e_3 are adjacent so $\rho(e_1, e_3)=|p(e_1)-p(e_3)|$ and the required inequality is the familiar triangle inequality in the complex plane.

The metric ρ defines a topology for the set $\mathcal{E}$. The following facts about the space $\mathcal{E}$ are obvious from the definition of ρ. The projection p of $\mathcal{E}$ onto the complex plane carries the ball about e of radius $R(e)$ homeomorphically onto the e-disk and even maps the ball of radius $\frac{1}{2}R(e)$ isometrically, for any two elements in the latter ball are adjacent. The functions $F^{(n)}$ are all continuous since locally at e they can be represented in the form $f^{(n)}\circ p$ where f is the function defined by e. The function R is continuous since $|R(e_1)-R(e_2)|\leqq\rho(e_1, e_2)$ by Lemma 1. From these facts it is clear that ρ defines the usual topology for $\mathcal{E}$.

SECOND DERIVATIVES ON LEVEL SURFACE ELEMENTS

SIDNEY F. MACK, The Pennsylvania State University

Let $F(x, y, z)$ denote a function of class C_1 in some domain D in space which is constant on a level surface element S contained in D. It is well known that at every interior point P of S the first derivative of F in the direction tangent to S at P vanishes. In this paper, with F of class C_2, we give some formulas for the second derivatives on level surfaces, which show how the derivatives are related to the geometric properties of the surfaces. In the following, the directions x and y are chosen tangent to S at P, k_x is the normal curvature of S at P in the x direction, T_x is the geodesic torsion of S at P in the x direction, K is the mean curvature of S at P, and ∇^2F is the Laplacian of F.

THEOREM. *Let F be of class C_2 in a domain D in space and let S be a level surface element of F which is contained in D. With x, y, n denoting a tangent-normal system of rectangular coordinates at the interior point P of S, we have*

$$\frac{\partial^2 F}{\partial x^2} = -k_x \frac{\partial F}{\partial n}, \tag{1}$$

$$\frac{\partial^2 F}{\partial x \partial y} = T_x \frac{\partial F}{\partial n}, \tag{2}$$

$$\frac{\partial^2 F}{\partial n^2} = 2K \frac{\partial F}{\partial n} + \nabla^2 F. \tag{3}$$

Proof. Let the surface element S be given by $z = f(x, y)$ with reference to the coordinates x, y in the tangent plane at P, and let α, β, ζ denote a tangent-normal system of rectangular coordinates at the arbitrary point Q of S. Since F is constant on S and S is of class C_2, we find that $(\partial F/\partial \alpha)_Q = (\partial F/\partial \beta)_Q = 0$. From this, the general expression

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \alpha} \cos(x, \alpha) + \frac{\partial F}{\partial \beta} \cos(x, \beta) + \frac{\partial F}{\partial \zeta} \cos(x, \zeta)$$

reduces on S to

$$\left(\frac{\partial F}{\partial x}\right)_Q = \left(\frac{\partial F}{\partial \zeta}\right)_Q \cos(x, \zeta), \qquad Q \in S. \tag{4}$$

Let M be a point of the x-axis and Q its projection on S parallel to the z-axis. Then

$$\left(\frac{\partial F}{\partial x}\right)_Q = \left(\frac{\partial F}{\partial x}\right)_M + \overline{MQ}\left(\frac{\partial^2 F}{\partial x \partial z}\right)_{M'}, \tag{5}$$

where M' is some point between M and Q.

Now, since $(\partial F/\partial x)_P = 0$, we have

$$\left(\frac{\partial^2 F}{\partial x^2}\right)_P = \lim_{M \to P} \frac{1}{\overline{PM}}\left[\left(\frac{\partial F}{\partial x}\right)_M - \left(\frac{\partial F}{\partial x}\right)_P\right] = \lim_{M \to P} \frac{1}{\overline{PM}}\left(\frac{\partial F}{\partial x}\right)_M \cdot$$

Making use of (4) and (5) and the fact that $\overline{MQ}$ vanishes like $\overline{PM}^2$ while $\partial^2 F/\partial x \partial z$ remains bounded, we arrive at

$$\left(\frac{\partial^2 F}{\partial x^2}\right)_P = \lim_{M \to P} \frac{1}{\overline{PM}}\left(\frac{\partial F}{\partial \zeta}\right)_Q \cos(x, \zeta).$$

But,

$$\lim_{M \to P} \frac{\cos(x, \zeta)}{\overline{MP}} = \left(\frac{\partial}{\partial x} \cos(x, \zeta)\right)_P$$

which, in view of

$$\cos(x, \zeta) = \frac{-(\partial f/\partial x)_Q}{\sqrt{1 + (\partial f/\partial x)_Q^2 + (\partial f/\partial y)_Q^2}},$$

$$\left(\frac{\partial f}{\partial x}\right)_P = \left(\frac{\partial f}{\partial y}\right)_P = 0, \qquad \left(\frac{\partial^2 f}{\partial x^2}\right)_P = k_x(P),$$

yields formula (1).

The derivation of the second formula follows from analogous considerations; we use here the relation $T_x(P) = -(\partial^2 f/\partial x \partial y)_P$. The third formula is obtained directly from Laplace's operator, upon making use of formulas for $(\partial^2 F/\partial x^2)_P$ and $(\partial^2 F/\partial y^2)_P$ in addition to the relation $k_x(P) + k_y(P) = 2K(P)$.

The tangent-normal derivative $\partial^2 F/\partial x \partial n$ does not seem to be related to the second-order geometric properties of the level surface element.

The author is greatly indebted to Professor Griffith C. Evans for suggesting the method employed in obtaining the formulas in this paper.

ON THE CAUCHY-LIPSCHITZ THEOREM

GIDEON PEYSER, Pratt Institute, Brooklyn

Our purpose is to present a proof of the classical Cauchy-Lipschitz theorem concerning the uniqueness and existence of the solution of the ordinary differential equation of first order. Our methods differ from well-established classical procedures. These methods lend themselves to various other fields in analysis. Uniqueness is proved by solving an integral inequality. For the existence we use the usual Picard iteration but introduce an exponential contracting factor to prove convergence. We also use the described methods to obtain an *a priori* estimate of the solution.

The Cauchy-Lipshitz theorem states:

If $f(x, y)$ is defined in the rectangle R, $x_0 \leqq x \leqq x_0 + a$, $y_0 - b \leqq y \leqq y_0 + b$, continuous in R with respect to x and satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leqq A |y_1 - y_2|$$

for all pairs y_1, y_2 uniformly in x, then there exists a uniquely determined function $y(x)$ with $y(x_0) = y_0$ satisfying the differential equation

$$dy/dx = f(x, y) \tag{1}$$

in the interval $x_0 \leqq x \leqq x_0 + h$ with $h = \min(a, b/M)$ where $M = \max_{(x,y)\in R} |f(x, y)|$.

Proof of uniqueness. If $u(x)$ and $w(x)$ are two solutions of (1) with $u(x_0) = w(x_0) = y_0$, then

$$|u(x) - w(x)| \leqq \int_{x_0}^{x} |f(\xi, u(\xi)) - f(\xi, w(\xi))| \, d\xi \leqq A \int_{x_0}^{x} |u(\xi) - w(\xi)| \, d\xi. \tag{2}$$

Denoting the right hand side of (2) by $W(x)$, we have $W(x) \geqq 0$ and $W(x_0)=0$ and $dW/dx \leqq AW$.

Therefore $e^{-Ax}dW/dx - Ae^{-Ax}W \leqq 0$ and $d/dx(e^{-Ax}W) \leqq 0$. Integrating from x_0 to x we have $0 \leqq e^{-Ax}W \leqq 0$ and therefore $W \equiv 0$. From (2) follows $u(x) \equiv w(x)$.

Proof of existence. Following the classical Picard iteration, we define

$$\begin{aligned} y_0(x) &= y_0, \\ y_{n+1}(x) &= y_0 + \int_{x_0}^{x} f(\xi, y_n(\xi))d\xi. \end{aligned} \tag{3}$$

The restriction $h \leqq b/M$ assures us that all operations take place in R and therefore the process can be carried out.

To establish uniform convergence of the iteration process we proceed as follows:

$$| y_{n+1} - y_n | \leqq \int_{x_0}^{x} | f(\xi, y_n) - f(\xi, y_{n-1}) | d\xi \leqq A \int_{x_0}^{x} | y_n - y_{n-1} | d\xi. \tag{4}$$

Multiply both sides of (4) by $e^{-\lambda x}$ where λ is a positive constant at our disposal. Integrate both sides from x_0 to x using integration by parts on the right. We get

$$\int_{x_0}^{x} e^{-\lambda\xi} | y_{n+1} - y_n | d\xi \leqq A/\lambda \int_{x_0}^{x} e^{-\lambda\xi} | y_n - y_{n-1} | d\xi. \tag{5}$$

Repeating the same argument with the right side of (5) we have, choosing λ such that $A/\lambda = q < 1$,

$$\int_{x_0}^{x} e^{-\lambda\xi} | y_{n+1} - y_n | d\xi \leqq q^n \int_{x_0}^{x} e^{-\lambda\xi} | y_1 - y_0 | d\xi \leqq Bq^n.$$

Therefore

$$\begin{aligned} | y_{n+1} - y_n | &\leqq \int_{x_0}^{x} | f(\xi, y_n) - f(\xi, y_{n-1}) | d\xi \leqq A \int_{x_0}^{x} | y_n - y_{n-1} | d\xi \\ &\leqq Ae^{\lambda(x_0+h)} \int_{x_0}^{x} e^{-\lambda\xi} | y_n - y_{n-1} | d\xi \leqq Cq^{n-1}. \end{aligned} \tag{6}$$

Therefore

$$| y_{n+m} - y_n | \leqq | y_{n+m} - y_{n+m-1} | + \cdots + | y_{n+1} - y_n | \leqq Cq^{n-1}/(1 - q)$$

and $y_n(x)$ approaches uniformly a continuous function $y(x)$ which satisfies

$$y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))d\xi.$$

This completes the existence proof.

A proof similar to the uniqueness proof yields an estimate of the solution of equation (1):

$$|y(x)-y_0| \leqq e^{Ax}\int_{x_0}^{x} e^{-A\xi}|f(\xi, y_0)|\,d\xi \qquad x_0 \leqq x \leqq x_0+h. \tag{7}$$

Proof.

$$\begin{aligned} |y(x)-y_0| &\leqq \int_{x_0}^{x} |f(\xi, y(\xi))|\,d\xi \\ &\leqq \int_{x_0}^{x} |f(\xi, y(\xi)) - f(\xi, y_0)|\,d\xi + \int_{x_0}^{x} |f(\xi, y_0)|\,d\xi \\ &\leqq A\int_{x_0}^{x} |y(\xi)-y_0|\,d\xi + \int_{x_0}^{x} |f(\xi, y_0)|\,d\xi. \end{aligned} \tag{8}$$

Denoting the right hand side of (8) by $V(x)$, we have $V(x) \geqq 0$ and $V(x_0)=0$ and $dV/dx - |f(x, y_0)| \leqq AV$. Therefore $e^{-Ax}dV/dx - Ae^{-Ax}V \leqq e^{-Ax}|f(x, y_0)|$ and $dV/dx(e^{-Ax}V) \leqq e^{-Ax}|f(x, y_0)|$. Integrating from x_0 to x,

$$e^{-Ax}V \leqq \int_{x_0}^{x} e^{-A\xi}|f(\xi, y_0)|\,d\xi$$

and from (8) follows

$$|y(x)-y_0| \leqq e^{Ax}\int_{x_0}^{x} e^{-A\xi}|f(\xi, y_0)|\,d\xi.$$

It is clear that the above methods can also be used for systems of first order ordinary differential equations.

NOTE ON THE CONVERGENCE OF SERIES OF ULTRASPHERICAL POLYNOMIALS

D. P. Gupta, University of Saugar, Sagar, India

1. For the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \tag{1.1}$$

Kolmogoroff ([1], p. 109) has proved the following theorem:

If $a_n \to 0$ and the sequence $\{a_n\}$ is quasiconvex, the series (1.1) *converges, save for $x=0$, to an integrable function $f(x)$, and is the Fourier series of $f(x)$.*

Zitarosa [2] has proved a similar theorem for the series of Legendre polynomials. In the present note we establish the following more general result, which includes Zitarosa's theorem as a particular case for $\lambda=1/2$.

THEOREM. *If* $a_n = o(n^\mu)$ *and the sequence*

$$\left\{\frac{a_n}{(n+\lambda)^\mu}\right\} \tag{1.2}$$

is of bounded variation for $0<\mu<\lambda$ *and* $0<\lambda\leqq 1/2$, *then the series*

$$\sum_{n=0}^{\infty} a_n P_n^{(\lambda)}(x) \tag{1.3}$$

converges uniformly in the interval $(-1+\delta,\ 1-\delta)$, $P_n^{(\lambda)}(x)$ *being the ultraspherical polynomial of order* λ.

In view of Exercise 1 ([1], p. 129), the above theorem can be stated in the following form:

If $\{a_n/(n+\lambda)^\mu\}$ *tends to* 0 *and is quasiconvex for* $0<\mu<\lambda$ *and* $0<\lambda\leqq 1/2$, *then the series* (1.3) *is uniformly convergent in the range* $(-1+\delta,\ 1-\delta)$.

2. *Proof*. From Szegö ([3], p. 82) we have

$$\sum_{\nu=0}^{n} (\nu+\lambda)P_\nu^{(\lambda)}(x) = \frac{1}{2}\,\frac{(n+2\lambda)P_n^{(\lambda)}(x) - (n+1)P_{n+1}^{(\lambda)}(x)}{1-x}\,. \tag{2.1}$$

Also, Szegö ([3], p. 166) has proved that for $0<\lambda\leqq 1/2$ and $\epsilon\leqq\theta\leqq\pi-\epsilon$,

$$\left| P_n^{(\lambda)}(\cos\theta)\right| < A(\epsilon)n^{\lambda-1} \tag{2.2}$$

where $A(\epsilon)$ denotes a constant for a fixed $\epsilon>0$, not necessarily the same at each occurrence.

Hence,

$$\begin{aligned}\left|\sum_{\nu=0}^{n}(\nu+\lambda)P_\nu^{(\lambda)}(x)\right| &< \frac{1}{2}\,\frac{(n+2\lambda)\left|P_n^{(\lambda)}(x)\right| + (n+1)\left|P_{n+1}^{(\lambda)}(x)\right|}{1-x}\\ &< \frac{1}{2}A(\epsilon)\left\{\frac{(n+2\lambda)n^{\lambda-1}+(n+1)^\lambda}{1-x}\right\}\\ &< A(\epsilon)\cdot n^\lambda.\end{aligned} \tag{2.3}$$

Consider now the series

$$\sum_{\nu=0}^{n}(\nu+\lambda)^\mu P_\nu^{(\lambda)}(x) \qquad \text{for } \mu<\lambda.$$

Let

$$\sum_{\nu=0}^{n}(\nu+\lambda)^\mu P_\nu^{(\lambda)}(x) = \sum_{\nu=0}^{n} c_\nu\phi_\nu(x),$$

where $c_\nu = (\nu+\lambda)^{\mu-1}$ and $\phi_\nu(x) = (\nu+\lambda)P_\nu^{(\lambda)}(x)$. Write

$$\Phi_n(x) = \sum_{\nu=0}^{n} \phi_\nu(x) = \sum_{\nu=0}^{n} (\nu+\lambda)P_\nu^{(\lambda)}(x).$$

Using Abel's transformation, we have

$$\left| \sum_{\nu=0}^{n} c_\nu \phi_\nu(x) \right| = \left| \sum_{\nu=0}^{n-1} \Phi_\nu(x)\{(\nu+\lambda)^{\mu-1} - (\nu+\lambda+1)^{\mu-1}\} + \Phi_n(x)(n+\lambda)^{\mu-1} \right|$$

$$\leqq \sum_{\nu=0}^{n-1} |\Phi_\nu(x)| \, |(\nu+\lambda)^{\mu-1} - (\nu+\lambda+1)^{\mu-1}| + |\Phi_n(x)| \, |(n+\lambda)^{\mu-1}|$$

$$(2.4) \qquad < A(\epsilon)\left\{ \sum_{\nu=0}^{n-1} \nu^\lambda |(\nu+\lambda)^{\mu-1} - (\nu+\lambda+1)^{\mu-1}| + n^\lambda(n+\lambda)^{\mu-1} \right\}$$

$$< A(\epsilon)\left\{ \sum_{\nu=0}^{\infty} \nu^\lambda(\nu+\lambda)^{\mu-2}\cdot B + O(1) \right\} \qquad (B \text{ being a constant}).$$

$$< A(\epsilon).$$

(1.3) is
$$\sum_{n=0}^{\infty} a_n P_n^{(\lambda)}(x) = \sum_{n=0}^{\infty} \frac{a_n}{(n+\lambda)^\mu}(n+\lambda)^\mu P_n^{(\lambda)}(x)$$

$$= \sum_{n=0}^{\infty} b_n\{c_n\phi_n(x)\}, \text{ say.}$$

The sequence $\{b_n\} \to 0$ and is of bounded total fluctuation, and $\sum c_n\phi_n(x)$ has its partial sums uniformly bounded in view of (2.4). Thus the theorem is proved by virtue of Abel's criterion for uniform convergence of series.

It should be interesting to solve the corresponding problem for Jacobi polynomials. I am thankful to Dr. B. N. Prasad for his kind help in the preparation of this note.

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

A CALCULUS PROBLEM WITH OVERTONES IN RELATED FIELDS

C. S. OGILVY, Hamilton College

Several elementary extremum problems in the calculus which seem at first glance to be quite distinct from one another can be linked together through considerations not usually presented in the texts.

I. One ship, A, is anchored 9 miles offshore. Opposite a point 6 miles down the coast a second ship, B, is anchored 3 miles offshore. A boat from A is to put a passenger ashore and then proceed to B (Fig. 1).

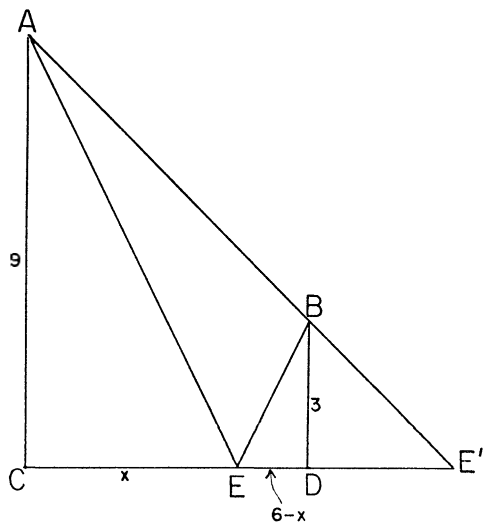

FIG. 1

(a) What is the shortest course for the boat?

(b) Suppose the passenger pays each member of the boat's crew \$1 per mile for every mile they carry him; but each member of the crew must forfeit \$1 for every mile travelled from shore to B. What is the course which will yield them the maximum profit, under the restriction that it must consist of two straight-line segments?

II. A man can row 4 miles per hour and run 5 miles per hour. He leaves ship B of problem I wishing to go to point Q on the shore 5 miles from D (Fig. 2).

(a) Where should he land in order to reach Q most quickly?

These do not look to the student like connected problems. I(a) seeks a minimum distance, I(b) a maximum profit, and II(a) a minimum time. We observe first that the two (a) parts are both solved by applying Fermat's principle of optics, where in I the shore is a reflector for light travelling from A to B and in II the shore is a refractor with light travelling from the slower medium into the faster. From this point of view, both problems do minimize the same variable, time. In the first instance, the physicist seeks two similar triangles so that the angle of incidence will equal the angle of reflection: $9/x=3/(6-x)$, or $x=CE=4.5$ miles. In II(a) the light must travel in conformance with Snell's Law: $\sin\alpha/\sin\beta=4/5$. But this is the limiting case where $\beta=\pi/2$, which means $\sin\alpha=4/5$, or $DS=4$ miles.

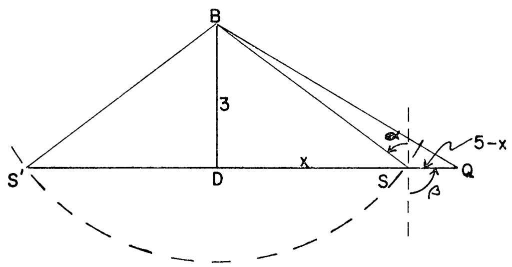

FIG. 2

We turn now to the calculus method of handling these two problems.

I(a). The distance function to be minimized is $\sqrt{81+x^2}+\sqrt{9+(6-x)^2}$. Setting the first derivative equal to zero gives immediately the equation

$$\frac{x}{\sqrt{81+x^2}}=\frac{6-x}{\sqrt{9+(6-x)^2}}. \tag{1}$$

The "solutions" of (1) are $x=4.5$, 9. The student throws away the second value because he feels pretty certain that $x<6$. This is the wrong reason for throwing it away. The function is not stationary at $x=9$ and indeed 9 does not satisfy (1). It is purely extraneous, introduced by squaring in the course of solving (1). But although "extraneous" roots may have no place in algebra, there is usually some reason for their appearance in the calculus. We now discover the meaning of this one.

I(b). The profit function to be maximized is $\sqrt{81+x^2}-\sqrt{9+(6-x)^2}$, which leads, on differentiation, to

$$\frac{x}{\sqrt{81+x^2}}=\frac{x-6}{\sqrt{9+(6-x)^2}}. \tag{2}$$

Observe that when we square both sides of (2) it is identical with the equation resulting from squaring both sides of (1). Hence solutions again appear to be 4.5 and 9; but this time 9 satisfies (2) and 4.5 does not. The crew should land the passenger at E'.

At this stage we can formulate a General Remark:

If two problems requiring different solutions lead at some stage to the same equation, then this equation must deliver at least two solutions, one of which will be extraneous to each problem.

II(a). The time function to be minimized is $\sqrt{9+x^2}/4+(5-x)/5$. Setting the first derivative equal to zero we get

$$\frac{x}{\sqrt{9+x^2}} = \frac{4}{5}, \tag{3}$$

or $x = \pm 4$. As before, only one of the two values of x is admissible. It is $+4$ which satisfies (3). And now our General Remark *predicts* the existence of a companion problem:

II(b). Suppose the man has to pay the owner of the boat \$1 for each hour he rows, but some altruistic soul pays him \$1 per hour for each hour he runs. What now is his most profitable course, assuming still that he runs only straight along the shore after landing?

The profit function to be maximized is $-\sqrt{9+x^2}/4+(5-x)/5$, which yields

$$\frac{x}{\sqrt{9+x^2}} = -\frac{4}{5}, \tag{4}$$

and only -4 satisfies (4). He should row to S'.

These results have interesting geometric interpretations. Equations (1) and (2) are proportions calling for similar right triangles. There are *two* pairs of similar right triangles formed by the ships and the shoreline, and E and E' give them both. Equations (3) and (4) ask for $\alpha=\sin^{-1}(\pm 4/5)=\cos^{-1}(3/5)$. This is the "ambiguous case" of high school trigonometry: triangles BQS and BQS' both satisfy the required conditions. Student reaction to these unexpected cross-connections with previously studied topics has been favorable.

We leave to the reader the statement of the more difficult problems III(a) and III(b) where the point Q is inland. Snell's Law this time provides no short-cut, and the solution is technically burdensome because the terms in x^3 and x^4 do not drop out in either the physics or the calculus method. It is believed that only two of the roots of the resulting equation are ever real, namely, the answers to III(a) and III(b) respectively.

I am indebted to Professor H. M. Gehman of the University of Buffalo for the statement of problem I(b).

sider the equation $x^2+2x+1=0$ with $r=-3/2$. In (B), the equation $x^2-5x+6=0$ with $r=4$ shows that we cannot conclude that roots will not exist in the interval $0 \leqq x < r$.

THE RAZOR'S EDGE?

C. S. OGILVY, Hamilton College AND N. G. GUNDERSON, University of Rochester

Most modern calculus texts are careful to point out that the vanishing of the first partial derivatives is a necessary *but not sufficient* condition for a maximum or minimum of a function of two variables. One must examine the second partials also. Failure to do so has led more than one writer to include in his textbook an apparently plausible problem which in fact has a strange solution.* Interestingly enough the same problem, with only slight modifications, has found its way into several of the prominent texts.

In one book the problem is stated as follows: "A manufacturer produces safety razors and blades at a cost of 20¢ per razor and 10¢ per dozen blades. If he charges x cents per razor and y cents per dozen blades, he finds that he can sell $100{,}000/xy$ razors and $400{,}000/xy$ dozen blades daily. How should he fix prices so as to maximize his profit?"

The profit z is readily expressed as a function of x and y: $z = f(x, y) = 100{,}000\,(1/y+4/x-60/xy)$. If one equates the two first partials to zero and solves the resulting two equations simultaneously, one arrives quite happily at $x=60$, $y=15$. These apparently reasonable figures yield a profit of \$66.67. But a closer examination of the profit function discloses some disturbing facts. The ordinary three-dimensional curved surface represented by $z=f(x, y)$ has negative Gaussian curvature at the point P: $(60, 15, 6666\frac{2}{3})$; that is, the surface is saddle-shaped at P, so that although the first partial derivatives are both zero, every neighborhood of P contains points where z has a greater value and points where z has a lesser value than at P. Furthermore the surface has two horizontal rulings through P, straight lines lying in the surface, one parallel to the X-axis and the other parallel to the Y-axis. This means that so long as $x=60$, z is independent of y, and if $y=15$, z is independent of x. The company could sell the blades for 15¢ a dozen and the razors at a penny apiece and make the same profit.

But we have worse in store. Let the blades be sold for more than 15¢ per dozen, and it matters not how little more—say at 16¢ per dozen. Now by selling the razors at a price arbitrarily close to zero, the profits can be made to soar indefinitely! We are drawn to the reluctant conclusion that there must be something wrong with this economic dream.

The trouble, of course, lies in the original data. The immediate concern of our manufacturer is not how best to fix prices, but how most quickly to put his sales

* This was initially called to our attention by Dr. Alfred W. Jones of Bell Telephone Laboratories.

manager on the carpet and try to find out where he dredged up this formula. No law of supply and demand ever behaved that way. The price of each commodity is varying inversely as the product of both together. This says that a change in the price of only one affects the sales of both of them in the same ratio. Such a relation can be realistically interpreted in only one way: no one buys razors without blades or blades without razors; and each sale of one razor carries with it the automatic sale of some fixed number of blades, and vice-versa. This is another way of saying that the price of the package is to be treated as a single variable. And if indeed one lets $y=kx$, so that the profit reduces to a function of one variable, the problem begins to make sense. Even so, however, $x=60$ and $y=15$ is not the best solution. It is the optimum for $k=\frac{1}{4}$; but because P is a saddle point *any* other k will produce a maximum profit higher than \$66.67. Now where did that sales manager disappear to?

The problem can be salvaged. A corrected version appears in a revised edition of one of the texts which formerly contained the faulty problem. In the revised problem, $100{,}000/xy$ is replaced by $100{,}000/x^2y$, and $400{,}000/xy$ is replaced by $400{,}000/xy^2$. Everything else remains the same. The surface represented by the new $z=f(x, y)$ has positive curvature in the region in question and a true maximum at the point $x=12$, $y=24$, an elliptic point. We now have the unexpected result that the razors are being sold below cost. This is not inconsistent; that it can happen to certain items of a line is well known to all manufacturers. But there is another objection. In the revised problem both the retail prices and the daily profit are too low to be realistic. If we write 1,000,000 in place of both the 100,000 and the 400,000 in the statement of the revised problem, we end up with more reasonable figures throughout. We leave the working of this final version to the reader or his calculus class.

A DEFINITE INTEGRAL

ALBERT WILANSKY, Lehigh University

To evaluate $\int_a^b x^p dx$ without use of the fundamental theorem of calculus, textbooks usually assume that the integral exists (appealing, say, to a general existence theorem) and then choose a definite sequence of partitions, using finally, in some cases, an identity involving $\sum k^p$. The following derivation will give the existence and value of the integral without assuming the existence first. In addition, an inequality, (3), is given connecting the integral and the Riemann sum for an arbitrary partition. The case $p=1$ is exceptionally easy and well motivated.

We assume that p is a positive integer. We begin with the identity

$$B^p(B-A) = \frac{B^{p+1}-A^{p+1}}{p+1} + (B-A)\left[B^p - \frac{B^p + B^{p-1}A + B^{p-2}A^2 + \cdots A^p}{p+1}\right].$$

A glance at the right-hand side verifies this. A suggestion for motivating it is

as follows: We consider a typical interval $[x_{k-1}, x_k]$, call it $[A, B]$. Then $B^p(B-A)$ is the area of a rectangle, $(B^{p+1}-A^{p+1})/(p+1)$ is the area under the curve $y=x^p$, and the remaining term is the area of the curvilinear triangle above the curve. This is no proof, of course. The identity is proved by checking it.

If we assume $A<B$, the quantity inside square brackets is less than

$$B^p - \frac{A^p + A^p + \cdots + A^p}{p+1} = B^p - A^p.$$

Thus we have

$$(1) \qquad B^p(B-A) < \frac{B^{p+1} - A^{p+1}}{p+1} + (B-A)(B^p - A^p) \qquad \text{if } A < B.$$

Similarly

$$(2) \qquad A^p(B-A) > \frac{B^{p+1} - A^{p+1}}{p+1} - (B-A)(B^p - A^p) \qquad \text{if } A < B.$$

Next consider a partition of $[a, b]$, $a=x_0<x_1<x_2 \cdots <x_n=b$, $x_{k-1}\leqq\xi_k\leqq x_k$, and the Riemann sum $\sum f(\xi_k)(x_k-x_{k-1}) = \sum \xi_k^p(x_k-x_{k-1})$ which we shall denote by $\sum$. Clearly

$$\sum x_{k-1}^p(x_k - x_{k-1}) \leqq \sum \leqq \sum x_k^p(x_k - x_{k-1}).$$

Apply (1), (2) with $A=x_{k-1}$, $B=x_k$; this yields

$$\frac{1}{p+1}\sum (x_k^{p+1} - x_{k-1}^{p+1}) - \sum (x_k - x_{k-1})(x_k^p - x_{k-1}^p)$$
$$\leqq \sum \leqq \frac{1}{p+1}\sum (x_k^{p+1} - x_{k-1}^{p+1}) + \sum (x_k - x_{k-1})(x_k^p - x_{k-1}^p),$$

i.e.,

$$\frac{x_n^{p+1} - x_0^{p+1}}{p+1} - \delta \sum (x_k^p - x_{k-1}^p) \leqq \sum \leqq \frac{x_n^{p+1} - x_0^{p+1}}{p+1} + \delta \sum (x_k^p - x_{k-1}^p),$$

where δ is the norm of the partition $(=\max (x_k-x_{k-1}))$. Hence

$$\frac{b^{p+1} - a^{p+1}}{p+1} - \delta(b^p - a^p) \leqq \sum \leqq \frac{b^{p+1} - a^{p+1}}{p+1} + \delta(b^p - a^p).$$

This yields

$$(3) \qquad \left| \sum - \frac{b^{p+1} - a^{p+1}}{p+1} \right| \leqq \delta(b^p - a^p)$$

and the result follows.

MATHEMATICAL EDUCATION NOTES

Edited by JOHN R. MAYOR, American Association for the Advancement of Science and the University of Maryland, and JOHN A. BROWN, University of Delaware

Contributions for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

NOTE ON DEGREES GRANTED IN 1956–57

A report issued recently by the Office of Education shows the number of bachelor's degrees in mathematics granted by 1,341 institutions of higher education in 1956–57 was 5,546, an increase of 19 per cent over the previous year. The total number of bachelor's degrees conferred in all fields showed an increase of 9 per cent for this period. The per cent increase in the other sciences was greater than the per cent increase in number of degrees in all fields but in no case as great as in mathematics. Mathematics was the only one of the sciences in which more than half (52.1 per cent) of the bachelor's degrees in 1956–57 were granted by publicly-controlled institutions as compared with privately-controlled institutions.

The report also includes data on second-level (master's) and doctorate degrees. Numbers of degrees at the three levels are tabulated by states and institutions of higher education within the states. The state and institutional tabulation is broken down into major fields of study. The 196-page report, prepared by the U. S. Department of Health, Education, and Welfare, Office of Education, can be obtained from the U. S. Government Printing Office, Washington 25, D. C. for $1.50.

SUNRISE SEMESTER AT NEW YORK UNIVERSITY*

WCBS-TV and New York University this fall will expand "Sunrise Semester," the early-morning educational television program, to include four college-credit courses. Beginning September 29, "Sunrise Semester" will be presented six hours a week, Monday through Friday from 6:30 to 7:30 and on Saturday from 7 to 8 A.M. The 1958–59 program, University spokesmen said, also will provide the opportunity for a closer relationship between the student and his instructor and will include optional discussion sessions on the NYU campus and in various suburban communities.

The four courses, the professors who will teach them, and their schedules are:

Classical Civilization H1, a history and literature course on the legacy of Greece and Rome, Dr. Casper J. Kraemer, Jr., professor of archaeology and classics, Monday, Wednesday, and Friday from 6:30 to 7 A.M.

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*Quoted from a News Release, July 16, 1958, Office of Information Services, New York University.

Mathematics N1, an introductory course, Dr. Hollis R. Cooley, professor of mathematics, Monday, Wednesday, and Friday from 7 to 7:30 A.M.

English H5, a "Great Books" course, Dr. David H. Greene, associate professor of English, Tuesday and Thursday from 6:30 to 7 A.M. and Saturday from 7 to 7:30 A.M.

Government S1, a study of the governmental process with particular emphasis on American theory and practice, Dr. Morley Ayearst, associate professor of government, Tuesday and Thursday from 7 to 7:30 A.M. and Saturday from 7:30 to 8 A.M.

Qualified undergraduate applicants can enroll for as many as four courses, earning three credits for each upon satisfactory completion of requirements that include home quizzes, term papers, and final examinations at NYU. Special arrangements will be made for physically-handicapped students. The courses, part of the basic requirements of the Program of Coordinated Liberal Studies of NYU's Washington Square College of Arts and Sciences, will be offered to television viewers under the supervision of Dr. Thomas Clark Pollock, dean of the College.

In addition to the home-television lectures, NYU will offer periodic discussion sessions at its Washington Square Center for each of the courses. Where registration warrants it, there also will be discussion sessions in suburban communities in Westchester County, New Jersey, Long Island, and Connecticut. The discussion groups will be led by the course professors or their departmental colleagues.

A student who takes all eight courses offered during the year can earn 24 academic credits toward a bachelor of arts degree. Persons who do not wish to enroll for degree credit, but who wish to take one or more courses and receive recognition for satisfactory completion, may register for certificate credit.

REPORT OF THE COMMISSION ON MATHEMATICS

ROBERT E. K. ROURKE, Executive Director, Commission on Mathematics

The Commission on Mathematics of the College Entrance Examination Board, appointed in 1955 for the purpose of improving the high school curriculum in college preparatory mathematics, published its report in December 1958.

The report is in two parts. The first part sets forth the proposals of the Commission and their reasons for making them; the second part consists of appendices designed to amplify certain of the Commission's recommendations and to provide useful materials for teachers.

To meet contemporary needs, the report presents the following nine-point program for college-capable students:

1. Strong preparation *both* in concepts *and* in skills for college mathematics at the level of calculus and analytic geometry.
2. Understanding of the nature and role of deductive reasoning—in algebra as well as in geometry.
3. Appreciation of mathematical structure ("patterns")—for example, properties of natural, rational, real, and complex numbers.

4. Judicious use of unifying ideas—set, variable, function, and relation.
5. Treatment of inequalities along with equations.
6. Incorporation with plane geometry of some coordinate geometry and also essentials of solid geometry and space perception.
7. Introduction in grade 11 of fundamental trigonometry—centered on coordinates, vectors, and complex numbers.
8. Emphasis in grade 12 on elementary functions (polynomial, exponential, circular).
9. Recommendation of additional alternative units for grade 12: *either* introductory probability with statistical applications *or* an introduction to modern algebra.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1341. *Proposed by A. A. Mullin, University of Illinois*

If $w=z^z$, $z=x+iy$, express $\operatorname{Re}(w)$ and $\operatorname{Im}(w)$ in terms of x and y.

E 1342. *Proposed by D. J. Newman, AVCO Research Division, Wilmington, Mass.*

If x and y are positive, prove that $x^y+y^x>1$.

E 1343. *Proposed by W. F. Cheney, University of Hartford, Connecticut*

Through an arbitrarily-selected point within a plane triangle, a straight line is drawn which bisects the area of the triangle. What is the probability of more than one solution for a given point?

E 1344. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

Let $\{x_n^{(1)}\}$, $\{x_n^{(2)}\}$, $\cdots$, $\{x_n^{(m)}\}$ be m sequences of nonzero real numbers. Prove that there exists an integer i $(1 \leqq i \leqq m)$ and an ascending sequence $\{n_r\}$

of positive integers ($r=1, 2, \cdots$) such that each of the sequences

$$\{x_{n_r}^{(i)}/x_{n_r}^{(1)}\}, \{x_{n_r}^{(i)}/x_{n_r}^{(2)}\}, \cdots, \{x_{n_r}^{(i)}/x_{n_r}^{(m)}\}$$

is convergent.

E 1345. *Proposed by I. J. Schoenberg, University of Pennsylvania*

There are given $2n$ points on the x-axis: $x_1<x_2< \cdots <x_{2n}$, $(n \geqq 1)$. For convenience we also write $x_0=x_1$, $x_{2n}=x_{2n+1}$ and determine, for each $k=1, \cdots, n$, a point ξ_k between x_{2k-1} and x_{2k} satisfying the linear equation

$$(\xi_k - x_{2k-2})(\xi_k - x_{2k-1}) = (\xi_k - x_{2k})(\xi_k - x_{2k-1}),$$

thus obtaining n points $\xi_1<\xi_2< \cdots <\xi_n$. Finally, let

$$\gamma_k = (x_{2k} + x_{2k+1})/2, \qquad k = 0, 1, \cdots, n.$$

Show that the relation

$$\sum_{k=1}^{n} (\gamma_k - \gamma_{k-1})f(\xi_k) = \int_{x_1}^{x_{2n}} f(x)dx$$

holds for any continuous function $f(x)$ which is linear in each of the intervals (x_k, x_{k+1}), $k=1, \cdots, 2n-1$.

SOLUTIONS

Covering a Square

E 1311 [1958, 284]. *Proposed by G. K. Wenceslas, Santa Monica, Calif.*

The dissection of a unit square into one $1/8\times 1$ and two $1/2\times 7/8$ rectangular pieces shows that it is possible to cover the unit square with three sets each having diameter $d=\sqrt{65}/8$. Prove that the unit square cannot be covered by three sets all of which have diameter less than d.

Solution by Joe Lipman, University of Toronto. Let the unit square (0, 0), (0, 1), (1, 1), (1, 0) be covered by three sets S_i of diameter $D_i<\sqrt{65}/8$, $i=1, 2, 3$. Since one of the S_i contains two vertices, suppose, without loss of generality, that S_1 contains (0, 0) and (0, 1). Then the region common to the circles of radius D_1 and having these vertices as centers contains S_1 but contains none of the points (1/8, 0), (1/8, 1), (1, 1/2). One of S_2, S_3, say S_2, must contain two of these points, and, D_2 being less than $\sqrt{65}/8$, the two points must be (1/8, 0) and (1/8, 1). The region common to the circles of radius D_2 and having these two points as centers contains S_2, but contains neither of the points (1/4, 0), (1, 1). These must then belong to S_3, but the distance between them is greater than $\sqrt{65}/8$, and so the situation is impossible.

Also solved by L. R. Bragg, Vern Hoggatt, A. R. Hyde, D. C. B. Marsh, C. F. Pinzka, L. A. Ringenberg, and the proposer.

Willie's Doodling

E 1312 [1958, 284]. *Proposed by C. F. Pinzka, University of Cincinnati*

"What kind of doodling is that?" I asked, noticing that Willie had the following steps on his paper:

(0) x_1y_2

(1) $x_3y_4y_5x_6$

(2) $x_7y_8y_9x_{10}y_{11}y_{12}y_{13}y_{14}$

"It's sequential doodling," replied Willie. "I get each new step from the last step by inserting an x after each y, a y after each x, and continuing the subscripts in order."

"Does your doodling have any interesting properties?" I asked in jest.

"Oh, yes," said Willie. "In step n you will find that the sum of the kth powers of the x subscripts equals the sum of the kth powers of the y subscripts for $k=0, 1, \cdots, n$. In fact, you can use any numbers in arithmetic progression for the subscripts and this property will still hold."

Prove that Willie's assertions hold for all (nonnegative integers) n.

Solution by I. C. Gentry, Wake Forest College, N. C. Let the subscripts be any numbers in arithmetic progression and let us employ mathematical induction. The cases $n=0$ and $n=1$ are easily verified. Assume Willie's statement is true for all cases up through the nth and consider the $(n+1)$th sequence

$$x_ay_{a+d}y_{a+2d}x_{a+3d} \cdots y_bx_{b+d}x_{b+2d}y_{b+3d} \cdots,$$

where $b=a+2^{n+1}d$ and the sequence contains 2^{n+2} terms. The induction hypothesis applies for $k=0, 1, \cdots, n$ to each half, and hence to the whole, of this sequence.

Now consider $k=n+1$. Set $D=2^{n+1}d$. Then for every x subscript s in the first half of the sequence there is a y subscript $s+D$ in the second half, and for every y subscript t in the first half there is an x subscript $t+D$ in the second half. Let S_x represent the sum of the $(n+1)$th powers of the x subscripts and let S_y represent the sum of the $(n+1)$th powers of the y subscripts. Then

$$S_x = \sum s^{n+1} + \sum (t + D)^{n+1}, \qquad S_y = \sum t^{n+1} + \sum (s + D)^{n+1}.$$

Expanding the terms in parentheses by the binomial theorem and using the fact that $\sum s^k = \sum t^k$ for $k \leqq n$, it now follows that $S_x = S_y$, and Willie's statement holds for the $(n+1)$th case.

Also solved by Peter Beisswanger, Joe Lipman, D. C. B. Marsh, Helen Marston, Benjamin Sapolsky, and the proposer.

Quadratic with Roots Exterior to the Unit Circle

E 1313 [1958, 274]. *Proposed by Roger Pinkham, Princeton University*

If $pz^2-qz+1=0$, then what values of p and q yield roots exterior to the unit circle?

Solution by Howard Eves, University of Maine. The assumption is that p and q are real. The given equation has its roots exterior to the unit circle if and only if the equation $z^2-qz+p=0$ has its roots interior to the unit circle. Denote the roots of this latter equation by α and β. Since $\alpha+\beta=q$, $\alpha\beta=p$, and p and q are real, it follows that α and β are both real or are non-real conjugate complex numbers.

Case 1. (α and β real)

Suppose $|\alpha|<1$, $|\beta|<1$. If α and β are both positive we have

$$(\alpha-1)(\beta-1)=\alpha\beta-(\alpha+\beta)+1>0, \qquad 0<\alpha\beta<1,$$

whence $0<p<1$ and $q<p+1$. If α and β are both negative we have

$$(\alpha+1)(\beta+1)=\alpha\beta+(\alpha+\beta)+1>0, \qquad 0<\alpha\beta<1,$$

whence $0<p<1$ and $-q<p+1$. If α, say, is positive and β negative we have

$$(\alpha-1)(\beta-1)=\alpha\beta-(\alpha+\beta)+1>0, \qquad -1<\alpha\beta<0,$$

whence $-1<p<0$ and $q<p+1$. If one root, say α, is zero, then $\beta=q$ and we have $p=0$, $|q|=|\beta|<1=p+1$. Summarizing the possible situations we see that if $|\alpha|<1$ and $|\beta|<1$, then $0\leqq|p|<1$ and $|q|<p+1$.

By a similar analysis, we can show that if we do not have both $|\alpha|<1$ and $|\beta|<1$, then either $|q|\geqq p+1$ or $|p|\geqq 1$. It follows that if $0\leqq|p|<1$ and $|q|<p+1$, then $|\alpha|<1$ and $|\beta|<1$.

Case 2. (α and β non-real conjugate complex numbers)

Set $\alpha=x+iy$, $\beta=\bar{\alpha}=x-iy$. Then $0<(|x|-1)^2+y^2=x^2+y^2+1-2|x|$, whence $|q|=|\alpha+\beta|=2|x|<x^2+y^2+1=p+1$. Now $|\alpha|=|\beta|<1$ if and only if $p=|\alpha\beta|=|\alpha|^2<1$.

The results established in cases 1 and 2 show that $pz^2-qz+1=0$ has its roots exterior to the unit circle if and only if $0\leqq|p|<1$ and $|q|<p+1$, that is, if and only if in the pq-plane the point (p, q) lies in the interior of the triangle whose vertices are $(1, 2)$, $(-1, 0)$, $(1, -2)$.

Also solved by D. S. Adorno, Merrill Barnebey, A. P. Boblétt, L. R. Bragg, J. L. Brown, Jr., A. G. Clark, E. L. Ellis and D. L. Muench (jointly), José Gallego-Diaz, Michael Goldberg, A. R. Hyde, D. C. B. Marsh, C. S. Ogilvy, J. D. Steben and H. W. Vayo (jointly), R. J. Wagner, Dale Woods, David Zeitlin, and the proposer.

In general these solutions derived various conditions on p and q necessary for the roots of the quadratic to be exterior to the unit circle, but failed to establish conditions which are both necessary and sufficient.

A Summation

E 1314 [1958, 284]. *Proposed by J. M. Gandhi, Jain Engineering College, Gurukul, Panchkoola, India*

Let $S(n, i)=\sum a_1\cdots a_i$, where the sum is taken over all possible integral choices of the a's such that $1\leqq a_1\leqq a_2\leqq\cdots\leqq a_i\leqq n$, and put $S(0, i)=0$, $S(n, 0)=1$. Prove that for all positive integral n

$$\sum_{k=0}^{n-1} (-1)^k (n-k)!S(n-k, k) = 1. \tag{1}$$

Solution by E. P. Starke, Rutgers University. Note first, from the formation of $S(n, i)$, the obvious relation

$$S(n, i) = nS(n, i-1) + S(n-1, i). \tag{2}$$

Now, in (1), replace each term $(n-j)!S(n-j, j)$ by its equal

$$(n-j+1)!S(n-j, j) - (n-j)!(n-j)S(n-j, j)$$

and recombine by taking the second part of each term with the first part of the following, obtaining

$$\begin{aligned}(n+1)!S(n, 0) - n!\{nS(n, 0) + S(n-1, 1)\} + \cdots \\ + (-1)^{j+1}(n-j)!\{(n-j)S(n-j, j) + S(n-j-1, j+1)\} \\ + \cdots + (-1)^n 1!S(1, n-1) = 1.\end{aligned}$$

But, by (2), this becomes at once

$$\begin{aligned}(n+1)!S(n+1, 0) - n!S(n, 1) + \cdots + (-1)^{j+1}(n-j)!S(n-j, j+1) \\ + \cdots + (-1)^n 1!S(1, n) = 1.\end{aligned}$$

Thus, since (1) is obvious for $n=1$, we have the proof by induction.

Also solved by E. L. Ellis, Joe Lipman, D. C. B. Marsh, C. F. Pinzka, and Benjamin Sapolsky. Late solution by J. H. Hodges.

A Composite Magic Square

E 1315 [1958, 285]. *Proposed by R. L. Caskey, Oklahoma State University*

Let M represent a magic square of order n, the elements of which are the positive integers from 1 to n^2. Let M_i represent the magic square obtained by replacing each element k of M by $k+(i-1)n^2$. Prove that the square array of elements obtained by replacing each element i of M by M_i is a magic square of order n^2.

Solution by C. F. Pinzka, University of Cincinnati. The row (column, diagonal) sum of M_i is $n(n^2+1)/2+(i-1)n^3=in^3-n(n^2-1)/2$. In the final square this sum is $[n(n^2+1)/2]n^3-n[n(n^2-1)/2]=n^2(n^4+1)/2$ as required. Furthermore, since $k+(i-1)n^2$ represents the consecutive integers from 1 to n^4 as k and i range from 1 to n^2, the resulting array is a magic square of order n^2.

If each element i of a magic square P of order p is replaced by M_i, the same argument shows the resulting array to be a magic square of order np. Kraitchik's *Mathematical Recreations* (1st ed., p. 170) discusses these composite magic squares.

Also solved by Tibor Bakos, Peter Beisswanger, Joe Lipman, D. C. B. Marsh, and D. A. Robinson. Late solution by J. H. Hodges.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4816. *Proposed by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass.*

Solve the integral equation

$$\int_0^\infty t^3\phi(x-t)dt = a\left\{\int_0^\infty t^2\phi(x-t)dt\right\}^b,$$

where a and b are independent of x.

4817. *Proposed by I. S. Gál and Stanley Kaplan, Cornell University*

Determine all entire functions $f(z)$ such that $|f(z)|=1$ whenever $|z|=1$.

4818. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let d_i be the sides of a complete quadrilateral, and A_{ij} be the vertex on d_i, d_j. Let t_i be the triangle formed by the sides other than d_i, and (O_i) denote the circumcircle of t_i. Denote the Simson line of a point S_i of (O_i) with respect to t_i by D_i.

Then prove that, if D_i and d_i are parallel for all i, (1) the line S_iO_p passes through the vertex A_{qr} (i, $p\neq q$, r), and (2) the points S_i all lie on the Miquel circle (O).

4819. *Proposed by P. T. Bateman, University of Illinois*

Suppose k is a given nonnegative integer. Show that every sufficiently large positive integer is a sum of five squares none of which is less than k^2 and no two of which are equal.

4820. *Proposed by K. Mahler, the University, Manchester, England*

Let p be a prime, K the p-adic field, and I the ring of p-adic integers. Further, let $f(x)$ be a function defined for all $x\in I$ with $f(x)\in K$, such that $f(n)=n!$ $(n=0, 1, 2, \cdots)$. Show that the points of discontinuity lie dense in I.

4821. *Proposed by M. P. Drazin, Trinity College, Cambridge, England*

A square matrix B is called normal if its elements are in the complex field

and if it commutes with its transposed conjugate matrix B^* (*i.e.*, if $BB^*=B^*B$); more generally, call a given $n\times n$ matrix, A, m-normal if A can be imbedded, as leading $n\times n$ submatrix, in some normal $m\times m$ matrix B. Find the greatest value of n for which every (complex) $n\times n$ matrix is $(n+1)$-normal, and show that every square matrix is m-normal for all sufficiently large m.

What are the corresponding results when A, B are restricted to be real?

SOLUTIONS

Vogt's Determinant

4771 [1958, 47]. *Proposed by Leonard Carlitz, Duke University*

Muir (*Contributions to the History of Determinants* 1900–1920) reproduces the assertion of Vogt that the determinant

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2^2 & \cdots & 2^n & 2^{n+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ n & n^2 & \cdots & n^n & n^{n+1} \\ \dfrac{n}{2} & \dfrac{n^2}{3} & \cdots & \dfrac{n^n}{n+1} & \dfrac{n^{n+1}}{n+2} \end{vmatrix}$$

vanishes for $n\geqq 2$ and even. Prove the truth of this assertion. Also show that $D_n\neq 0$ for n odd.

Solution by Gabor Szegö, Stanford University. We show that the associated linear equations

$$x_0\nu + x_1\nu^2 + \cdots + x_n\nu^{n+1} = 0, \qquad 1 \leqq \nu \leqq n,$$

$$x_0\frac{n}{2} + x_1\frac{n^2}{3} + \cdots + x_n\frac{n^{n+1}}{n+2} = 0,$$

have nontrivial solutions for $n\geqq 2$ and even, and only the trivial solution for n odd.

Writing $x_0z+x_1z^2+\cdots+x_nz^{n+1}=f(z)$, these equations read:

$$f(\nu) = 0,\ 1 \leqq \nu \leqq n; \qquad \int_0^n f(z)dz = 0,$$

so that $f(z)=x_nz(z-1)\cdots(z-n)$. Thus the assertion is equivalent to the fact that

$$I_n = \int_0^n \binom{z}{n+1} dz$$

is 0 for $n\geqq 2$ and even, and $I_n\neq 0$ for n odd.

Now

$$\binom{z}{n+1} = (-1)^{n+1}\binom{n-z}{n+1},$$

so that $I_n = -I_n$ if n is even.

If n is odd, we use

$$\binom{z}{n+1} + \binom{z}{n+2} = \binom{z+1}{n+2},$$

so that

$$I_n = \int_0^n \binom{z+1}{n+2} dz - \int_0^n \binom{z}{n+2} dz = \left(\int_1^{n+1} - \int_0^n\right)\binom{z}{n+2} dz$$

$$= -\int_0^1 \binom{z}{n+2} dz + \int_n^{n+1} \binom{z}{n+2} dz = -2\int_0^1 \binom{z}{n+2} dz,$$

in view of

$$\binom{z}{n+2} = -\binom{n+1-z}{n+2}.$$

Thus I_n is negative.

Also solved by A. C. Aitken, Robert Breusch, D. R. Brillinger, A. G. Konheim and J. A. Navarro, Joe Lipman, F. D. Parker, F. W. Ponting, Suja Ram, W. E. Roth, E. M. Wright, J. Van Yzeren, and the proposer.

Consecutive Prime Triplets

4772 [1958, 47]. *Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.*

It is easy to show that there exist consecutive prime pairs such that their difference is arbitrarily large. Do there exist consecutive prime triplets P_1, P_2, P_3 such that min (P_2-P_1, P_3-P_2) is arbitrarily large?

Solution by P. T. Bateman, University of Illinois. The question of the problem was answered affirmatively by Sierpiński [*Colloquium Math.* vol. 1, 1948, pp. 193–194]. The following stronger results have been since obtained. Erdös [*Publicationes Math. Debrecen*, vol. 1, 1949, pp. 33–37] proved that for any positive number C there exist consecutive prime triplets such that $\min(P_2-P_1, P_3-P_2) > C \log P_3$. Walfisz [*Doklady Akad. Nauk SSSR* (N. S.) vol. 90, 1953, pp. 711–713] proved that for almost all primes p the distance to the closest prime on either side is greater than $(\log p)/(\log\log\log p)^2$. Prachar [*Monatsh. Math.* vol. 58, 1954, pp. 114–116] showed that Walfisz's result is still true if $(\log\log\log p)^2$ is replaced by any function of p which tends to infinity with p.

The following proof is similar to Sierpiński's but differs somewhat in detail.

Let q be any prime greater than 2. Then $(q-1)!-1$ and $q!$ are relatively prime. In fact, the prime factors of $q!$ are the primes not exceeding q. Clearly $(q-1)!-1$ is not divisible by any of the primes less than q, while $(q-1)!-1\equiv -2 \pmod{q}$ by Wilson's theorem. Since $(q-1)!-1$ and $q!$ are relatively prime, Dirichlet's theorem guarantees the existence of a prime p such that

$$p \equiv (q-1)! - 1 \pmod{q!}.$$

Now the q integers following p and the $q-2$ integers preceding p are composite, since

$$p \pm k + 1 \equiv p + 1 \equiv (q-1)! \equiv 0 \pmod{k}$$

if $2 \leqq k \leqq q-1$, and since $p+2\equiv(q-1)!+1\equiv 0 \pmod{q}$. But q may be taken as large as desired and so the assertion of Sierpiński is established.

Also solved by Robert Breusch, W. E. Briggs and D. Hawkins, N. J. Fine and Emil Grosswald, John B. Kelly, Leo Moser, and Suja Ram.

A Class of Irrational Numbers

4773 [1958, 125]. *Proposed by Paul Erdös.*

Let $n_1 \leqq n_2 \leqq \cdots$ be a sequence of positive integers. Assume $n_k^{1/2^k} \to \infty$. Show that $\sum_{k=1}^{\infty} 1/n_k$ is irrational. (It is easy to construct $\sum 1/n_k$ rational with $n_k > A^{2^k}$ for every fixed A.)

Solution by Rimhak Ree, University of British Columbia. If $\sum_{k=1}^{\infty} 1/n_k = b/a$, where a and b are relatively prime positive integers, then we would have

$$n_1 n_2 \cdots n_{k-1} \sum_{t=0}^{\infty} 1/n_{k+t} \geqq 1/a \tag{1}$$

for all $k=1, 2, \cdots$. We shall derive a contradiction.

Let $n_k^{1/2^k} = \alpha_k$. Then $\alpha_k \to \infty$ implies the existence of an integer $r>0$ such that $\alpha_{r+t} > a+1$ for $t=0, 1, 2, \cdots$ and $\alpha_i \leqq \alpha_r$ for $i=1, 2, \cdots, r-1$. We have

$$\begin{aligned} n_1 n_2 \cdots n_{r-1}/n_{r+t} &\leqq \alpha_r^{2+2^2+\cdots 2^{r-1}}/n_{r+t} \\ &\leqq \alpha_{r+t}^{2^t(2^r-2)}/\alpha_{r+t}^{2^{r+t}} = \alpha_{r+t}^{-2^{t+1}} < (a+1)^{-(t+1)}, \end{aligned}$$

since $n_r \leqq n_{r+t}$ implies $\alpha_r \leqq \alpha_{r+t}^{2^t}$. Therefore we have

$$n_1 n_2 \cdots n_{r-1} \sum_{t=0}^{\infty} 1/n_{r+t} < \sum_{t=0}^{\infty} (a+1)^{-(t+1)} = 1/a,$$

which contradicts (1).

Also solved by Robert Breusch.

Two Summations

4775 [1958, 125]. *Proposed by Leonard Carlitz, Duke University*

Prove the formulas

$$\sum_{r=0}^{m} \frac{A_r A_{m-r} A_{n-r}}{A_{m+n-r}} \frac{2m+2n-4r+1}{2m+2n-2r+1} = 1 \tag{1}$$

$$\sum_{r=0}^{m} (-2)^r \frac{A_r A_{m-r} A_{n-r} A_{k-r}}{A_{m+n-r}} \frac{4k-4r+1}{4k-2r+1} = A_{m/2} A_{n/2}, \tag{2}$$

where $m \leqq n$, $A_r = 1 \cdot 3 \cdot 5 \cdots (2r-1)/r!$, $A_0 = 1$, $A_x = 0$ for x not an integer, and, in (2), $m+n=2k$.

Solution by A. E. Danese, Mount Morris, N. Y. The proposed formulas are special cases of a familiar result due to Adams (Whittaker and Watson, *A Course of Modern Analysis*, p. 331, 11):

$$P_m(z)P_n(z) = \sum_{r=0}^{m} \frac{A_{m-r} A_r A_{n-r}}{A_{n+m-r}} \frac{2n+2m-4r+1}{2n+2m-2r+1} P_{n+m-2r}(z)$$

where $P_n(z)$ is the Legendre polynomial of order n, m and n are positive integers with $m \leqq n$, and $A_r = 1 \cdot 3 \cdots (2r-1)/r!$

Now using $P_n(1) = 1$, $P_{2n}(0) = (-2)^{-n} A_n$, $P_{2n+1}(0) = 0$, for all positive integers n (*ibid.*, p. 303, ex. 1), the results are immediate.

Also solved by P. Henrici, B. S. Popov, G. Szegö, Chih-yi Wang, and the proposer.

A Summation Representing an Exponential Function

4776 [1958, 451]. *Proposed by D. J. Newman, A VCO Research and Development, Lawrence, Mass.*

If $|\alpha| < 1/e$, then

$$\sum_{n=0}^{\infty} \frac{(z+\alpha n)^n}{n!}$$

represents an entire function. Prove in fact that it is a simple exponential, $Ae^{\lambda z}$.

Solution by Leonard Carlitz, Duke University. It is known (see, *e.g.*, Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, p. 302, no. 214) that

$$\sum_{n=0}^{\infty} \frac{(n+\beta)^n \alpha^n}{n!} = \frac{e^{\beta\mu}}{1-\mu}, \qquad (|\alpha| < 1/e),$$

where $\mu e^{-\mu} = \alpha$. With $z = \alpha\beta$ and $\mu = \log \lambda$, this becomes immediately

$$\sum_{n=0}^{\infty} \frac{(z+\alpha n)^n}{n!} = \frac{e^{ze^{\mu}}}{1-\mu} = \frac{e^{z\lambda}}{1-\log \lambda}.$$

Also solved by Robert Breusch, James Clunie, N. J. Fine, M. Ganeshavijaranayengar, Emil

Grosswald, A. H. M. Levelt, Immanuel Marx, and the proposer. Late solutions by Norman Greenspan and Robert Weinstock.

The original form of the problem [1958, 125] was also solved by P. T. Bateman, D. R. Brillinger, C. N. Campopiano, Michael Goldberg, J. W. Haake, J. D. E. Konhauser, R. L. London, D. C. B. Marsh, Burton Randol, and Chih-yi Wang.

Sums of Powers of Binomial Coefficients

4777 [1958, 125]. *Proposed by R. P. Pakshirajan, Indian Statistical Institute, Calcutta*

Prove that

$$\lim_{n\to\infty} \frac{n}{2^{3n}}\left\{\binom{n}{0}^3 + \binom{n}{1}^3 + \cdots + \binom{n}{n}^3\right\} = \frac{2}{\pi\sqrt{3}}\cdot$$

Editorial Note. As pointed out by several readers, this is the case $k=3$ of II, 40, pp. 42, 201 in Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. I.

Klamkin points out that it is also the case $f(x)=x^p$ of II, 190, pp. 76, 242 in the same work, vol. I. Wang refers to a formula involving $\binom{n}{0}^3-\binom{n}{1}^3+-\cdots\pm\binom{n}{n}^3$ by A. C. Dixon, *Messenger of Mathematics*, vol. 20 (1891), pp. 79–80.

Other solutions by Robert Breusch, Yoshio Matsuoka, M. R. Mickey, G. W. Petrie III, Chih-yi Wang, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

College Algebra (6th ed.). By Henry L. Rietz, Arthur R. Crathorne, and J. William Peters. Holt, New York, 1958. xiv+385 pp. $4.50.

This text differs from the fifth edition (reviewed this MONTHLY, 1952, vol. 59, pp. 192–194) in about one-fourth of the exercise sets and in a couple of the sections. Representative of the difference is the increase in the price of coffee by twenty-five cents. The changes do little more than require that class work be restricted to one of the editions, unless the course covers only the last half of the book. For, starting with Chapter 11, the only differences seem to be on pages 254, 264, 274 and 275.

JAMES H. MCKAY
Seattle University

Differential Equations: Geometric Theory. By Solomon Lefschetz. Interscience, New York and London, 1957. vi+364 pp. $9.50.

The twelve chapters of this book can be divided into four parts. The first part would consist of the first four chapters, which contain a review and develop-

ment of the material on topology, vectors, matrices and functions of several variables necessary to the rest of the book. This section has chapters on stability and linear systems. In the second part, consisting of four chapters, three chapters are devoted to point stability of systems and the treatment of Poincare, Liapunov and Perron. The last chapter of this part deals with the stability of general periodic systems. In the third part, of two chapters, attention is turned to the results of Poincare and Bendixson on two dimensional systems. The results of Andronov, Pontrjagin and De Baggis on structural stability are discussed. The last section, of two chapters, discusses equations of the second order and the contributions of Cartwright, Littlewood, Levinson and Smith. This covers the self-oscillations of van der Pol, the method of perturbation and applications to Mathieu's equation. There is an appendix on matrices and on the index of a curve. The book is in the form of a monograph in that there are no applications or exercises. However, nineteen problems of a serious nature are listed at the end for the reader's further investigation.

This work forms an important and welcome addition to the literature. Previously part of the material was available in the Annals of Mathematics Studies, in *Nonlinear Vibrations in Mechanical and Electrical Systems*, by H. H. Stoker, and in the translation by Lefschetz of Andronov and Chaikin's *Theory of Oscillations*. Much of the material was not in book form or available to the non-Russian speaking reader. The typography is excellent and many difficult diagrams have been clearly reproduced.

The selection of the material reflects the individuality and wide interests of the author. This is a volume by one of the most prominent workers in the subject and is a volume which anyone interested in the theory of nonlinear ordinary differential equations will want to own.

G. M. PETERSEN
University of New Mexico

Paper Folding for the Mathematics Class. By Donovan A. Johnson. National Council of Teachers of Mathematics, Washington, D. C., 1958, 32 pp. $0.75.

This interesting little booklet gives a fresh viewpoint to geometry. Starting with basic assumptions such as "paper can be folded so that the crease passes through one or two given points," most of the constructions of plane Euclidean geometry can be performed by folding and creasing. They range from simple constructions such as "a line perpendicular to a given straight line," through circle relationships, and even such products as $(x+y)(x-y)=x^2-y^2$.

Some of the later chapters like "Polygons Constructed by Tying Paper Knots," and "Conic Sections," are ingenious but rather tricky to follow. The illustrations are numerous and excellent. It should appeal to students in all levels of high school and beginning college and, of course, to their teachers. Such a fresh outlook on a routine subject is very welcome.

JOSEPHINE P. ANDREE
University of Oklahoma

Advanced Calculus. By Louis Brand. Wiley, New York, 1955. 574 pp. $8.50.

This book, as the author points out in his preface, serves as an excellent introduction to real variable theory. Following an introductory chapter on the development of the number system from an axiomatic standpoint, the author begins to develop his function theory and the theory of limits in a chapter on sequences and series. It is refreshing to find a text at this level which includes several tests for convergence of series in addition to the more familiar ratio, root and comparison tests, although the author properly postpones until later the discussion of the integral test.

With this introduction, the author proceeds to chapters on functions of a real variable, functions of several variables, vectors, the Riemann integral, improper integrals, line integrals, and multiple integrals. Here follows a complete chapter on uniform convergence and its implications in the reversal of order in limiting processes. This chapter includes a discussion of gamma and beta functions. The penultimate chapter in the text proper concerns functions of a complex variable, including the use of contour integration for the evaluation of definite integrals. The last chapter is an introduction to Fourier series, including a brief discussion of the Gibbs phemonenon. The annexation of four appendices, on cluster points, difference equations, difference calculus, and dimensional checks, subjects not usually found in such books and not apropos to any particular chapter, is well worth while and quite commendable.

There are many excellent characteristics of this book. The proofs are clearly written. In many cases, introduction of special devices simplifies proofs and clarifies theorems which more standard procedures may leave obscure. For example, Euler's theorem on homogeneous functions and its converse (page 161), and the proof of uniform continuity of a continuous function over a closed interval and related theorems are proved without recourse to a Heine-Borel theorem (pages 91 ff.). The summarization of the content of each chapter at its close is excellent, and should serve as an aid in teaching from this book as a text.

Despite the high caliber of the presentation, it is marred at times by unfortunate errors. Some of these may be typographical, although their repetition in relatively close juxtaposition might lead one to believe they appeared in the manuscript. One particular blunder which stands out appears on the bottom of page 121, where the author uses $\lim_{x\to 0} x^{1/2}/\log x$ as an indeterminate form to illustrate a portion of his theory. Again, there is a repetition of the error of omitting the radical in the integral for $\sin^{-1} b - \sin^{-1} a$ on page 265; and on pages 316–18, the differential dx is omitted in several integrals.

With the correction of errors, typographical and otherwise, this book will be an excellent text for an introduction to mathematics at the graduate level. There are an ample number of exercises, theoretical and numerical, with answers.

ROBERT E. LOWNEY
Montana State College

Plane Geometry for Colleges. By L. J. Adams. Holt, New York, 1958. v+213 pp. $3.50.

This book is designed for a one-semester, three-hour course in plane geometry for college freshmen. An effort has been made to write a more mature book than that found in some available texts in this field. Objectional statements of some geometry texts, such as "axioms are self-evident truths" and the formulation of definitions for undefined objects are conspicuous by their absence here. Although many plane geometries cover far more theorems and corollaries than this book, it is evident that the author has stressed quality rather than quantity.

The development of material is along essentially three lines: (i) nature of geometry as a logical system, (ii) constructions and mensuration, (iii) meaning and content of theorems and their corollaries. A last chapter points out applications of geometry.

Perhaps, for college students, more on rigor in geometric proofs might have been done, and, although vectors are introduced about one-fourth of the way through this book, they seem to be placed more in the category of a curiosity rather than put to any real geometric use.

These criticisms seem rather finicky when compared with the book's good points. The book appears teachable, and merits consideration as a text for a first course in plane geometry at the college level.

BEN T. GOLDBECK, JR.
Texas Christian University

Introduction to Statistical Reasoning. By Philip J. McCarthy. McGraw-Hill, New York, 1957. xiv+402 pp. $5.75.

Dr. McCarthy's purpose is to present to lower division majors in social sciences, with only high school mathematics, the "elementary methods common to all statistical investigations, regardless of the field" (p. 2).

He accomplishes this aim satisfactorily, and without undue originality, but so have many recent texts such as Sprowls, Croxton and Crowden, and Mills.

His introductory remarks take up the preface and the first two chapters—more space than is necessary. For instance, on page 11 he devotes an entire paragraph to an elaboration of the variability of an individual's responses at various times, an obvious truth requiring merely a sentence.

Chapter 3 presents effectively the rectangular, normal, Poisson, and some irregular density functions, although, oddly enough, he does not name them.

Chapters 4 and 5 discuss location and scale parameters and their estimates. He uses $N-1$ as the denominator of the variance estimate with only a footnote of explanation: " . . . simplifies . . . expressions . . . used in more advanced statistical work" (p. 109). However, elementary students find that dividing the sum of N squared differences from the mean by $N-1$ is a confusing procedure. They are entitled to a fuller explanation of the concept, "bias."

Chapter 6 on random sampling is excellent but not new. Chapter 7 covers

elementary probability very well, using some interesting examples.

Chapter 8 on the binomial distribution demonstrates how that function arises, its meaning and employment in statistical inference, and the precise significance of "95 per cent confidence." Chapter 9 is good on large-sample techniques, as is Chapter 10 on systematic, stratified, and random sampling.

The two final chapters cover chi-square tests and linear regression. His symbol y_{x_i} for "estimated value of the dependent parameter" is unconventional as well as awkward. The usual $\hat{y}_i$ is preferred. He discusses chi-square exclusively in connection with an "$m\times n$" contingency table, relegating to a footnote (p. 310) any other use of that versatile statistic.

A bad omission is the reduction of all discussion of Student's t-distribution to two footnotes, pages 265, 355. Few will agree that this important idea "can best be originated in a second course" (p. ix).

EUGENE H. LEHMAN, JR.
San Diego College for Women

Dynamic Programming. By Richard Bellman. Princeton University Press, Princeton, 1957. xxv+342 pp. $6.75.

This book brings under one cover the introduction and development of the theory of dynamic programming, which to a great extent has appeared previously in many papers scattered throughout many journals and pamphlets.

The class of problems which gave impetus to the development of the dynamic programming approach are multi-stage decision processes as they appear in economics, game theory, logistics, *etc.*

To give a greatly over-simplified idea of the subject, suffice it to say that "n" successive one-dimensional decisions are substituted for one "n" dimensional decision.

The author's aim of providing "an introduction to the mathematical theory" certainly has been accomplished. However, his suggestion to use the book as the text for "a course on the advanced calculus level" seems to be an application of the author's "principle of wishful thinking," since throughout the book there is a disturbing disconnectedness. Due to the scope of the subject covered this may have been unavoidable.

There are an abundant number of exercises at the end of each chapter. These, as well as the examples in the text, cover the wide range of the applicability of dynamic programming. Since the answers to most problems are not given, the designation of the problem sections as "Exercises and Research Problems" should prove rather confusing to the reader as to which is exercise and which is research problem.

Topics covered include: Chapter I: A discussion of a multi-stage allocation process, including existence and uniqueness theorems; the properties of the solution. Chapter II: A similar discussion of a stochastic multi-stage decision process. Chapter III: A more general discussion of dynamic programming proc-

esses; mathematical formulation of these. Chapter IV: Existence and uniqueness theorems, convergence, and stability. Chapter V: Formulation and discussion of the optimal inventory equation and its ramification. Chapters VI and VII: Bottleneck problems, the dual problem, and discussion of several examples. Chapter VIII: A continuous stochastic decision process; several approaches to its solution. Chapter IX: Dynamic programming approaches to several classical and other calculus-of-variation problems. Chapter X: Multi-stage games, including some discussion of nonzero-sum games. Chapter XI: Markovian decision processes.

This book will certainly prove extremely useful to the mathematicians, economists, statisticians, engineers, and operation analysts alike. The mathematician especially will look forward to the "contemplated second volume on a higher mathematical level" in which the author promises to rectify some omissions of the present volume.

ALBERT NEWHOUSE
University of Houston

BRIEF MENTION

Analysis of Research in the Teaching of Mathematics. By Kenneth E. Brown, Specialist for Mathematics, U. S. Department of Health, Education, and Welfare. No. 4. 73 pp. $0.25.

Dr. Brown has done a masterful job summarizing the recent educational research done in the teaching of mathematics. Perhaps one of the most interesting features in this brief booklet is the inclusion of seventy-one questions which are still unanswered and which, in the opinion of the research workers, would be significant topics for further research. Let us hope that each university having a college of education will forward this list to the proper persons.

Man and Number. By Donald Smeltzer. Emerson Books, New York, 1958. viii +114 pp. $2.50.

Perhaps the table of contents is as indicative as any review could be: Chapter One, Man Learns to Count; Two, Number Recording; Three, Early Calculating Devices; Four, The Modern Number System. One should not conclude that Chapter Four contains a rational development of the real numbers or anything as modern as this.

Plane Trigonometry. By John J. Corliss and Winifred V. Berglund, Second Ed. Houghton Mifflin, Boston, 1958. xii+397 pp. $4.00.

This is a book from which it would be quite possible to give a reasonably modern course in trigonometry. It would also be possible to give an old-fashioned course with the emphasis mainly on triangle solving. The study of the graphs of the trigonometric functions before the solution of the right triangle is interesting.

Energy for Man. Windmills to Nuclear Power. By Hans Thirring. Indiana University Press, 1958. 409 pp. $6.95.

Operational Mathematics. By Ruel V. Churchill. Second Ed. McGraw-Hill, New York, 1958. ix+337 pp. $7.00.

The revised second edition of this book on operational mathematics and integral transformations is a welcome addition to the growing literature on advanced mathematics. An advanced calculus background is assumed along with work on partial differential equations. A number of the proofs have been replaced with better versions, and several chapters have apparently been entirely rewritten.

Switching Circuits and Logical Design. By Samuel H. Caldwell. Wiley, New York, 1958. xvii+686 pp. $14.00.

The Russian Literature of Satellites, Parts I and II. International Physical Index, New York, 1958. Part I, $10.00, Part II, $12.50.

The translation into English of a series of Russian papers on the subject of satellites is indeed timely and welcome.

A Comprehensive Bibliography on Operations Research through 1956 with supplement for 1957. Wiley, New York, 1958. xi+188 pp. $6.50.

Finite Queuing Tables. By L. G. Peck and R. N. Hazelwood. Operations Research and Mathematics Group, Cambridge, Mass. Wiley, New York, 1958. xvi+210 pp. $8.50.

Extensive, legible tables of the queuing function. This book was produced by photo-offset from machine printing from Univac One, thus requiring no human editing of the numerical portions.

Physics Express, Automation Express, Electronics Express. International Physical Index, Inc., New York. Any one Express $57.50 a year; any two Expresses $100 a year; all three Expresses $150 a year. (Free samples on request.)

Three new periodicals each appearing ten times a year with comprehensive digests of current Russian periodical literature based on the contents of 73 Russian journals

Arithmetic for Colleges. By Harold D. Larsen. Macmillan, New York, 1958. xiii+286 pp. $5.50.

Professor Larsen has rewritten the exercises in his earlier, 1950, text designed primarily for "elementary school teachers who have had no contact with arithmetic since their own elementary school days." It is sad to report that there are many elementary school teachers in this situation, but facts must be faced even when they are unpleasant. Professor Larsen's book has already helped alleviate the situation.

Figurets. By J. A. H. Hunter. Oxford University Press, New York, 1958. x+116 pp. $3.50.

The author of *Fun with Figures* brings us another collection of mathematical and near-mathematical puzzles—some old, some new.

Mathematical Excursions. By Helen A. Merrill. Dover, New York, 1958. 145 pp. $1.00.

This reprint of Professor Merrill's earlier, 1933, book is apparently unchanged from the original edition. It contains 90 interesting puzzles woven into a fabric of mathematical excursions. Your bookstore should carry this on its shelf of paperbacks.

An Introduction to the Dynamics of Airplanes. By H. Norman Abramson. Ronald Press, New York, 1958. viii+225 pp. $4.50.

An undergraduate text in aeronautical engineering which includes the use of matric applications to vibration problems.

Applications of Tensor Analysis. By A. J. McConnell. Dover, New York, 1958. 318 pp. $1.85.

This is a reprint, with a more apt title, of the author's *Applications of the Absolute Differential Calculus* published in 1931.

The Theory of Functions of a Real Variable. Vols. I and II. By E. W. Hobson. Dover, New York, 1958. xv+736 pp. in Vol. I and x+780 pp. in Vol. II. $6.00.

Thanks to Dover Publications for making available an inexpensive reprint of Hobson's well-known work.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Lloyd J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

NATIONAL ACADEMY OF SCIENCES—NATIONAL RESEARCH COUNCIL DIVISION OF MATHEMATICS FELLOWSHIP AND RESEARCH OPPORTUNITIES

The Division of Mathematics calls attention to the fact that several foundations and offices offer financial support for research in mathematics during the year 1959–60.

A number of fellowships will be made available, as well as opportunities for mathematicians to engage in basic research. A partial list, with comments, is given below.

1. *National Science Foundation.* The National Science Foundation sponsors various fellowship programs in the sciences, including mathematics.

Predoctoral fellowships are awarded annually at the first-year, intermediate, and terminal-year levels of graduate study. Applications for 1959–60 will be available in October 1958 from the National Academy of Sciences—National Research Council until the closing date in early January 1959; Award date—March 16, 1959.

Science Faculty fellowships for college science teachers (including mathematics), who plan to continue teaching and who wish to increase their competence as teachers, are offered semiannually at the present time. Eligibility requirements include a baccalaureate degree and three years of full-time experience in teaching natural science subjects at the collegiate level. Awarded annually. The program will be open from May to October. Awards will be announced in early December. Address all inquiries for information and applications to National Science Foundation, Division of Scientific Personnel and Education, Washington 25, D. C.

Postdoctoral fellowships (in making inquiry about postdoctoral awards specify program).

(1) *Regular* postdoctoral fellowships—primarily for recipients of the doctoral degree; awarded semiannually. Program for 1959–60 concurrent with predoctoral program (see above) except that program closes in December. Information and applications will be available from NAS-NRC. The program will also be open from July to early September 1959. Awards are announced in March and October.

(2) *Senior* postdoctoral fellowships—are open to persons who have held a doctoral degree in one of the basic fields of science for a minimum of five years at time of application, or who have had equivalent training and experience. Awarded annually. Applications are available from the National Science Foundation, Division of Scientific Personnel and Education, Washington 25, D. C. The program will be open from May to October. Awards will be announced in early December.

Research Grants. The National Science Foundation also supports basic research in the mathematical sciences by means of grants. While proposals for such support are accepted at any time, individuals desiring support to begin in the summer or at the beginning of a fall semester should submit their proposals in the mathematical sciences preferably by November 1; persons desiring support to begin in the spring semester should submit their proposals preferably by May 1. Instructions for the preparation of proposals, contained in a booklet entitled Grants for Scientific Research, may be obtained upon request from the Program Director for Mathematical Sciences, National Science Foundation, Washington 25, D. C.

2. *Office of Naval Research.* The Office of Naval Research, through contracts with universities and other organizations, supports basic research in broadly-selected fields of mathematics. Proposals should be directed to the Mathematics Branch, Office of Naval Research, Washington 25, D. C. In addition, postdoctoral research associateships in pure mathematics are being established under contracts with the ONR at selected universities. For details and application forms write to the above address.

3. *Air Force Office of Scientific Research.* The Air Force Office of Scientific Research supports research in mathematics directly through contracts with colleges, universities, foundations, and industrial laboratories. Such organizations are encouraged to submit proposals for research in mathematical fields in which they specialize. Proposals should be mailed to the Commander, Air Force Office of Scientific Research, Attn: Mathematics Division, Washington 25, D. C.

4. *Office of Ordnance Research, U. S. Army*. Among the functions of the Office of Ordnance Research is the support of basic research in mathematics. Proposals for projects are ordinarily made by individual scientists or groups of scientists in a form which leads to a contract between the Office of Ordnance Research and a university or research laboratory. For further information write to Commanding Officer, Office of Ordnance Research, Box CM, Duke Station, Durham, North Carolina.

5. *Fulbright Awards—Public Law 584 (79th Congress)*. Approximately 400 awards are offered annually for university lecturing and postdoctoral research in all academic fields in Argentina, Australia, Brazil, Burma, Chile, Colombia, Ecuador, India, New Zealand, Pakistan, Paraguay, Peru, the Philippines, and Thailand (competition for the preceding countries closes April 15, 1959); Austria, Belgium-Luxembourg, Republic of China, Denmark, Finland, France, Germany, Greece, Iceland, Iran, Ireland, Israel, Italy, Japan, the Netherlands, Norway, Turkey, and the United Kingdom, including colonial dependencies (competition for the latter countries closes October 1, 1959). In both cases awards are for the academic year 1960–61 (the 1959–60 competition for Europe closes October 1, 1958), but in the former group of countries the academic year begins in the spring or summer instead of the autumn. Awards are payable in foreign currency and usually include travel for the grantee, but not for members of his family, and a maintenance allowance, which may be adjusted in relation to the number of accompanying dependents up to four. Requests for information should be addressed to the Committee on International Exchange of Persons, Conference Board of Associated Research Councils, 2101 Constitution Avenue, Washington 25, D. C.

6. *National Bureau of Standards. Naval Research Laboratory. Air Research and Development Command*. Postdoctoral resident research associateships are available in a variety of sciences including mathematics and are tenable at the Washington, D. C. and Boulder, Colorado laboratories of the National Bureau of Standards; at the Naval Research Laboratory in Washington, D. C.; and at selected development and research centers of the Air Research and Development Command. Necessary facilities and equipment incident to the research of the associate will be provided. For further information write to Fellowship Office, National Academy of Sciences—National Research Council, 2101 Constitution Avenue, Washington 25, D. C. Applications for the 1959–60 program must be filed on or before January 19, 1959.

7. *Atomic Energy Commission*. The Division of Research of the Atomic Energy Commission, through contracts with universities and other organizations, supports research in the fields of numerical analysis, digital-computer design, programming research, and related topics. Proposals should be submitted to the Division of Research, Atomic Energy Commission, Washington 25, D. C.

Brookhaven National Laboratory. Brookhaven National Laboratory, operated by Associated Universities, Inc., under contract with the Atomic Energy Commission, offers postdoctoral research appointments in mathematics. Appointments are for one year, and may be renewed for one additional year. U. S. citizenship is not required, although Atomic Energy Commission approval is a prerequisite. The appointee may work in numerical analysis, digital computing, mathematical physics, differential equations, probability and statistics, and various specialized branches, including reactor theory, hydrodynamics, and orbit theory. Computational facilities are available. Letters from candidates should give details of personal history, scientific background, and qualifications; two letters of recommendation, one from the applicant's research professor, are required. Applications should be directed to M. E. Rose, Head, Applied Science Division, Brookhaven National Laboratory, Upton, Long Island, New York.

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PERSONAL ITEMS

Professor Jewell H. Bushey of Hunter College represented the Association at the inauguration of Richard Heathcote Heindel as President of Wagner Lutheran College, and the observance of the 75th Anniversary of the Founding of the College on the afternoon of Sunday, October 19, 1958.

Professor H. D. Brunk, University of Missouri, will be a Fulbright lecturer at the University of Copenhagen, Copenhagen, Denmark, from July 1958 until May 1959.

Miss Dorothy I. Koehler, University of Kentucky, is a Fulbright Award teacher at Cornwall College, Montego Bay, Jamaica, British West Indies.

University of California, Berkeley: Professor G. P. Hochschild, University of Illinois, and Associate Professor M. A. Rosenlicht, Northwestern University, have been appointed to professorships; Assistant Professor H. J. Bremermann, University of Washington, and Associate Professors S. R. Fary, University of Montreal, and R. J. De Vogelaere, University of Notre Dame, have been appointed to associate professorships; Assistant Professors H. O. Cordes, University of Southern California, R. L. Vaught, University of Washington, J. W. Woll, Jr., Lehigh University, and Dr. J. F. Treves of Paris, have been appointed to assistant professorships; Dr. Charles Ballantine, Dr. C. W. Clark, University of Washington, Dr. Denis Rutovitz and Dr. Roy Takenaga have been appointed to instructorships; Professor Andrzej Mostowski, University of Warsaw, has been appointed Visiting Professor, 1958–59; Professor Emeritus George Pólya, Stanford University has been appointed Visiting Professor for the Spring term 1959; Professor J. C. Shepherdson, University of Bristol, England, will be Visiting Associate Professor, 1958–59; Dr. S. R. Foguel, New York University, Dr. Avner Friedman, Indiana University, Dr. Helmut Klingen, University of Gottingen, Germany, Dr. Anil Nerode, University of Chicago, and Assistant Professor C. J. Titus, University of Michigan, will be Visiting Assistant Professors during 1958–59; Assistant Professor J. R. Shoenfield, Duke University will be Visiting Assistant Professor in the Spring term 1959; Dr. E. C. Zeeman, Cambridge University, England, is Visiting Assistant Professor for the Fall term 1958; Associate Professors L. A. Henkin, H. D. Huskey, M. H. Protter, and Abraham Seidenberg, have been promoted to professorships; Assistant Professors Harley Flanders and Henry Helson have been promoted to associate professorships; Assistant Professor W. G. Bade has been awarded a Senior Postdoctoral National Science Foundation Fellowship for study at Yale, 1958–59; Dr. Leo Breiman, who was a Visiting Assistant Professor 1957–58, has been awarded a Postdoctoral National Science Foundation Fellowship for study in Paris, 1958–59; Assistant Professor James Eells, Jr., will be Visiting Assistant Professor at Columbia University 1958–59; Professor M. H. Protter has accepted an invitation to spend the Spring term 1959 at the Institute for Advanced Study, Princeton; Assistant Professor Emery Thomas has been awarded a Postdoctoral National Science Foundation Fellowship for study at Oxford, England, 1958–59.

The University of Connecticut: Associate Professor J. C. Montgomery has been appointed Acting Head of the Department for the academic year 1958–59; Professor C. H. W. Sedgewick has retired, and has accepted a position as Statistician at the Bureau of the Census, Washington, D. C.; Assistant Professor R. P. Gosselin has been promoted

to Associate Professor; Dr. Geraldine A. Coon, Taylor Instrument Companies, Rochester, New York, and Assistant Professor Chester Feldman, University of New Hampshire, have been appointed Assistant Professors; Mrs. Grazyna Gross has been appointed Instructor at the Stamford Branch; Mrs. Irene S. Yerger has been appointed Instructor at the Waterbury Branch.

Grinnell College, Grinnell, Iowa: Dr. Roscoe Woods, University of Iowa, has been appointed Visiting Professor for 1958–59; Assistant Professor W. A. Small has been promoted to Associate Professor.

University of Nebraska: Dr. Bernard Harris, Stanford University, Dr. John Kimber, Tufts University, and Mr. Hubert Hunzeker, University of Michigan, have been appointed Assistant Professors; Mr. J. M. Anderson, Iowa State College, has been appointed Instructor; Dr. D. L. Guy has been promoted to Assistant Professor.

• *Ohio Wesleyan University:* Professor S. A. Rowland, after thirty-two years as Department Head, has retired as Professor Emeritus and will start the instruction in mathematics for Ohio State University in the new branch college at Mansfield, Ohio; Assistant Professor G. L. Thompson, Dartmouth College, has been appointed Professor.

University of South Carolina: Associate Professors Simon Green and W. A. Rutledge, University of Tulsa, and Dr. Johann Sonner, Wright-Patterson Air Force Base, Dayton, Ohio, have been appointed to associate professorships.

Mr. Eugene Albert, Union College, Schenectady, New York, has been appointed Junior Instructor at the University of Virginia.

Dr. J. F. Andrus, University of Florida, has accepted a position as Senior Mathematical Engineer with the Lockheed Aircraft Corporation, Marietta, Georgia.

Mr. P. G. Archer, University of Buffalo, has been appointed Research Assistant at Johns Hopkins University.

Professor Emil Artin, Princeton University, will be on leave of absence during 1958–59, and will be at the University of Hamburg, Hamburg, Germany.

Dr. Ward Barnes, University of North Carolina, has been appointed Instructor at the University of Massachusetts.

Mr. Martin Berman, on leave from the University of Cincinnati, is a Graduate Assistant at the University of Illinois.

Dr. M. R. Bottaccini, State University of Iowa, has been appointed Professor at the University of Arizona.

Mr. J. R. Boyd, Chance Vought Aircraft, Dallas, Texas, has been appointed Assistant Professor at Lamar State College.

Mrs. Carolina Brennan, University of California, Computer Center, Berkeley, has been appointed Instructor at the University of the Phillipines.

Dr. A. R. Brown, Jr., Air Proving Ground Center, has accepted a position as Operations Analyst with Headquarters, Air Defense Command, Colorado Springs, Colorado.

Associate Professor R. G. Buschman, University of Wichita, has been appointed Assistant Professor at Oregon State College.

Dr. John Christopher, Electro Data Corporation, Pasadena, California, has been appointed Assistant Professor at Sacramento State College.

Dr. L. W. Cohen, formerly Program Director for the Mathematical Sciences, National Science Foundation, has been appointed Professor of Mathematics and Head of the Department of Mathematics at the University of Maryland.

Professor Richard Courant has retired as scientific director of New York University's Institute of Mathematical Sciences and head of the University's mathematics department. He will become Professor Emeritus of Mathematics and Science Advisor to the University.

Dr. H. B. Curtis, Jr., Rice Institute, has been appointed Assistant Professor at the University of Texas.

Mr. C. L. Davis, General Motors Research Staff, has been appointed Associate Professor at Tri-State College, Angola, Indiana.

Dr. Martinus Esser, University of Maryland, has been appointed Assistant Professor at the University of Dayton.

Dr. J. R. Foote. Alamagordo, New Mexico, has accepted a position as Professor of Mathematics and Director of the Graduate Center, Holloman Air Force Missile Development Center, for the University of New Mexico.

Assistant Professor Ilse N. Gâl, Cornell University, will be on leave at Yale University for the academic year 1958–59.

Assistant Professor I. S. Gâl, on leave from Cornell University for the academic year 1958–59, has been appointed Research Associate at Yale University.

Professor José Gallego-Diaz, Escuela Especial de Ingenieros Agronomos, Madrid, Spain, has been appointed Visiting Professor at the University of Puerto Rico.

Professor R. W. Gardner, Olivet Nazarene College, has been appointed Dean of Students and Professor at the Eastern Nazarene College.

Associate Professor J. W. Hamblen, Oklahoma State University, has been employed as full-time Director of the new Computing Center at the University of Kentucky.

Mr. R. W. Hartop, University of Maine, has been awarded a Master of Science Fellowship at the Hughes Aircraft Company, Culver City, California.

Mr. J. R. Hatcher has accepted a position as Research Engineer with North American Aviation, Inc., Los Angeles, California.

Reverend F. A. Homann, University of Pennsylvania, has been appointed Instructor at Loyola College, Baltimore, Maryland.

Professor Aughtum S. Howard, Kentucky Wesleyan College, has been appointed Associate Professor at Eastern Kentucky State College.

Associate Professor L. K. Jackson, University of Nebraska, is a Visiting Associate Professor at the University of Oregon.

Miss Diane M. Johnson, University of Toronto, has been appointed Lecturer at the University of Manitoba.

Mr. W. B. Jones, Vanderbilt University, has accepted a position as Mathematician with the National Bureau of Standards, Boulder, Colorado.

Mr. E. H. Kanning III, University of Minnesota, has accepted a position as Associate Scientist with Lockheed Missile Systems Division, Sunnyvale, California.

Assistant Professor D. A. Kearns, University of Maine, has been appointed Professor and Head of the Department of Mathematics, Merrimack College.

Assistant Professor R. J. Kohlmeyer, Pratt Institute, has been appointed Associate Professor at Albright College.

Dr. George Kolettis, Jr., Northwestern University, has been appointed O.N.R. Research Associate at the University of Notre Dame.

Professor A. V. Kozak, Concord State College, has been appointed Professor of Mathematics Education, School of Education, Pennsylvania State University.

Mr. F. A. Kros, Sperry Rand, Univac Division, St. Paul, Minnesota, has been appointed Manager at Sperry Rand, Univac Division, New York.

Associate Professor Stephen Kulik, University of South Carolina, has been appointed Professor at Utah State University.

Dr. R. B. Leipnik, University of Washington, has accepted a position as Mathematician Consultant at the United States Naval Ordnance Testing Station, China Lake, California.

Assistant Professor W. D. Lindstrom, Iowa State College, has been appointed Associate Professor at Kenyon College.

Professor H. W. Linscheid, Southwestern State College, has been appointed Associate Professor at the University of Wichita.

Mr. F. J. Lorenzen, Jr., Union College, has been appointed Instructor at the University of Florida.

Mr. R. A. Lufburrow, Woods Hole Oceanographic Institute, has been appointed Assistant Professor at St. Lawrence University.

Mr. Ransom Van B. Lynch, Phillips Exeter Academy, is a Visiting Lecturer at Princeton University for one year.

Mr. B. L. McAllister, University of Wisconsin, has been appointed Assistant Professor at the South Dakota School of Mines and Technology.

Dr. G. H. Meisters, Iowa State College, has been appointed Instructor at Duke University.

Associate Professor Josephine M. Mitchell, University of Pittsburgh, has been appointed Associate Professor at Pennsylvania State University.

Mr. R. P. Mitchell, Naval Ordnance Laboratory, Corona, California, has accepted a position as Scientist with the Lockheed Aircraft Corporation, Palo Alto, California.

Mr. J. G. Moser, Purdue University, has been appointed Instructor at the Rose Polytechnic Institute.

Miss Elsie C. Muller, Morningside College, has been appointed Instructor at Iowa State College.

Mr. D. E. Myers, University of Illinois, has been appointed Associate Professor at Millikin University.

Professor Abba V. Newton, on leave from Vassar College for the academic year 1958–59, is at the University of Michigan on a National Science Foundation Science Faculty Fellowship.

Mr. R. D. Oberg, University of Minnesota, has accepted a position as Mathematician with the National Security Agency, Fort Meade, Maryland.

Mr. Enuenwemba Obi, University of Kansas City, has been appointed Assistant Professor at Bethany College.

Mr. R. K. Otnes, University of Nebraska, has accepted a position as a Computer Analyst at the Santa Monica Plant of Douglas Aircraft.

Mr. R. T. Pegis, University of Toronto, has accepted a position as Mathematician with Bausch & Lomb Optical Company, Rochester, New York. He is also an Assistant Lecturer at the University of Rochester.

Associate Professor E. J. Polak, Bucknell University, is at Princeton University on a National Science Foundation Full Faculty Fellowship.

Dr. H. J. Renggli, Tulane University of Louisiana, has been appointed Assistant Professor at Rutgers, The State University.

Assistant Professor J. D. Riley, Iowa State College, has accepted a position as Mathematician with the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. C. D. Robbins, University of Oklahoma, has accepted a position as Research Engineer with Douglas Aircraft Company, Santa Monica, California.

Professor Henry Scheffe, University of California, will be Visiting Professor at Princeton University for the academic year 1958–59, where his work with the Statistical Techniques Research Group will be partially supported by a grant from the National Science Foundation.

Dr. E. C. Schlesinger, Yale University, has been appointed Assistant Professor at Wesleyan University.

Mr. R. L. Schwaller, Marquette University, has been appointed Instructor at the College of St. Thomas.

Mr. R. F. Shortt, Clarkson College, has been appointed Assistant Professor at the South Dakota School of Mines and Technology.

Associate Professor R. J. Silverman, Illinois Institute of Technology, is Visiting Professor at the College of Agriculture and Mechanical Arts, University of Puerto Rico.

Mr. W. E. Smith, University of California, Los Angeles, has been appointed Assistant Professor at Occidental College.

Associate Professor W. K. Smith, Antioch College, has been appointed Associate Professor at Bucknell University.

Professor S. M. Spencer, Jr., Louisiana College, has been appointed Head of the Mathematics Department at McNeese State College.

Associate Professor R. A. Struble, Illinois Institute of Technology, has been appointed Associate Professor at the State College of Agriculture and Engineering, University of North Carolina.

Mr. C. J. Struth, East Texas State Teachers College, is now Assistant Instructor at the University of Kansas.

Mr. B. K. Swartz, Massachusetts Institute of Technology, is now Research Assistant at the University of California, Los Alamos Scientific Laboratory.

Dr. W. C. Swift, Bell Telephone Laboratories, Murray Hill, New Jersey, has been appointed Assistant Professor at Rutgers, The State University.

Associate Professor Choy-tak Taam, The Catholic University of America, has been appointed Professor at Georgetown University.

Mr. Peter Terwey, Jr., on leave from Lamar State College of Technology, is at the Agricultural and Mechanical College of Texas on a National Science Foundation Science Faculty Fellowship.

Mr. R. J. Thomas, University of Illinois, has been appointed Instructor at DePauw University.

Assistant Professor Nura D. Turner, State University of New York, College for Teachers at Albany, has been promoted to Associate Professor.

Mr. R. G. Vinson, University of Tennessee, has been appointed Instructor at the University of Alabama.

Dr. Vaughan Weston, Defense Research Board, Toronto, Canada, is now Research Associate with the Radiation Laboratory, University of Michigan.

Mrs. L. M. Wheeler, Astoria High School, Astoria, Oregon, is now a mathematics teacher at Edmonds Senior High School, Edmonds, Washington.

Professor Leo Zippin, Queens College, has been appointed Professor of Mathematics at Yeshiva University's newly-created Graduate School of Mathematics.

Professor Emeritus Harry Birchenough, State University of New York, College for Teachers at Albany, died on August 20, 1958 at the age of 74. He was a charter member of the Association.

Miss Patience B. Klopp, University of Connecticut, Hartford Branch, died July 22, 1958.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The seventeenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Hofstra College, Hempstead, Long Island, April 19, 1958. Dr. J. T. Johnson, Vice-President of Hofstra College gave the address of welcome. Professor E. R. Stabler, Vice-Chairman for Colleges, chaired the morning session which was devoted to papers in applied mathematics. In the afternoon,

Dr. R. N. Walter, Vice-Chairman for High Schools, was moderator for the panel discussion on the topic "What a High School Teacher Should Know About Subjects and Techniques." There were 110 persons in attendance, including 81 members of the Association.

Professor J. N. Eastham, Chairman of the Section, presided at the business meeting. Reports were given by the Governor, Professor Jewell H. Bushey, the Treasurer, Mr. Aaron Shapiro, and the Committee on Contests and Awards. Professor E. R. Stabler resigned his office of Vice-Chairman for Colleges; no appointment was made at the meeting to fill the vacancy.

A resolution, proposed by the Treasurer, to table an amendment to the By-Laws to raise the dues of the Section to one dollar was adopted.

A resolution that a Speaker's Bureau composed of college and high school teachers be established by the Section to furnish speakers to high school mathematics clubs was proposed by the Chairman. The members of the Section approved the establishment of the Bureau on an experimental basis for one year.

It was moved and unanimously carried that a letter be addressed to the New York State Commissioner of Education (1) citing the article "New Requirements for Certification of High School Teachers" by Dr. E. K. Fretwell in the New York State Mathematics Teachers' Journal, Vol. 8, No. 2, April 1958 and (2) requesting that the two sections of The Mathematical Association of America, The Metropolitan New York Section and The Upper New York Section, be invited to present its views at all hearings concerning certification requirements of High School Mathematics Teachers in New York State.

The following papers were presented at the meeting:

1. *The nature of applied mathematics*, by Professor J. B. Keller, Institute of Mathematical Sciences, New York University, introduced by the Secretary.

Applied mathematics was defined as that science of which mathematics is a branch. This viewpoint was supported by historical evidence. Of the three main subdivisions of mathematics, it was pointed out that analysis is most important in applications. Then it was explained why an applied mathematician must know the field of application in which he works and why he must formulate the problems himself. The significance of properly-posed problems was described next and examples of such problems were presented. Once a problem is formulated and found to be properly posed, its properties must be deduced. Finally, expressions for the solution must be found.

2. *The mathematics of operations research*, by Dr. A. W. Jones, Bell Telephone Laboratories, New York City.

Papers 3, 4, and 5 were given in the panel discussion on the topic "What a High School Teacher Should Know About Subjects and Techniques."

3. *Commission on Mathematics of the College Entrance Examination Board*, by Professor M. F. Rosskopf, Teachers College, Columbia University.

The program of the Commission on Mathematics of the College Entrance Examination Board is not a radical program. However, it does imply that secondary school teachers will need to know something about mathematics that may not have been included in their preparatory program. The Commission program does use the language of sets and vectors, and does suggest that probability and statistics be included in the college preparatory program. Much emphasis is put upon clear, precise, and careful use of mathematical language. The Commission report will include not only some discussion of the way in which the program can be put into effect but also will include some suggested classroom units.

4. Panel Discussion, by Mr. Frank Hawthorne, Supervisor of Mathematics Education, The State Education Department, Albany.

Certain conditions are necessary. These include a knowledge of mathematics beyond anything he will be required to teach, an interest in and love for his subject, a knowledge of some of the related fields and a real interest in young people and their intellectual growth. Among the desirable conditions are: a knowledge of the usual sequence in analysis, certainly through the calculus; at least one college course in synthetic geometries and one in statistics. The most valuable "pedagogical" work is probably practice teaching under a skilled and experienced teacher. The existence of sufficient conditions is denied.

5. Panel Discussion, by Mrs. Roxee W. Joly, President, Mathematics Chairmen's Association of New York City, introduced by the Secretary.

As chairman of a mathematics department in a large high school, Mrs. Joly based her ideas on desirable factors of subject and technique which new teachers did not know, even though they had a regular mathematics major in college and graduate school. One neglected area was awareness of actual new syllabi, their philosophy and content. Another serious need was ability to prepare Key Lesson Plans to make the subject alive and exciting. All should take Professionalized Subject matter; a laboratory in Math Recreations, clubs, teams, assemblies; a course tying the anecdotes of history of mathematics to units in the secondary school syllabi; observation and teaching in schools where a dynamic, carefully-selected leader can help them grow in all these directions.

AZELLE B. WALTCHER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Valparaiso University, May 2, 1959.

IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.

KANSAS

KENTUCKY, Centre College of Kentucky, Danville, April, 1959.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 6, 1958.

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University of Agriculture and Applied Science, East Lansing, March 28, 1959.

MINNESOTA

MISSOURI, Lindenwood College, St. Charles, April 25, 1959.

NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, Stanford University, January 17, 1959.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.

PHILADELPHIA

ROCKY MOUNTAIN, Utah State University of Agriculture and Applied Science, Logan, May 8–9, 1959.

SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.

SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.

SOUTHWESTERN, Arizona State College, Tempe, Spring, 1959.

TEXAS, University of Texas, Austin, April, 1959.

UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

WISCONSIN, Wisconsin State College, Platteville, May 2, 1959.

ACKNOWLEDGEMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: C. B. Allendoerfer, Emil Artin, Howard Campbell, L. Cesari, R. R. Christian, R. V. Churchill, Paul Civin, N. A. Court, H. F. Davis, D. Derry, D. J. Dickinson, Trevor Evans, G. E. Forsythe, J. E. Freund, C. Froese, H. W. Gould, Leon Henkin, E. Hewitt, S. A. Jennings, M. S. Klamkin, Emma Lehmer, E. Leimanis, A. E. Livingston, N. Macon, N. S. Mendelsohn, Karl Menger, Richard Montague, L. Moser, B. N. Moyls, M. E. Munroe, S. W. Nash, R. A. Restrepo, R. A. Rosenbaum, I. J. Schoenberg, Nathan Schwid, D. E. Spencer, D. J. Struik, Olga Taussky-Todd, H. A. Thurston, L. Tornheim, R. M. Winger, R. J. Wisner, H. J. Zassenhaus.

Mathematical Notes: T. M. Apostol, J. H. Barrett, R. A. Beaumont, R. E. Bellman, R. P. Boas, Samuel Borofsky, Louis Brand, S. J. Bryant, Leonard Carlitz, L. R. Ford, W. A. Guenther, Frank Harary, Edwin Hewitt, V. E. Hoggatt, B. W. Jones, T. C. Kipps, M. S. Klamkin, R. E. Langer, D. H. Lehmer, Harold Levine, D. C. Murdoch, C. D. Olds, Sam Perlis, E. R. Rainville, W. T. Reid, Herbert Robbins, Walter Rudin, R. G. Selfridge, D. J. Struik, G. Szegö, A. E. Taylor, Leonard Tornheim, M. S. Webster, H. S. Wilf, Peter Yff.

Classroom Notes: A. N. Aheart, Michael Aissen, H. R. Beverage, J. C. Brixey, J. W. Cell, S. Chowla, A. B. Coble, W. W. Comfort, N. A. Court, Nathan Fine, Philip Franklin, Michael Goldberg, Morton Hellman, Fritz Herzog, A. O. Hickson, Robert Jackson, Paul Johnson, P. W. Ketchum, J. M. Kingston, T. L. Koehler, John Lamperti, Mary Landers, Rose Lariviere, Walter Leighton, Aaron Lemonick, R. G. Long, V. O. McBrien, Neal McCoy, W. T. Moore, A. A. Mullin, M. E. Munroe, D. A. Norton, Roger Osborn, Hans Rademacher, Earl Rainville, Rothwell Stephens, G. R. Strohl, W. J. Swartz, Marvin Tomber, D. H. Wagner, G. C. Webber, A. H. Wilson, R. J. Wisner, J. W. Woll.

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GENERAL MATHEMATICAL PAPERS

ALGEBRA, NUMBER THEORY

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.